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HARMONIC ANALYSIS ON MEASURES SPACES ATTACHED TO SOME DUNKL OPERATORS ON $\mathbb{R}^d$ AND APPLICATIONS

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Abstract. In this paper we define and study the Dunkl transform and the Dunkl convolution product on measures spaces associated to some Dunkl operators on $\mathbb{R}^d$. Next, as applications of the results obtained, we prove a Levy's continuity theorem and central limit theorem for probability measures on $\mathbb{R}^d$.

1. Introduction

In this paper we consider the root system $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ given by

$$\mathcal{R} = \{\pm\alpha_i, \pm 2\alpha_i, i = 1, 2, ..., d\},$$

with

$$\mathcal{R}_+ = \{\alpha_i, 2\alpha_i, i = 1, 2, ..., d\},$$

the set of positive roots. We suppose that the vectors $\alpha_1, \alpha_2, ..., \alpha_d$ are linearly independent, and the multiplicity function k associated with $\mathcal{R}$ satisfies

$$\forall \alpha \in \mathcal{R}_+, \quad k(\alpha) + k(2\alpha) > 0.$$

We consider the Dunkl operators attached to this root system. We define the eigenfunction $K(x, \lambda)$, $x \in \mathbb{R}^d$, $\lambda \in \mathbb{C}^d$ of these operators, called Dunkl kernel, and give its properties. The results obtained have permitted to define and study the Dunkl transform and the Dunkl convolution product on spaces of functions and measures on $\mathbb{R}^d$. By using this harmonic analysis, we establish a Levy's continuity theorem and a central limit theorem for probability measures on $\mathbb{R}^d$.

2. The Dunkl operators on $\mathbb{R}^d$ and their Dunkl kernel (see [11])

We consider $\mathbb{R}^d$ with the standard basis $\{e_1, e_2, ..., e_d\}$, and the inner product $\langle , \rangle$ for which this basis is orthonormal.

We take the root system $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ given by

$$\mathcal{R} = \{\pm\alpha_i, \pm 2\alpha_i, i = 1, 2, ..., d\}, \tag{2.1}$$

with

$$\mathcal{R}_+ = \{\alpha_i, 2\alpha_i, i = 1, 2, ..., d\}. \tag{2.2}$$

For $\alpha \in \mathcal{R}$, we consider

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$$r_\alpha(x) = x - \langle \breve{\alpha}, x\rangle\alpha, \quad \text{with } \breve{\alpha} = \frac{2}{||\alpha||^2}\alpha, \tag{2.3}$$

the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α. The reflections $r_\alpha, \alpha \in \mathcal{R}$ generate a finite group $W \subset O(d)$ called the Weyl group associated with $\mathcal{R}$.

The multiplicity function associated with $\mathcal{R}$ is a function $k : \mathcal{R} \to]0, +\infty[$, which is invariant under the action of the group W.

Let $\mathbb{R}^d_{reg} = \mathbb{R}^d \backslash \cup_{\alpha\in\mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha). \tag{2.4}$$

Let ω_k denote the weight function

$$\forall x \in \mathbb{R}^d, \quad \omega_k(x) = \prod_{\alpha\in\mathcal{R}_+} |\langle\alpha, x\rangle|^{2k(\alpha)}. \tag{2.5}$$

This function can also be written in the form

$$\forall x \in \mathbb{R}^d, \quad \omega_k(x) = (\prod_{i=1}^{d} 2^{2k(2\alpha_i)}) \prod_{i=1}^{d} |\langle\alpha, x\rangle|^{2(k(\alpha_i)+k(2\alpha_i))}. \tag{2.6}$$

2.1. **The Dunkl operators.**

The Dunkl operators $\mathcal{T}_j, j = 1, 2, ..., d$ on $\mathbb{R}^d$ associated to the root system $\mathcal{R}$, the reflection group W and the multiplicity function k, are defined for f of class C^1 on $\mathbb{R}^d$ and $x \in \mathbb{R}^d_{reg}$ by

$$\mathcal{T}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in\mathcal{R}_+} \frac{k(\alpha)\alpha^j}{\langle\alpha, x\rangle}\{f(x) - f(r_\alpha x)\}, \tag{2.7}$$

where

$$\alpha^j = \langle\alpha, e_j\rangle. \tag{2.8}$$

By using the fact that from (2.3) we have

$$\forall i \in \{1, 2, ..., d\}, \quad r_{2\alpha_i} x = r_{\alpha_i} x,$$

we obtain

$$\mathcal{T}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{i=1}^{d} \frac{(k(\alpha_i) + k(2\alpha_i))\alpha^j}{\langle\alpha_i, x\rangle}\{f(x) - f(r_{\alpha_i} x)\}. \tag{2.9}$$

The Dunkl operators form a commutative system of differential-difference operators.

We like to emphasize on some studies on the commutants and nonlocal operational calculi for the Dunkl operator ($d = 1$, see $\mathcal{T}_1$, (2.11)) by Professor Ivan Dimovski et al., as for example [1], [2], etc.

2.2. **The Dunkl kernel.**

We denote by $K(x, \lambda)$, $x \in \mathbb{R}^d$, $\lambda \in \mathbb{C}^d$ the eigenfunction of the operators $\mathcal{T}_j, j = 1, 2, ..., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} \mathcal{T}_j K(x, \lambda) & = -i\lambda_j K(x, \lambda), \quad j = 1, 2, ..., d, \quad x \in \mathbb{R}^d, \\ K(0, \lambda) & = 1. \end{cases} \tag{2.10}$$

It is called the Dunkl kernel.

