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AN EXAMPLE OF NONLOCAL BOUNDARY VALUE PROBLEM WITH COMPLEX EIGENVALUES

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Abstract. An initial-boundary value problem for one-dimensional heat equation with a nonlocal boundary condition is studied. This boundary condition ensures both real eigenvalues with double multiplicity and simple complex eigenvalues of the corresponding one-dimensional spectral problem. Applying spectral projectors, we find a series solution of the problem for a special choice of the initial function. Then, using the operational calculus approach of Dimovski, we obtain an explicit representation of the solution in the general case. The expression obtained contains a non-classical convolution product of the particular solution and an arbitrary initial function. This result is an extension of the classical Duhamel principle, but for the space variable.

1. Introduction

Here a nonlocal boundary value problem for the classical heat equation with a integral boundary functional is considered. A special feature of the problem is the presence both of real and complex eigenvalues of the corresponding one-dimensional spectral problem. For example, in Ionkin [4] all the eigenvalues are real with double multiplicity. Here we propose for the first time explicit representation in a case of both of real and complex eigenvalues.

Let us consider the following initial-boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = F(x,t), \quad 0 < x < 2,\ 0 < t, \tag{1.1}$$

$$u(x,0) = f(x), \quad 0 \le x \le 2, \tag{1.2}$$

$$u(0,t) = 0, \quad 0 \le t, \tag{1.3}$$

$$\int_0^1 3u(\xi,t) - u(2\xi,t)d\xi = 0, \quad 0 \le t. \tag{1.4}$$

The last boundary condition is nonlocal one of Ionkin type. The original Ionkin condition would looks as: $\int_0^1 u(\xi,t)d\xi = 0$. It is introduced in [4].

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2. A spectral problem, connected with BVP (1.1)–(1.4)

In order to find a series solution of (1.1)-(1.4), we are to consider the following non-local eigenvalue problem in $C^2[0,2]$:

$$\frac{d^2y}{dx^2}+\lambda^2 y=0, \quad 0<x<2, \quad y(0)=0, \quad \int_0^1 3y(\xi)-y(2\xi)d\xi=0. \tag{2.1}$$

We denote

$$\Phi_\xi\{y(\xi)\}=2\int_0^1 3y(\xi)-y(2\xi)d\xi, \tag{2.2}$$

and have $\Phi_\xi\{\xi\}=1$.

A special feature of the problem is the presence of simple complex eigenvalues along with real double eigenvalues.

Indeed, as it is easy to see, each eigenvalue $-\lambda_n^2$ of (2.1) could be obtained from a zero λ_n of the sine-indicatrix $E(\lambda)$ of the functional Φ. The sine-indicatrix of the functional (2.2) $\Phi_\xi\{y(\xi)\}=2\int_0^1 3y(\xi)-y(2\xi)d\xi$ is

$$E(\lambda)=\Phi_\xi\left\{\frac{\sin\lambda\xi}{\lambda}\right\}=\frac{1}{\lambda^2}\left(5-6\cos\lambda+\cos 2\lambda\right). \tag{2.3}$$

After simplification we obtain

$$E(\lambda)=\frac{4}{\lambda^2}(2-\cos\lambda)\sin^2\frac{\lambda}{2}.$$

We have $E(0)=\lim_{\lambda\to 0}E(\lambda)=1$.

The zeros of $E(\lambda)$ determined from

$$\sin^2\frac{\lambda}{2}=0$$

are $\lambda_n=2n\pi$, $n\in\mathbb{Z}\setminus 0$ (we have $E(0)=1$) and they are with multiplicity two. The corresponding eigenvalues are $-\lambda_n^2=-(2n\pi)^2$, $n\in\mathbb{N}$ (here we take only a half of the zeros $\lambda_n=2n\pi$, $n\in\mathbb{Z}\setminus 0$).

The zeros of $E(\lambda)$ determined by

$$2-\cos\lambda=0$$

are non-real numbers. They are

$$\mu_n=2n\pi-i\ln(2-\sqrt{3}) \quad \text{and} \quad \gamma_n=2n\pi-i\ln(2+\sqrt{3}),$$

$n=...,-2,-1,0,1,2,....$ Using that

$$\ln(p+\sqrt{p^2-1})=-\ln(p-\sqrt{p^2-1}), \quad p\ge 1,$$

we obtain that the eigenvalues μ_n and γ_n are complex conjugate, i.e. $\mu_n=\overline{\gamma_n}$.