Example 2.1. We consider for $d = 1$ the root system $\mathcal{R} = \{\pm\alpha, \pm 2\alpha\}$ with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$ and the reflection group is $W = \mathbb{Z}_2$. The multiplicity function ia a single parameter denoted by k.

The Dunkl operator T_1 is defined for f of class C^1 on $\mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$ by

$$\mathcal{T}_1 f(x) = \frac{d}{dx} f(x) + \frac{k}{x}(f(x) - f(-x)). \tag{2.11}$$

The Dunkl kernel is given by

$$K(x, \lambda) = j_{k-\frac{1}{2}}(\lambda x) + \frac{i\lambda x}{2k+1} j_{k+\frac{1}{2}}(\lambda x), \tag{2.12}$$

where for $\beta > -\frac{1}{2}$, and j_β is the normalized Bessel function defined by

$$j_\beta(z) = 2^\beta \Gamma(\beta + 1) \frac{J_\beta(z)}{z^\beta} = \Gamma(\beta + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(\beta + n + 1)}, \tag{2.13}$$

with J_β being the Bessel function of first kind and of index β (see [11]).

2.3. **Properties of the Dunkl kernel.**
The function $K(x, \lambda)$ satisfies the following properties:
i) For all $\lambda \in \mathbb{C}^d$ the function $x \longmapsto K(x, \lambda)$ is of class C^∞ on $\mathbb{R}^d$.
For all $x \in \mathbb{R}^d$ the function $\lambda \longmapsto K(x, \lambda)$ is entire on $\mathbb{C}^d$.
ii) For all $x \in \mathbb{R}^d$, $\lambda \in \mathbb{C}^d$ and $c \in \mathbb{R}$, we have

$$K(x, \lambda) = K(\lambda, x), \tag{2.14}$$

and

$$K(cx, \lambda) = K(x, c\lambda). \tag{2.15}$$

iii) For all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\overline{K(x, \lambda)} = K(x, -\overline{\lambda}). \tag{2.16}$$

iv) For all $x \in \mathbb{R}^d$, we have

$$K(x, 0) = 1. \tag{2.17}$$

v) For all $\lambda \in \mathbb{R}^d$, we have

$$\sup_{x \in \mathbb{R}^d} K(x, \lambda) = 1. \tag{2.18}$$

vi) The function $K(x, \lambda)$, $\lambda \in \mathbb{C}^d$, admits the following Laplace type representations

$$1)\ \forall x \in \mathbb{R}^d, \quad K(x, \lambda) = \int_{\mathbb{R}^d} e^{-i\langle \lambda, y \rangle} d\mu_x^D(y), \tag{2.19}$$

where μ_x^D is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(0, ||x||)$ of center 0 and radius $||x||$ (see [7]):

$$2)\ \forall x \in \mathbb{R}^d_{reg}, \quad K(x, \lambda) = \int_{\mathbb{R}^d} \mathcal{K}^D(x, y) e^{-i\langle \lambda, y \rangle} dy, \tag{2.20}$$

where $\mathcal{K}(x, y)$ is a positive integrable function on $\mathbb{R}^d$, with respect to the Lebesgue measure on $\mathbb{R}^d$, and we have

$$\forall x \in \mathbb{R}^d_{reg}, \quad \int_{\mathbb{R}^d} \mathcal{K}(x, y) dy = 1, \tag{2.21}$$

(see [13]).

Proposition 2.1. *The function $K(x, \lambda)$, $\lambda \in \mathbb{R}^d$, possesses the following product formulas:*

$$1)\ \forall x, u \in \mathbb{R}^d, \quad K(x, \lambda) K(u, \lambda) = \int_{\mathbb{R}^d} K(z, \lambda) d\eta^D_{x,u}(z), \tag{2.22}$$

where $\eta^D_{x,u}$ is a probability measure on $\mathbb{R}^d$, with support contained in the set

$$\{z \in \mathbb{R}^d,\ |||x|| - ||u||| \leq ||z|| \leq ||x|| + ||u||\};$$

$$2)\ \forall x, u \in \mathbb{R}^d_{reg}, \quad K(x, \lambda) K(u, \lambda) = \int_{\mathbb{R}^d} K(z, \lambda) \nu^D(x, u, z) \omega_k(z) dz, \tag{2.23}$$

where $\nu^D(x,u,z)$ is a positive integrable function on $\mathbb{R}^d$, with respect to the measure $\omega_k(z)dz$, and we have

$$\int_{\mathbb{R}^d} \nu^D(x,u,z)\omega_k(z)dz = 1.$$

Proof.

1) By using the relation

$$\lim_{\epsilon\to 0} G_{\epsilon^{-1}\lambda}(\epsilon x) = K(x,\lambda),$$

with $G_\lambda(x)$ being the Opdam-Cherednik kernel, and the product formula for the function $G_\lambda(x)$ (see [9],[17]), and by using the same arguments as in the proof of the result given at the end of the paper [8], we obtain the relation (2.22).

2) By using the relation (2.22) and by applying the method given in the paper [14], we deduce the relation (2.23).

Remark 2.1.

1) We obtain the relations (2.19),(2.20) by using the relations (4.23),(4.29),(5.16) of [15] and by applying a proof analogous to the proof of Proposition 2.1.

2) By using [16] and the proof of Proposition 2.1, we obtain the product formulas (2.22),(2.23) for the Dunkl kernel associated to the root system of type BC_2.

3. The Dunkl intertwining operator and the Dunkl transform of functions

Notation. We denote by
- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.
- $\mathcal{D}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$, with compact support.

We equip these spaces with the classical topology.
- $PW_a(\mathbb{C}^d)$, $a>0$, the space of entire functions g on $\mathbb{C}^d$ satisfying

$$\forall m \in \mathbb{N},\quad q_m(f) = \sup_{\lambda\in\mathbb{C}^d}(1+||\lambda||)^m e^{-a||\Im m\lambda||}|f(\lambda)| < +\infty.$$

The topology of $PW_a(\mathbb{C}^d)$ is defined by the semi-norms $q_m, m\in\mathbb{N}$.

- The space $PW(\mathbb{C}^d) = \cup_{a>0} PW_a(\mathbb{C}^d)$. is called the Paley-Wiener space. It is equipped with the inductive limit topology.

- $L^p_k(\mathbb{R}^d)$, $p=1,\infty$, the spaces of measurable functions f on $\mathbb{R}^d$ satisfying

$$\begin{aligned}\|f\|_{k,1} &= \int_{\mathbb{R}^d}|f(x)|\omega_k(x)dx < +\infty,\\ \|f\|_{k,\infty} &= ess\sup_{x\in\mathbb{R}^d}|f(x)| < +\infty.\end{aligned}$$

3.1. The Dunkl intertwining operator V_k^D (see [11]).

Definition 3.1. The Dunkl intertwining operator V_k^D is defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall\, x\in\mathbb{R}^d,\; V_k^D(f)(x) = \int_{\mathbb{R}^d} f(y)d\mu_x^D(y), \tag{3.1}$$

where μ_x^D is the measure given by the relation (2.19).

Remark 3.1. From the relations (2.19) and (3.1) we obtain

$$\forall\, x\in\mathbb{R}^d,\;\forall\,\lambda\in\mathbb{C}^d, K(x,\lambda) = V_k^D(e^{-i\langle\lambda,.\rangle})(x). \tag{3.2}$$

Theorem 3.1. *The operator V_k^D is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the permutation relations*

$$\forall\, x \in \mathbb{R}^d,\ \mathcal{T}_j V_k^D(f)(x) = V_k^D(\frac{\partial}{\partial x_j} f)(x),\ j = 1, 2, ..., d, \quad f \in \mathcal{E}(\mathbb{R}^d), \tag{3.3}$$

$$V_k^D(f)(0) = f(0). \tag{3.4}$$

3.2. **The Dunkl transform of functions (see [12]).**

Definition 3.2. The Dunkl transform of a function f in $L_k^1(\mathbb{R}^d)$ is given by

$$\forall\, \lambda \in \mathbb{R}^d,\ \mathcal{F}_D(f)(\lambda) = \int_{\mathbb{R}^d} f(x)K(x,\lambda)\omega_k(x)dx. \tag{3.5}$$

Remark 3.2. For f in $L_k^1(\mathbb{R}^d)^W$ we have

$$||\mathcal{F}_D(f)||_{k,\infty} \leq ||f||_{k,1}. \tag{3.6}$$

Theorem 3.2. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism, from $\mathcal{D}(\mathbb{R}^d)$ onto $PW(\mathbb{C}^d)$. The inverse transform is given by*

$$\forall\, x \in \mathbb{R}^d, \quad (\mathcal{F}_D)^{-1}(g)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} g(\lambda)K(x,-\lambda)\omega_k(\lambda)d\lambda, \tag{3.7}$$

where c_k is the Mehta-type constant

$$c_k = \Big(\int_{\mathbb{R}^d} e^{-||x||^2}\omega_k(x)dx\Big)^{-1}.$$

Theorem 3.3. *Let f be in $L_k^1(\mathbb{R}^d)$ such that $\mathcal{F}_D(f)$ belongs to $L_k^1(\mathbb{R}^d)$. Then we have the following inversion formula for the transform $\mathcal{F}_D$:*

$$f(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(\lambda)K(x,-\lambda)\omega_k(\lambda)d\lambda, \quad a.e. \tag{3.8}$$

4. The Dunkl translation operator and the Dunkl convolution product of functions

4.1. **The Dunkl translation operator τ_x (see [12]).**

Definition 4.1. The Dunkl translation operator τ_x, $x \in \mathbb{R}^d$, is defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\tau_x(f)(y) = (V_k)_x(V_k)_y[(V_k)^{-1}(f)(x+y)]. \tag{4.1}$$

Proposition 4.1. *The operator $\tau_x, x \in \mathbb{R}^d$, satisfies the following properties:*

i) For all $x \in \mathbb{R}^d$, the operator τ_x is continuous from $\mathcal{E}(\mathbb{R}^d)$ into itself.

ii) For all $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have the product formula

$$\tau_x K(y,\lambda) = K(x,\lambda)K(y,\lambda). \tag{4.2}$$

iii) For all f in $\mathcal{E}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\tau_x(f)(0) = f(x) \quad and \quad \tau_x(f)(y) = \tau_y(f)(x). \tag{4.3}$$

iv) For all f, g in $\mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \tau_x(f)(y)g(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\tau_{-x}(g)(y)\omega_k(y)dy. \tag{4.4}$$

Proposition 4.2. *For all f in $\mathcal{E}(\mathbb{R}^d)$ the Dunkl translation operator τ_x admits the integral representations:*

1) $$\forall\, x, y \in \mathbb{R}^d, \tau_x(f)(y) = \int_{\mathbb{R}^d} f(z)d\eta^D_{x,y}(z), \tag{4.5}$$

where $\eta^D_{x,y}$ is the measure given in Proposition 2.1;

2) $$\forall\, x, y \in \mathbb{R}^d, \tau_x(f)(y) = \int_{\mathbb{R}^d} f(z)\mathcal{V}^D(x, y, z)\omega_k(z)dz, \tag{4.6}$$

where $\mathcal{V}^D(x, y, .)$ is the function given in Proposition 2.1.