Next we use

$$E''(\lambda_k)=\frac{2}{\lambda_k^2}, \qquad E'''(\lambda_k)=-\frac{12}{\lambda_k^3}. \tag{2.4}$$

Using that $\sin\mu_n=\sqrt{3}i$ and $\sin\gamma_n=-\sqrt{3}i$, we obtain

$$E'(\mu_n)=-\frac{2\sin\mu_n}{\mu_n^2}=-\frac{2\sqrt{3}i}{\mu_n^2}, \tag{2.5}$$

$$E'(\gamma_n)=\frac{2\sin\gamma_n}{\gamma_n^2}=\frac{2\sqrt{3}i}{\gamma_n^2}, \tag{2.6}$$

where $\lambda_k=2k\pi$, $k=1,2,...$, $\mu_n=2n\pi-i\ln(2-\sqrt{3})$ and $\gamma_n=2n\pi-i\ln(2+\sqrt{3})$, $n=...,-3,-2,-1,0,1,2,3,....$

The resolvent operator $R_{-\lambda^2}$ of the spectral problem (2.1) is determined by the solution $y = R_{-\lambda^2} f(x)$ of the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda^2 y = f(x), \quad 0 < x < 2, \quad y(0) = 0, \quad \int_0^1 3y(\xi) - y(2\xi)d\xi = 0. \tag{2.7}$$

Explicitly,

$$R_{-\lambda^2} f(x) = \frac{1}{\lambda}\int_0^x f(\xi)\sin\lambda(x-\xi)d\xi - \frac{\sin\lambda x}{\lambda E(\lambda)}\Phi_\xi\left\{\frac{1}{\lambda}\int_0^\xi f(\eta)\sin\lambda(\xi-\eta)d\eta\right\}.$$

$R_{-\lambda^2} f(x)$ is determined for all $\lambda \in \mathbb{C}$, except for the zeros of $E(\lambda)$. The resolvent operator $R_{-\lambda^2}$ is defined for $\lambda = 0$ since $E(0) = 1$. Further, for the aims of our operational calculus, it is convenient to use only the operator R_0. Then, denoting $L_x f(x) = R_0 f(x)$, we have

$$\begin{aligned} L_x f(x) &= \int_0^x (x-\xi)f(\xi)d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi-\eta)f(\eta)d\eta\right\} \\ &= \int_0^x (x-\xi)f(\xi)d\xi - x\int_0^1 (1-\eta)^2\left(3f(\eta) - 4f(2\eta)\right)d\eta. \end{aligned}$$

$y = L_x f(x)$ is the right inverse operator of $\frac{d^2}{dx^2}$ (i.e. $\frac{d^2}{dx^2} y = f(x)$), determined by boundary conditions $y(0) = 0$, $\int_0^1 3y(\xi) - y(2\xi)d\xi = 0$.

3. Spectral projectors

3.1. **The case of real eigenvalues.** Let us consider the eigenvalues $-\lambda_n^2$, where $\lambda_n = 2n\pi$, $n = 1, 2, \ldots$. Then the spectral Riesz' projectors $P_{\lambda_n} : C[0,2] \to \operatorname{span}\{\sin\lambda_n x\,,\, x\cos\lambda_n x\}$ are

$$\begin{aligned} &P_{\lambda_n}\{f\} = \frac{1}{\pi i}\int_{\Gamma_{\lambda_n}} R_{-\lambda_n^2} f(x)\lambda d\lambda \qquad (3.1) \\ &= 4\left(\frac{3E''(\lambda_n) + \lambda_n E'''(\lambda_n)}{3\lambda_n^2 E''^2(\lambda_n)}\Phi_\xi\left\{\int_0^\xi f(\eta)\sin\lambda_n(\xi-\eta)d\eta\right\}\right. \\ &\qquad \left. - \frac{1}{\lambda_n E''(\lambda_n)}\Phi_\xi\left\{\int_0^\xi f(\eta)(\xi-\eta)\cos\lambda_n(\xi-\eta)d\eta\right\}\right)\sin\lambda_n x \\ &\qquad - \frac{4}{\lambda_n E''(\lambda_n)}\Phi_\xi\left\{\int_0^\xi f(\eta)\sin\lambda_n(\xi-\eta)d\eta\right\} x\cos\lambda_n x, \end{aligned}$$

where Γ_{λ_n} is a simple contour containing the zero λ_n only.