Proposition 4.3. *For all f in $\mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $y \mapsto \tau_x(f)(y)$ belongs to $\mathcal{D}(\mathbb{R}^d)$ and we have*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\tau_x f)(\lambda) = K(x, \lambda)\mathcal{F}_D(f)(\lambda). \tag{4.7}$$

Proposition 4.4. *The Gauss kernel α_t, $t > 0$, defined by*

$$\forall\, x \in \mathbb{R}^d, \quad \alpha_t(x) = \frac{c_k}{(4t)^{\gamma+\frac{d}{2}}} e^{-\frac{||x||^2}{4t}}, \tag{4.8}$$

satisfies the following properties:

i) The function $(x, t) \mapsto \alpha_t(x)$ is a solution of the heat equation:

$$\frac{\partial u}{\partial t}(x, t) - \triangle_k u(x, t) = 0, \quad on \ \ \mathbb{R}^d \times (0, \infty), \tag{4.9}$$

where $\triangle_k$ is the Dunkl Laplacian given by

$$\triangle_k = \sum_{j=1}^{d} \mathcal{T}_j^2. \tag{4.10}$$

ii) We have

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\alpha_t)(\lambda) = e^{-t||\lambda||^2}. \tag{4.11}$$

iii) We have

$$\forall\, t > 0, \quad \int_{\mathbb{R}^d} \alpha_t(x)\omega_k(x)dx = 1. \tag{4.12}$$

4.2. **The Dunkl convolution product of functions (see [12]).**

Definition 4.2. The Dunkl convolution product of two functions f and g of $D(\mathbb{R}^d)$ is the function $f *_D g$ defined by

$$\forall\, x \in \mathbb{R}^d, \ f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)\omega_k(y)dy. \tag{4.13}$$

Theorem 4.1.

i) The convolution product $*_D$ is commutative and associative.

ii) For all f, g in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(f *_D g)(\lambda) = \mathcal{F}_D(f)(\lambda)\mathcal{F}_D(g)(\lambda). \tag{4.14}$$

5. The Dunkl transform and the Dunkl convolution product of measures

Notations. We denote by

-$C_K(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$, with compact support.

-$C_b(\mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d$.

We equip these spaces with their classical topology.

-$\mathcal{M}_b(\mathbb{R}^d)$ the space of positive bounded Borel measures on $\mathbb{R}^d$.

-$\mathcal{M}^1(\mathbb{R}^d)$ the subset of probability measures on $\mathbb{R}^d$.

5.1. **The Dunkl transform of measures.**

Definition 5.1. The Dunkl transform of a measure η in $\mathcal{M}_b(\mathbb{R}^d)$ is the function $\mathcal{F}_D(\eta)$ given by

$$\forall\, \lambda \in \mathbb{R}^d,\ \mathcal{F}_D(\eta)(\lambda) = \int_{\mathbb{R}^d} f(x)K(x,\lambda)d\eta(x). \tag{5.1}$$

Proposition 5.1.

i) For η in $\mathcal{M}_b(\mathbb{R}^d)$, the function $\mathcal{F}_D(\eta)$ is continuous on $\mathbb{R}^d$.

ii) For η in $\mathcal{M}_b(\mathbb{R}^d)$, we have

$$\forall\, \lambda \in \mathbb{R}^d,\ |\mathcal{F}_D(\eta)| \leq ||\eta||. \tag{5.2}$$

Proof.

i) For all $x \in \mathbb{R}^d$, the function $\lambda \mapsto K(x,\lambda)$ is continuous on $\mathbb{R}^d$ and from the relation (2.18) it satisfies

$$\forall\, \lambda \in \mathbb{R}^d, \quad |K(x,\lambda)| \leq 1.$$

Then the dominated convergence theorem implies the continuity of the function $\mathcal{F}_D(\eta)(\lambda)$ on $\mathbb{R}^d$.

ii) We deduce (5.2) from (5.1) and (2.18).

Proposition 5.2. *Let η, ν be two measures in $\mathcal{M}_b(\mathbb{R}^d)$ such that*

$$\forall\, \lambda \in \mathbb{R}^d,\ \mathcal{F}_D(\eta)(\lambda) = \mathcal{F}_D(\nu)(\lambda). \tag{5.3}$$

Then

$$\eta = \nu. \tag{5.4}$$

Proof. We denote by σ the measure of $\mathcal{M}_b(\mathbb{R}^d)$ given by

$$\sigma = \eta - \nu.$$

We have

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\sigma)(\lambda) = 0. \tag{5.5}$$

Let f be in $\mathcal{D}(\mathbb{R}^d)$. From Theorem 3.2 we have

$$\forall\, x \in \mathbb{R}^d, \quad f(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(\lambda)K(x,-\lambda)\omega_k(\lambda)d\lambda.$$

This relation can also be written in the form

$$\forall\, x \in \mathbb{R}^d, \quad f(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(-\lambda)K(x,\lambda)\omega_k(\lambda)d\lambda. \tag{5.6}$$

By using the relations (5.1), (5.6), (5.5), and Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)d\sigma(x) &= \int_{R^d}(\int_{\mathbb{R}^d} \mathcal{F}_D(f)(-\lambda)K(x,\lambda)\omega_k(\lambda)d\lambda)d\sigma(x)\\ &= \int_{R^d} \mathcal{F}_D(f)(-\lambda)\mathcal{F}_D(\sigma)(\lambda)\omega_k(\lambda)d\lambda = 0.\end{aligned}$$