3.2. **The case of complex eigenvalues.** Next we consider the complex eigenvalues $-\mu_n^2$, where $\mu_n = \alpha_n + i\beta$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$. Then the spectral Riesz' projectors $P_{\mu_n} : C[0,2] \to E_{\mu_n} = \text{span}\{\sin \mu_n x\}$ are

$$\begin{aligned} P_{\mu_n}\{f\} &= P_{\overline{\gamma}_n}\{f\} = \frac{1}{\pi i}\int_{\Gamma_{\mu_n}} R_{-\mu_n^2} f(x)\lambda d\lambda \\ &= \frac{-2}{\mu_n E'(\mu_n)}\Phi_\xi\left\{\int_0^\xi f(\eta)\sin\mu_n(\xi-\eta)d\eta\right\}\sin\mu_n x \\ &= \frac{4}{\sin\mu_n}\left(\int_0^1 3f(\eta)\sin^2\frac{\mu_n}{2}(1-\eta) - f(2\eta)\sin^2\mu_n(1-\eta)d\eta\right)\sin\mu_n x. \end{aligned} \tag{3.2}$$

Analogically, we find

$$\begin{aligned} P_{\gamma_n}\{f\} &= P_{\overline{\mu}_n}\{f\} \\ &= \frac{-2}{\gamma_n E'(\gamma_n)}\Phi_\xi\left\{\int_0^\xi f(\eta)\sin\gamma_n(\xi-\eta)d\eta\right\}\sin\gamma_n x \\ &= \frac{4}{\sin\gamma_n}\left(\int_0^1 3f(\eta)\sin^2\frac{\gamma_n}{2}(1-\eta) - f(2\eta)\sin^2\gamma_n(1-\eta)d\eta\right)\sin\gamma_n x. \end{aligned} \tag{3.3}$$

3.2.1. *The real valued projectors.* We consider real valued functions and it will be convenient to use real valued projectors too. Next we consider the following projectors:

$$\begin{aligned} \mathbf{P}_{\mu_n} &= \frac{1}{2}\left(P_{\mu_n} + P_{\overline{\mu}_n}\right) = \Re(P_{\mu_n}), \\ \mathbf{P}_{\gamma_n} &= \frac{1}{2}\left(P_{\gamma_n} + P_{\overline{\gamma}_n}\right) = \Re(P_{\gamma_n}), \\ \mathbf{Q}_{\mu_n} &= \frac{1}{2i}\left(P_{\mu_n} - P_{\overline{\mu}_n}\right) = \Im(P_{\mu_n}), \\ \mathbf{Q}_{\gamma_n} &= \frac{1}{2i}\left(P_{\gamma_n} - P_{\overline{\gamma}_n}\right) = \Im(P_{\gamma_n}). \end{aligned}$$

Here $\Re(P_{\mu_n})$ is the real part of P_{μ_n} and $\Im(P_{\mu_n})$ is the complex part of P_{μ_n}. From $\mu_n = \overline{\gamma}_n$ we have

$$\begin{aligned} \mathbf{P}_{\mu_n} &= \mathbf{P}_{\gamma_n}, \\ \mathbf{Q}_{\mu_n} &= \mathbf{Q}_{\gamma_n}. \end{aligned}$$

Using that

$$\sin(\alpha_n + i\beta)x = \cosh\beta x\sin\alpha_n x + i\sinh\beta x\cos\alpha_n x,$$

and after some calculation we obtain

$$\begin{aligned} \mathbf{P}_{\mu_n}\{f\} = \frac{2}{3}\ \Phi_\xi&\left\{\int_0^\xi f(\eta)\cosh\beta(\xi-\eta)\sin\alpha_n(\xi-\eta)d\eta\right\}\cosh\beta x\sin\alpha_n x \\ -\frac{2}{3}\ \Phi_\xi&\left\{\int_0^\xi f(\eta)\sinh\beta(\xi-\eta)\cos\alpha_n(\xi-\eta)d\eta\right\}\sinh\beta x\cos\alpha_n x, \\ \mathbf{Q}_{\mu_n}\{f\} = \frac{2}{3}\ \Phi_\xi&\left\{\int_0^\xi f(\eta)\cos\alpha_n(\xi-\eta)\sinh\beta(\xi-\eta)d\eta\right\}\cosh\beta x\sin\alpha_n x \\ +\frac{2}{3}\ \Phi_\xi&\left\{\int_0^\xi f(\eta)\cosh\beta(\xi-\eta)\sin\alpha_n(\xi-\eta)d\eta\right\}\sinh\beta x\cos\alpha_n x. \end{aligned}$$

3.3. **The formal spectral expansion of a function.** The projectors $P_{\lambda_k}\{f(x)\}$, $\mathbf{P}_{\mu_n}\{f(x)\}$ and $\mathbf{Q}_{\mu_n}\{f(x)\}$, considered together, form a total system of projectors, i.e. such that $P_{\lambda_k}\{f(x)\} = 0$ for all $k = 1, 2, ...$, $\mathbf{P}_{\mu_n}\{f(x)\} = 0$ and $\mathbf{Q}_{\mu_n}\{f(x)\} = 0$ for all $n = ..., -2, -1, 0, 1, 2, ...$, then $f \equiv 0$ (see Bozhinov [1]).

Definition 3.1. Let $f \in C[0, 2]$. The formal spectral expansion of $f(x)$ for eigenvalue problem (2.1) is the correspondence

$$f(x) \sim \sum_{k=1}^{\infty} P_{\lambda_k}\{f\} + \sum_{n=-\infty}^{\infty} \mathbf{P}_{\mu_n}\{f\} + \sum_{n=-\infty}^{\infty} \mathbf{Q}_{\mu_n}\{f\}. \tag{3.4}$$

Example 3.1. If

$$f(x) = L_x x = \frac{x^3}{6} + \frac{x}{12},$$

then

$$\frac{d^2}{dx^2} f(x) = x, \quad f(0) = 0, \quad \int_0^1 3f(\xi) - f(2\xi) d\xi = 0.$$

Let us consider the eigenvalues $-\lambda_n^2$ with $\lambda_n = 2n\pi$, $n = 1, 2, 3, ...$ Then

$$P_{\lambda_n}\{f\} = \frac{2x \cos \lambda_n x}{\lambda_n^2} - \frac{4 \sin \lambda_n x}{\lambda_n^3}. \tag{3.5}$$

Let us consider the eigenvalues $-\mu_n^2$ with $\mu_n = 2n\pi - i \ln(2 - \sqrt{3}) = \alpha_n + i\beta$, $n = ..., -2, -1, 0, 1, 2, ...$, here we denote $\alpha_n = 2n\pi$ and $\beta = -\ln(2 - \sqrt{3})$. Then

$$P_{\mu_n}\{f\} = -\frac{\sin \mu_n x}{\mu_n^2 \sin \mu_n}. \tag{3.6}$$

Let us consider the eigenvalues $-\gamma_n^2$ with $\gamma_n = 2n\pi - i \ln(2 + \sqrt{3})$, $n = ..., -2, -1, 0, 1, 2, ...$ Then

$$P_{\gamma_n}\{f\} = -\frac{\sin \gamma_n x}{\gamma_n^2 \sin \gamma_n}. \tag{3.7}$$

We obtain

$$\begin{aligned} f(x) &= \frac{x^3}{6} + \frac{x}{12} = \sum_{n=1}^{\infty} P_{\lambda_n}\{f\} + \sum_{n=-\infty}^{\infty} \mathbf{P}_{\mu_n}\{f\} + \sum_{n=-\infty}^{\infty} \mathbf{Q}_{\gamma_n}\{f\} \\ &= \sum_{n=1}^{\infty} \left(\frac{2x \cos \lambda_n x}{\lambda_n^2} - \frac{4 \sin \lambda_n x}{\lambda_n^3} \right) \\ &\quad + \frac{1}{\sinh \beta} \sum_{n=-\infty}^{\infty} \frac{(\alpha_n^2 - \beta^2) \cosh \beta x \sin \alpha_n x + 2\alpha_n \beta \sinh \beta x \cos \alpha_n x}{(\alpha_n^2 + \beta^2)^2} \\ &\quad + \frac{1}{\sinh \beta} \sum_{n=-\infty}^{\infty} \frac{2\alpha_n \beta \cosh \beta x \sin \alpha_n x - (\alpha_n^2 - \beta^2) \sinh \beta x \cos \alpha_n x}{(\alpha_n^2 + \beta^2)^2}, \end{aligned} \tag{3.8}$$

where $\alpha_n = 2n\pi$ and $\beta = -\ln(2 - \sqrt{3})$.