Thus for all $f \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} f(x)d\sigma(x) = 0.$$

Then

$$\sigma = 0 \Leftrightarrow \eta = \nu.$$

5.2. **The Dunkl convolution product of measures.**

Definition 5.2. The Dunkl convolution product $\eta *_D \nu$ of the two measures η and ν of $\mathcal{M}_b(\mathbb{R}^d)$ is defined by

$$\eta *_D \nu(f) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tau_x f(y)d\eta(x)d\nu(y), \quad f \in \mathcal{D}(\mathbb{R}^d). \tag{5.7}$$

In the following we give some properties of the Dunkl convolution product $*_D$ of measures.

Proposition 5.3.
i) We have

$$\forall\, x, y \in \mathbb{R}^d, \quad \delta_x *_D \delta_y(f) = \tau_x f(y), \tag{5.8}$$

where δ_x is the Dirac measure at the point x.
ii) Let η, ν be in $\mathcal{M}_b(\mathbb{R}^d)$. Then the measure $\eta *_D \nu$ belongs to $\mathcal{M}_b(\mathbb{R}^d)$ and we have

$$||\eta *_D \nu|| \leq ||\eta||\, ||\nu||. \tag{5.9}$$

For all η, ν in $\mathcal{M}^1(\mathbb{R}^d)$, the measure $\eta *_D \nu$ belongs to $\mathcal{M}^1(\mathbb{R}^d)$.

Proposition 5.4. *Let η, ν be in $\mathcal{M}_b(\mathbb{R}^d)$. Then we have*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\eta *_D \nu)(\lambda) = \mathcal{F}_D(\eta)(\lambda)\mathcal{F}_D(\nu)(\lambda). \tag{5.10}$$

Proof. We deduce (5.10) from the relations (5.7),(5.1),(4.2).

Corollary 5.1. *The Dunkl convolution product $*_D$ of measures is commutative and associative.*
Proof. We deduce the results of this corollary from Proposition 5.4.

6. Central limit theorem for probability measures on $\mathbb{R}^d$

6.1. **Levy's continuity theorem for propability measures.**

Proposition 6.1. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M}^1(\mathbb{R}^d)$, which converges vaguely to the measure μ of $\mathcal{M}_b(\mathbb{R}^d)$. Then*

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \lim_{n\to\infty} \mathcal{F}_D(\mu_n)(\lambda) = \mathcal{F}_D(\mu)(\lambda). \tag{6.1}$$

Proof. From the relations (2.14), (2.20) we have

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad K(\lambda, x) = \int_{\mathbb{R}^d} \mathcal{K}^D(\lambda, y) e^{-i\langle x,y\rangle} dy. \tag{6.2}$$

As the function $y \to \mathcal{K}^D(\lambda, y)$ is integrable on $\mathbb{R}^d$, with respect to the Lebesgue measure, then we have

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \lim_{x\to\infty} K(\lambda, x) = 0.$$

Thus

$$\forall\, \varepsilon > 0,\ \exists\, A > 0 / ||x|| \geq A \Rightarrow \quad |K(\lambda, x)| < \varepsilon. \tag{6.3}$$

Let a function ψ in $C_K(\mathbb{R}^d)$ such that $\psi(x) = 1$ for all x in the closed ball $B(0, A)$ of center 0 and radius A, and $|\psi(x)| \leq 1$, for all x in the complementary $B^c(0, A)$ of the ball $B(0, A)$.

Then for all $\lambda \in \mathbb{R}^d_{reg}$, we have

$$|\mathcal{F}_D(\mu_n)(\lambda) - \mathcal{F}_D(\mu)(\lambda)| \leq |\int_{B(0,A)} K(\lambda, x)\psi(x) d\mu_n(x) - \int_{B(0,A)} K(\lambda, x)\psi(x) d\mu(x)| \leq \int_{B^c(0,A)} |K(\lambda, x)(1 - \psi(x))| d\mu_n(x) + \int_{B^c(0,A)} |K(\lambda, x)(1 - \psi(x))| d\mu(x).$$

By using the relation (6.3), we obtain

$$|\mathcal{F}_D(\mu_n)(\lambda) - \mathcal{F}_D(\mu)(\lambda)| \leq |\int_{B(0,A)} K(\lambda, x)\psi(x) d\mu_n(x) - \int_{B(0,A)} K(\lambda, x)\psi(x) d\mu(x)| + 2\varepsilon(1 + \int_{B^c(0,A)} d\mu(x)).$$

We deduce (6.1) from this relation and the fact that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges vaguely to the measure μ.

Theorem 6.1. (Levy's continuity theorem) *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M}^1(\mathbb{R}^d)$ such that*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \mathcal{F}_D(\mu_n)(\lambda) = f(\lambda), \tag{6.4}$$

where f is a function on $\mathbb{R}^d$. Then the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges vaguely to a nonnegative measure $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, with mass not larger than 1, satisfying

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu)(\lambda) = f(\lambda). \tag{6.5}$$

Furthermore if μ is a probability measure, then the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges weakly to the measure μ.