Example 3.2. Analogically, for $f(x) = x$ we obtain:

$$P_{\lambda_n}\{x\} = -2x \cos \lambda_n x, \qquad P_{\mu_n}\{x\} = \frac{\sin \mu_n x}{\sin \mu_n}, \qquad P_{\gamma_n}\{x\} = \frac{\sin \gamma_n x}{\sin \gamma_n}.$$

$$x = -\sum_{n=1}^{\infty} 2x \cos \lambda_n x + \frac{1}{\sinh \beta} \sum_{n=-\infty}^{\infty} \sinh \beta x \cos \alpha_n x - \frac{1}{\sinh \beta} \sum_{n=-\infty}^{\infty} \cosh \beta x \sin \alpha_n x,$$

where $\alpha_n = \text{Re } \mu_n = 4n\pi$, $\quad \beta = \text{Im } \mu_n = -2 \ln(2 - \sqrt{3})$.

4. Convolutions

The Duhamel convolution. The operation

$$(\varphi \overset{t}{*} \psi)(t) = \int_0^t \varphi(t-\tau)\psi(\tau)d\tau, \quad \varphi, \psi \in C([0,\infty)), \tag{4.1}$$

bears the name of Duhamel, but sometimes it is called also either Borel, or Laplace convolution. It is related to the Volterra integration operator

$$l_t\varphi(t) = \int_0^t \varphi(\tau)d\tau, \tag{4.2}$$

since $l_t\varphi = \{1\} \overset{t}{*} \varphi$. To say it differently, l_t is the convolution operator $\{1\} \overset{t}{*}$, i.e. $l_t = \{1\} \overset{t}{*}$.

A non-classical convolution.

Theorem 4.1. (Dimovski [2], p.119) *Let $f, g \in C[0,2]$. Then the operation* $\overset{x}{*}$

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\Phi_\xi \left\{ \int_0^\xi h(x,\eta)d\eta \right\}, \tag{4.3}$$

where

$$h(x,\eta) = \int_x^\eta f(x+\eta-\zeta)g(\zeta)d\zeta - \int_{-x}^\eta f(|\eta - x - \zeta)g(|\zeta|)sgn(\zeta(\eta - x - \zeta))d\zeta$$

is a bilinear, commutative and associative operations in $C[0,2]$, such that

$$R_{-\lambda^2}\{f\} = \left\{ \frac{\sin\lambda x}{\lambda E(\lambda)} \right\} \overset{x}{*} f, \tag{4.4}$$

with $E(\lambda) = \Phi_\xi \left\{ \dfrac{\sin\lambda\xi}{\lambda} \right\}$.

For $\lambda = 0$, we get

$$L_x f = \{x\} \overset{x}{*} f. \tag{4.5}$$

In [2] the corresponding theorem is stated for an arbitrary linear functional Φ in $C[0,a]$. Next we will combine both the Duhamel convolution (4.1) and the Dimovski convolution (4.3) into a two-dimensional convolution in $C(D) = C([0,2] \times [0,\infty))$.

Theorem 4.2. *Let $u, v \in C(D)$. Then the operation*

$$(u * v)(x,t) = \int_0^t u(x,t-\tau) \overset{x}{*} v(x,\tau)d\tau,$$

where $\overset{x}{*}$ is operation (4.3)*, is a bilinear, commutative and associative operation in $C(D)$, such that*

$$l_t L_x u(x,t) = \{x\} * u(x,t), \tag{4.6}$$

where $D = [0,2] \times [0,\infty)$.

For a proof, see [3], where more general case is considered.

5. Ring of the multiplier fractions of $(C(D), *)$

We consider the convolution algebra $(C, *)$, where $C = C(D)$ and $D = [0,2] \times [0,\infty)$. Our direct operational calculus approach is based on the notion of a multiplier of the convolution algebra $(C(D), *)$ (see Larsen [6]).

Definition 5.1. (Larsen [6]) An operator $M : C(D) \to C(D)$ is said to be a multiplier of the convolution algebra $(C(D), *)$ iff $M(f * g) = (Mf) * g$ for all $f, g \in C(D)$.

Here we will remind only some specific notations. The multipliers of the form $\{u(x,t)\}*$ will be denoted by $\{u\}$ or u and the result of the application of the operator $u*$ to a function $F \in C(D)$ will be denoted simply by $\{u\}F$ or uF.

Lemma 5.1. A) *Let f be a function from $C[0,2]$. The convolution operator $f \overset{x}{*}$, defined in $C(D)$ by $(f \overset{x}{*})u = f \overset{x}{*} u$ is a multiplier of the convolution algebra $(C(D), *)$.*

B) *Let φ be a function from $C[0,\infty)$. The convolution operator $\varphi \overset{t}{*}$, defined in $C(D)$ by $(\varphi \overset{t}{*})u = \varphi \overset{t}{*} u$ is a multiplier of the convolution algebra $(C(D), *)$.*

For a proof in a more general situation, see [3].

Definition 5.2. Let $f \in C[0,2]$ and $\varphi \in C([0,\infty))$, but both considered as functions of $C(D)$. The operator $[f]_x$ defined by $[f]_x u = f \overset{x}{*} u$ is said to be a *partial numerical operator* with respect to t and the operator $[\varphi]_t u = \varphi \overset{t}{*} u$ is said to be *partial numerical operator* with respect to x.

In these notations we have $L_x = [x]_t$ and $l_t = [1]_x$. The notion of numerical operator for Duhamel convolution is introduced in [7].

Lemma 5.2. (Larsen [6]) *The set of all multipliers of the convolution algebra $(C, *)$ is a commutative ring $\mathfrak{M}$.*

The multiplicative set $\mathfrak{N}$ of the non-zero non-divisors of 0 in $\mathfrak{M}$ is non-empty, since at least the operators $\{x\} \overset{x}{*} = [x]_t$ and $\{1\} \overset{t}{*} = [1]_x$ are non-divisors of 0.

Next we introduce the ring $\mathcal{M} = \mathfrak{N}^{-1}\mathfrak{M}$ of the multiplier fractions of the form $\dfrac{A}{B}$ where $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$. The standard algebraic procedure of constructing of this ring, named "localization", is described, e.g. in Lang [5]. Basic for our construction are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $s_t = \dfrac{1}{l_t}$ in $\mathcal{M}$, of the multipliers L_x and l_t correspondingly. If $u \in C^2(D)$, then u_{xx} and u_t are connected with $S_x u$ and $s_t u$ but they in general, are different from them.

Lemma 5.3. *Let $u \in C(D)$ have continuous partial derivatives u_{xx}, u_t on D. Then*

$$u_{xx} = S_x u + S_x\{(x\Phi\{1\} - 1)u(0,t)\} - \Phi_\xi\{u(\xi,t)\}, \tag{5.1}$$

$$u_t = s_t - [u(x,0)]_t. \tag{5.2}$$

Proof. Relation (5.2) is similar to a corresponding relation in Mikusinski [7]. Let us prove (5.1). It is easy to verify the identity

$$L_x\{u_{xx}\} = u(x,t) + (x\Phi\{1\} - 1)u(0,t) - x\Phi_\xi\{u(\xi,t)\}.$$

It remains to multiply it by S_x and to use $S_x\{x\} = S_x L_x = 1$, in order to get (5.1). □

6. Algebraization of boundary value problem (1.1)–(1.4)

Relations (5.1) and (5.2) allow to reduce both equation (1.1) and BVCs (1.2)-(1.4) to a single linear algebraic equation $\mathcal{M}$ for u. Indeed, substituting u_t and u_{xx} from ((5.1) and (5.2) in the equation $u_t(x,t) - u_{xx}(x,t) = F(x,t)$, we get

$$s_t u - [u(x,0)]_t - S_x u - S_x\{(x\Phi\{1\} - 1)u(0,t)\} + \Phi_{xi}\{u(\xi,t)\} = F(x,t).$$

Now using initial condition (1.2) and boundary value conditions (1.3) and (1.4), we obtain

$$(s_t - S_x)u = \{F(x,t)\} + [f(x)]_t. \tag{6.1}$$

Thus we reduced BVP (1.1)-(1.4) to the single linear algebraic equation (6.1) for u in $\mathcal{M}$. It is reasonable to introduce the notation of a *weak solution* of BVP (1.1) - (1.4), along with the classical solution.