Proof. We extract from the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ a subsequence $\{\mu_{n_j}\}_{n_j\in\mathbb{N}}$ which converges vaguely to a nonnegative measure μ, with mass not larger than 1. From Proposition 6.1, we have

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \lim_{n_j\to\infty} \mathcal{F}_D(\mu_{n_j})(\lambda) = \mathcal{F}_D(\mu)(\lambda).$$

This relation and (6.4) imply

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \mathcal{F}_D(\mu)(\lambda) = f(\lambda). \tag{6.6}$$

Let $\{\mu_{n'_j}\}_{n'_j\in\mathbb{N}}$ be another subsequence of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ which converges vaguely to a nonnegative measure μ', with mass not larger than 1. As for the measure μ, we obtain for the measure μ' the following relation

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \mathcal{F}_D(\mu')(\lambda) = f(\lambda). \tag{6.7}$$

Then by using (6.6),(6.7), we obtain

$$\forall\, \lambda \in \mathbb{R}^d_{reg}, \quad \mathcal{F}_D(\mu)(\lambda) = \mathcal{F}_D(\mu')(\lambda).$$

As the functions $\mathcal{F}_D(\mu)$ and $\mathcal{F}_D(\mu')$ are continuous on $\mathbb{R}^d$, then

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu)(\lambda) = \mathcal{F}_D(\mu')(\lambda).$$

This result and Proposition 5.2 imply that

$$\mu = \mu'.$$

Then, as all subsequence of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ has the same limit μ, this result shows that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges vaguely to the measure μ, and we have

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu)(\lambda) = f(\lambda).$$

If, furthermore the measure μ is a probability measure, then the vague convergence will be a weak convergence.

6.2. **Dispersions of probability measures on $\mathbb{R}^d$.**

Proposition 6.2.

1. For all $x, \lambda \in \mathbb{R}^d$, we have

$$\begin{aligned} K(x,\lambda) &= 1 - \sum_{p=1}^{d} \lambda_p \frac{\partial}{\partial \lambda_p} K(x,\lambda)_{|\lambda=0} + \sum_{p=1}^{d} \lambda_p^2 \frac{\partial}{\partial \lambda_p^2} K(x,\lambda)_{|\lambda=0} \\ &+ \sum_{q=1}^{d-1}\sum_{p=1}^{d-q} \lambda_p \lambda_{p+q} \frac{\partial^2}{\partial \lambda_p \partial \lambda_{p+q}} K(x,\lambda)_{|\lambda=0} + R(x,\lambda), \end{aligned} \tag{6.8}$$

with

$$R(x,\lambda) = \frac{i}{6} \int_{B(0,||x||)} (\langle \lambda, y\rangle)^3 e^{-i\theta\langle \lambda, y\rangle} d\mu_x^D(y), \tag{6.9}$$

with $\theta \in (0,1)$.

2. We have

$$\forall\, x, \lambda \in \mathbb{R}^d, \quad |R(x,\lambda)| \leq \frac{||x||^3\, ||\lambda||^3}{6}. \tag{6.10}$$

Proof.

1. For all $x, y \in \mathbb{R}^d$, we have from Taylor's formula

$$e^{-i\langle \lambda, y\rangle} = 1 - i\langle \lambda, y\rangle - \frac{1}{2}(\langle \lambda, y\rangle)^2 + \frac{i}{6}(\langle \lambda, y\rangle)^3 e^{-i\theta\langle \lambda, y\rangle}, \;\; 0 < \theta < 1. \tag{6.11}$$

Thus

$$e^{-i\langle \lambda, y\rangle} = 1 - i(\sum_{p=1}^{d} \lambda_p y_p) - \frac{1}{2}(\sum_{p=1}^{d} \lambda_p y_p)^2 + \frac{i}{6}(\langle \lambda, y\rangle)^3 e^{-i\theta\langle \lambda, y\rangle}, \;\; 0 < \theta < 1. \tag{6.12}$$

But we have

$$(\sum_{p=1}^{d} \lambda_p y_p)^2 = \sum_{p=1}^{d} \lambda_p^2 y_p^2 + 2\sum_{q=1}^{d-1}(\sum_{p=1}^{d-q} \lambda_p \lambda_{p+q} y_p y_{p+q}). \tag{6.13}$$

By using the relations (6.10),(6.11),(6.12), we obtain

$$\begin{aligned} e^{-i\langle\lambda,y\rangle} &= 1-i\sum_{p=1}^{d}\lambda_p\Big(\frac{\partial}{\partial y_p}e^{-i\langle\lambda,y\rangle}\Big)_{/\lambda=0}-\frac{1}{2}\sum_{p=1}^{d}\lambda_p^2\Big(\frac{\partial^2}{\partial y_p^2}e^{-i\langle\lambda,y\rangle}\Big)_{/\lambda=0}\\ &+ 2\sum_{q=1}^{d-1}(\sum_{p=1}^{d-q}\lambda_p\lambda_{p+q}\Big(\frac{\partial^2 e^{-i\langle\lambda,y\rangle}}{\partial y_p\partial y_{p+q}}\Big)_{/\lambda=0}+\frac{i}{6}(\langle\lambda,y\rangle)^3e^{-i\theta\langle\lambda,y\rangle},\ 0<\theta<1. \end{aligned} \tag{6.14}$$

We deduce (6.8) from the relation (6.14), the following integral representation of the function $K(x,\lambda)$:

$$K(x,\lambda)=\int_{B(0,||x||)}e^{-i\langle\lambda,y\rangle}d\mu_x^D(y),$$

and by making derivations under the integral sign.