Definition 3. A function $u \in C(D)$ is said to be weak solution of BVP (1.1)-(1.4), if it is a solution of (6.1).

Next, let us consider the problem of uniqueness of the solution of (1.1)-(1.4). Equation (6.1) reduces it to the algebraic question, whether $s_t - S_x$ is a divisor of zero in $\mathcal{M}$ or not.

Theorem 6.1. *The element $s_t - S_x$ is a nondivisor of zero in $\mathcal{M}$.*

The proof is analogical of the proof of Theorem 3 from [8].

The solution of (6.1) in $\mathcal{M}$ is

$$u = \frac{1}{s_t - S_x}[f(x)]_t + \frac{1}{s_t - S_x}\{F(x,t)\}. \tag{6.2}$$

We may call (6.2) the formal (generalized) solution of problem (1.1)-(1.4).

7. Interpretation of the formal (generalized) solution of (1.1)-(1.4) as a function

Our next task is to interpret (if it is possible) (6.2) as a function of $C(D)$. To this end, we consider a special case of problem (1.1)-(1.4) for $F(x,t) \equiv 0$ and $f(x) = L_x\{x\} = \frac{1}{S_x^2} = \frac{x^3}{6} + \frac{5x}{12}$. We denote its solution, if it exists, by $\Omega = \Omega(x,t)$. Having in mind that $L_x\{x\} = \frac{1}{S_x^2}$, we have the following algebraic representation of this solution:

$$\Omega = \frac{1}{s_t - S_x}\left[\frac{x^3}{6} + \frac{5x}{12}\right]_t = \frac{1}{S_x^2(s_t - S_x)}. \tag{7.1}$$

As for the special solution $\Omega(x,t)$, it can be found in an explicit series form, using the spectral projectors (3.1), (3.2) and (3.3) and representation (3.8). Thus we obtain the following lemma.

Lemma 7.1. *If* $f(x) = L_x x = \dfrac{x^3}{6} + \dfrac{5x}{12}$ *and* $F(x,t) \equiv 0$*, then the solution* $\Omega(x,t)$ *of the BVP (1.1)-(1.4) is:*

$$\begin{aligned}\Omega(x,t) &= \sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\left(\frac{2}{\lambda_n^2} x\cos\lambda_n x - \left(\frac{4}{\lambda_n^3} + \frac{4t}{\lambda_n}\right)\sin\lambda_n x\right) \\ &- \sum_{n=-\infty}^{\infty} \frac{e^{-\mu_n^2 t}}{\mu_n^2 \sin\mu_n}\sin\mu_n x.\end{aligned} \tag{7.2}$$

The proof may be accomplished by a direct check, too.

The generalized solution of problem (1.1)-(1.4) for arbitrary $f(x)$ and $F(x,t)$ can be written in the form:

$$\begin{aligned}u &= \frac{1}{s_t - S_x}[f(x)]_t + \frac{1}{s_t - S_x}F(x,t) \\ &= S_x^2\left(\frac{1}{S_x^2(s_t - S_x)}[f(x)]_t + \frac{1}{S_x^2(s_t - S_x)}F(x,t)\right).\end{aligned}$$

Under corresponding assumptions for smoothness of the functions $f(x)$ and $F(x,t)$ it can be written as a function of the form

$$u = \frac{\partial^4}{\partial x^4}\left[\Omega \overset{x}{*} f(x) + \Omega * F(x,t)\right]. \tag{7.3}$$

We can find another representation of the solution of (1.1)-(1.4). We consider a special case of problem (1.1)-(1.4) for $F(x,t) \equiv 0$ and $f(x) = x = L_x = \dfrac{1}{S_x} = x$. We denote its solution, if it exists, by $V = V(x,t)$. We have the following algebraic representation of this solution:

$$V = \frac{1}{s_t - S_x}[x]_t = \frac{1}{S_x(s_t - S_x)}. \tag{7.4}$$

As for the special solution $V = V(x,t)$, it can be found in an explicit series form, using the spectral projectors (3.1), (3.2) and (3.3). We obtain

Lemma 7.2. *If* $f(x) = x$ *and* $F(x,t) \equiv 0$*, then the solution* $V(x,t)$ *of the BVP (1.1)-(1.4) is:*

$$V(x,t) = 2\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}(2t\lambda_n \sin\lambda_n x - x\cos\lambda_n x) + \sum_{n=-\infty}^{\infty}\frac{e^{-\mu_n^2 t}}{\sin\mu_n}\sin\mu_n x. \tag{7.5}$$

The proof may be accomplished by a direct check, too.