2. We obtain (6.10) from the relation (6.9) and the fact that the measure μ_x^D is a probability measure.

Definition 6.1. Let μ be in $\mathcal{M}^1(\mathbb{R}^d)$. We call dispersion of the measure μ the following quantities:

1. Dispersion of the first order

$$D_p^{(1)}(\mu)=\frac{\partial}{\partial y_p}\mathcal{F}_D(\mu)(\lambda)_{|\lambda=0},\quad p=1,...,d. \tag{6.15}$$

2. Dispersion of the second order

$$D_p^{(2)}(\mu)=-\frac{1}{2}\frac{\partial^2}{\partial y_p^2}\mathcal{F}_D(\mu)(\lambda)_{|\lambda=0},\quad p=1,...,d. \tag{6.16}$$

$$D_{p,q}^{(2)}(\mu)=-\frac{\partial^2}{\partial y_p\partial y_{p+q}}\mathcal{F}_D(\mu)(\lambda)_{|\lambda=0},\begin{cases} p=1,...,d-q,\\ q=1,...,d-1.\end{cases} \tag{6.17}$$

Proposition 6.3. *Let μ,ν be two measures in $\mathcal{M}^1(\mathbb{R}^d)$ such that*

$$D_p^{(1)}(\mu)=0\quad and\quad D_p^{(1)}(\nu)=0,\quad p=1,...,d. \tag{6.18}$$

Then

$$D_p^{(2)}(\eta *_D^W \nu)=D_p^{(2)}(\eta)+D_p^{(2)}(\nu),\quad p=1,...,d. \tag{6.19}$$

Proof. From the relation (5.10) we have

$$\forall\,\lambda\in\mathbb{R}^d,\quad \mathcal{F}_D(\eta *_D \nu)(\lambda)=\mathcal{F}_D(\eta)(\lambda)\mathcal{F}_D(\nu)(\lambda).$$

We deduce the relation (6.19) from this relation and the relations (6.16), (6,18).

Proposition 6.4. *The Gaussian distributions are the measures α_t, $t>0$, of $\mathcal{M}^1(\mathbb{R}^d)$ satisfying*

$$\forall\,\lambda\in\mathbb{R}^d,\quad \mathcal{F}_D(\alpha_t)(\lambda)=e^{-t||\lambda||^2}. \tag{6.20}$$

Remark 6.1. The Gaussian distributions are the measures α_t, $t>0$, of $\mathcal{M}^1(\mathbb{R}^d)$ defined by

$$\alpha_t=\alpha_t(x)\omega_k(x)dx,$$

where $\alpha_t(x)$ is the function given by the relation (4.8).

Proposition 6.5. *For the Gaussian distributions α_t, $t>0$, we have*

$$1.\ D_p^{(1)}(\alpha_t)=0,\quad p=1,...,d. \tag{6.21}$$

$$2.\ D_p^{(2)}(\alpha_t) = t, \quad p = 1, ..., d. \tag{6.22}$$

6.3. **Central limit theorem for probability measures on $\mathbb{R}^d$.**

Let $\{\mu_{n,j}\}$, $n \in \mathbb{N}$, $1 \leq j \leq k_n$, be a sequence of measures of $\mathcal{M}^1(\mathbb{R}^d)$. We consider the sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ given by $\mu_n = \sum_{j=1}^{k_n} \mu_{n,j}$.

Theorem 6.2. *We suppose that the measures $\mu_{n,j}$, $n \in \mathbb{N}$, $1 \leq j \leq k_n$, satisfy the following conditions:*

$$1.\ \lim_{n\to+\infty} \sum_{j=1}^{k_n} \int_{\mathbb{R}^d} ||x||^3 d\mu_{n,j}(x) = 0. \tag{6.23}$$

2. For all $j \in \{1, ..., k_n\}$, we have

$$D_p^{(1)}(\mu_{n,j}) = 0, \quad p = 1, ..., d. \tag{6.24}$$

$$D_{p,q}^{(2)}(\mu_{n,j}) = 0, \begin{cases} p = 1, ..., d-q, \\ q = 1, ..., d-1. \end{cases} \tag{6.25}$$

$$3.\ \lim_{n\to+\infty} \sup_{1\leq j\leq k_n} D_p^{(2)}(\mu_{n,j}) = 0, \quad p = 1, ..., d. \tag{6.26}$$

$$4.\ \lim_{n\to+\infty} D_p^{(2)}(\mu_n) = t,\ t > 0, \quad p = 1, ..., d. \tag{6.27}$$

Then the sequence of measures $\{\mu_n\}$ converges weakly to the Gaussian distributions α_t, $t > 0$.

Proof. From Proposition 6.2, Definition 6.1 and the relation (6.25) we have

$$\forall\, \lambda \in \mathbb{R}^d, \quad 1 - \mathcal{F}_D(\mu_{n,j})(\lambda) = \sum_{p=1}^{d} \lambda_p^2 D_p^{(2)}(\mu_{n,j}) + \int_{\mathbb{R}^d} R(x,\lambda) d\mu_{n,j}(x). \tag{6.28}$$

By using the relation (6.10) we obtain

$$\forall\, \lambda \in \mathbb{R}^d, \quad |1 - \mathcal{F}_D(\mu_{n,j})(\lambda)| \leq \sum_{p=1}^{d} \lambda_p^2 D_p^{(2)}(\mu_{n,j}) + \frac{||\lambda||^3}{6} \int_{\mathbb{R}^d} ||x||^3 d\mu_{n,j}(x),$$

$$\forall\, \lambda \in \mathbb{R}^d, \quad |1 - \mathcal{F}_D(\mu_{n,j})(\lambda)| \leq \sum_{p=1}^{d} \lambda_p^2 \sup_{1\leq j\leq k_n} D_p^{(2)}(\mu_{n,j}) + \frac{||\lambda||^3}{6} \int_{\mathbb{R}^d} ||x||^3 d\mu_{n,j}(x).$$