Then the generalized solution of problem (1.1)-(1.4) for arbitrary $f(x)$ and $F(x,t)$ can be written in the form:

$$\begin{aligned}u &= \frac{1}{s_t - S_x}[f(x)]_t + \frac{1}{s_t - S_x}F(x,t) \\ &= S_x\left(\frac{1}{S_x(s_t - S_x)}[f(x)]_t + \frac{1}{S_x(s_t - S_x)}F(x,t)\right).\end{aligned}$$

Under corresponding assumptions for smoothness of the functions $f(x)$ and $F(x,t)$ it can be written as a function of the form

$$u = \frac{\partial^2}{\partial x^2}\left[V \overset{x}{*} f(x) + V * F(x,t)\right]. \tag{7.6}$$

Then (7.3) and (7.6) give the following Duhamel-type representations of the solution of (1.1) - (1.4) using $\Omega(x,t)$ (7.1) and $V(x,t)$ (7.2) respectively:

Theorem 7.3. *Let us* $f(x) \in C^1[0,2]$, $f(0) = 0$ *and* $2\int_0^1 3f(\xi) - f(2\xi)d\xi = 0$. *Then,*

$$\begin{aligned} u &= \frac{\partial^4}{\partial x^4}(\Omega(x,t) \overset{x}{*} f(x)) \\ &= 2\int_0^1 \Big((3\,(\Omega_x(x+1-\xi,t) - \Omega_x(1-x-\xi,t))\, f'(\xi) \\ &\qquad - (\Omega_x(x+1-2\xi,t) - \Omega_x(1-x-2\xi,t))\, f'(2\xi)\Big) d\xi \end{aligned} \tag{7.7}$$

is a weak solution of (1.1)-(1.4) *for* $F(x,t) \equiv 0$.

Theorem 7.4. *Let us* $f(x) \in C[0,2]$, $f(0) = 0$ *and* $2\int_0^1 3f(\xi) - f(2\xi)d\xi = 0$. *Then,*

$$\begin{aligned} u &= \frac{\partial^2}{\partial x^2}(V(x,t) \overset{x}{*} f(x)) \\ &= 2\int_0^1 \Big(3\,(V(x+1-\xi,t) - V(1-x-\xi,t))\, f(\xi) \\ &\qquad - (V(x+1-2\xi,t) - V(1-x-2\xi,t))\, f(2\xi)\Big) d\xi \end{aligned} \tag{7.8}$$

is a weak solution of (1.1)-(1.4) *for* $F(x,t) \equiv 0$.

The proof may be accomplished by a direct check, too.

References

[1] N. S. Bozhinov, *On theorems of uniqueness and completeness of expansion on eigen and associated eigenfunctions of the nonlocal Sturm-Liouville operator on a finite interval*, Diferenzialnye Uravnenya, **26**, 5, (1990) 741–453 (Russian).

[2] I. H. Dimovski, *Convolutional Calculus*. Kluwer, Dordrecht (1990).

[3] I. H. Dimovski, *Nonlocal boundary value problems*, Mathematics and Math. Education, Proc. **38** Spring Conf. UBM (2009) 31–40.

[4] N. I. Ionkin, *Numerical solution of a boundary value problem for the heat equation with a nonclassical boundary condition*, Differencialnie Uravenia **13**, 2 (1977) 294–304 (Russian).

[5] S. Lang, *Algebra*, Addison - Wesley, Reading, Mass. (1965).

[6] R. Larsen, *An Introduction to the Theory of Multipliers*. Springer, Berlin-New York-Heidelberg (1971).

[7] J. Mikusinski, *Operational Calculus*. Oxford-Warszawa (1959).

[8] Y. T. Tsankov, *Explicit solution of a nonlocal boundary value problem for heat equation*, Comptes rendus de l'Acad. Bulgare des Sciences **66**, 7 (2013) 941–950.

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