Thus,

$$\forall\, \lambda \in \mathbb{R}^d, \quad \sup_{1\leq j\leq k_n} |1 - \mathcal{F}_D(\mu_{n,j})(\lambda)| \leq \sum_{p=1}^{d} \lambda_p^2 \sup_{1\leq j\leq k_n} D_p^{(2)}(\mu_{n,j}) + \frac{||\lambda||^3}{6} \int_{\mathbb{R}^d} ||x||^3 d\mu_{n,j}(x).$$

From the relations (6.23), (6.26) we get

$$\forall\, \lambda \in \mathbb{R}^d, \quad \lim_{n\to+\infty} \sup_{1\leq j\leq k_n} |1 - \mathcal{F}_D(\mu_{n,j})(\lambda)| = 0, \tag{6.29}$$

and we have

$$\mathcal{F}_D(\mu_{n,j})(0) = 1.$$

These relations imply that for n large, we have

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu_{n,j})(\lambda) \neq 0, \quad \text{for all } \ j \in \{1, ..., k_n\}.$$

Then

$$\forall\,\lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu_n)(\lambda) = \prod_{j=1}^{k_n} \mathcal{F}_D(\mu_{n,j})(\lambda) \neq 0.$$

For such n, the function $Log\mathcal{F}_D(\mu_n)$ is well defined and we have

$$\forall\,\lambda \in \mathbb{R}^d, \quad Log\mathcal{F}_D(\mu_n)(\lambda) = \sum_{j=1}^{k_n} Log\mathcal{F}_D(\mu_{n,j})(\lambda).$$

We write this relation in the following form:

$$\forall\,\lambda \in \mathbb{R}^d, \quad Log\mathcal{F}_D(\mu_n)(\lambda) = \sum_{j=1}^{k_n} Log(1-\beta_{n,j}(\lambda)) = -\sum_{j=1}^{k_n}\sum_{m=1}^{\infty} \frac{(\beta_{n,j}(\lambda))^m}{m}, \quad (6.30)$$

where

$$\forall\,\lambda \in \mathbb{R}^d, \quad \beta_{n,j}(\lambda) = 1 - \mathcal{F}_D(\mu_n)(\lambda).$$

From the relations (6.28),(6.24) and Proposition 6.3, we have

$$\forall\,\lambda \in \mathbb{R}^d, \quad \sum_{j=1}^{k_n} \beta_{n,j}(\lambda) = \sum_{p=1}^{d} \lambda_p^2 D_p^{(2)}(\mu_n) + \sum_{j=1}^{k_n} \int_{\mathbb{R}^d} R(x,\lambda) d\mu_{n,j}(x),$$

then from the relations (6.10), (6.23), (6.27), we obtain

$$\forall\,\lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \sum_{j=1}^{k_n} \beta_{n,j}(\lambda) = t||\lambda||^2. \quad (6.31)$$

On the other hand, we have

$$\forall\,\lambda \in \mathbb{R}^d, \quad 0 \leq \sum_{j=1}^{k_n}\sum_{m=2}^{\infty} \frac{(\beta_{n,j}(\lambda))^m}{m} \leq \frac{1}{2}\sum_{j=1}^{k_n}\sum_{m=2}^{\infty} (\beta_{n,j}(\lambda))^m. \quad (6.32)$$

We have

$$\forall\,\lambda \in \mathbb{R}^d, \quad \textstyle\sum_{j=1}^{k_n} (\beta_{n,j}(\lambda))^2 \leq \sup_{1\leq j\leq k_n} \beta_{n,j}(\lambda) \sum_{j=1}^{k_n} \beta_{n,j}(\lambda),$$

then from the relations (6.29), (6.31), we deduce that

$$\forall\,\lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \sum_{j=1}^{k_n} (\beta_{n,j}(\lambda))^2 = 0,$$

thus

$$\forall\,\lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \sum_{j=1}^{k_n} (\beta_{n,j}(\lambda))^m = 0, \quad \text{for all} \quad m \geq 3,$$

and then from the relation (6.32), we obtain

$$\forall\,\lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \sum_{j=1}^{k_n}\sum_{m=2}^{\infty} (\frac{(\beta_{n,j}(\lambda))^m}{m} = 0. \quad (6.33)$$

From the relation (6.30), we get

$$\forall\,\lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\mu_n)(\lambda) = \exp(-\sum_{j=1}^{k_n} \beta_{n,j}(\lambda)) \exp(-\sum_{j=1}^{k_n}\sum_{m=2}^{\infty} (\frac{(\beta_{n,j}(\lambda))^m}{m}).$$

Thus by using the relations (6.31), (6.32), we obtain

$$\forall\,\lambda \in \mathbb{R}^d, \quad \lim_{n\to\infty} \mathcal{F}_D(\mu_n)(\lambda) = \exp(-t||\lambda||^2).$$

This relation and Theorem 6.1 imply that the sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ converges vaguely to a measure α_t, $t > 0$, which belongs to $\mathcal{M}_b(\mathbb{R}^d)$ such that

$$\forall\,\lambda \in \mathbb{R}^d, \quad \mathcal{F}_D(\alpha_t)(\lambda) = e^{-t||\lambda||^2}.$$

As we have

$$\mathcal{F}_D(\alpha_t)(0) = \int_{\mathbb{R}^d} d\alpha_t(x) = 1,$$

then the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges weakly to the Gaussian distributions α_t, $t > 0$.

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