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SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS BY USING SUMUDU TRANSFORM AND MIKUSIŃSKI CALCULUS

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Abstract. In this paper the Sumudu transformation is analyzed and compared as operational tool with the Mikusiński calculus. The procedure of solving differential equations of integer order and fractional differential equations is considered in the frames of the Sumudu transform and Mikusiński calculus. The exact and the approximate solutions of a class of integer and fractional order differential equations are constructed and their characters are analyzed. The described procedure of the construction of solutions is illustrated by an example.

1. Introduction

The theory of Mikuśinski operators belongs to field of Professor Ivan Dimovski's scientific research. The Sumudu integral transformation is considered as one of the alternative tools of the operational calculus. In this paper the comparison between the Sumudu transforms and Mikusinński calculus is analyzed within the frame of determining the exact and the approximate solution of the $n-$th order differential equations and fractional differential equations of the form:

$$\mathcal{D}_t^{\beta_n} x(t) + \sum_{i=1}^{n-1} A_i \mathcal{D}_t^{\beta_i} x(t) + A_0 x(t) = f(t), \quad t > 0, \tag{1.1}$$

where β_i, $i = 1, 2, \ldots, n$, are rational exponents, such that $i-1 \le \beta_i \le i$, and A_i, $i = 1, 2, \ldots, n$, are real coefficients. Corresponding initial conditions are considered and the fractional derivatives are considered in the Caputo sense.

The Mikusiński operators have been introduced and analyzed by Professor Jan Mikusiński ([14], [15]) in the early fifties of the last century. The theory of Mikusiński operational calculus was modern and very attractive theory within the frame of generalized functions, at that time. Professor Ivan Dimovski ([4], [5], [6]) from Sofia, Professor Bogoljub Stanković ([20]) from Novi Sad, and other well known mathematicians, joined Professor Jan Mikusiński's scientific research and successfully

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contributed to the development of the Mikusiński theory and generalized functions. Mikusiński calculus can be applied for the construction of the solution of differential equations ([21]), fractional differential equations ([22]) and even of fuzzy fractional differential equations ([23]).

The Sumudu transformation has been introduced and studied by Watugala 1993 in the paper [25]. The properties of the Sumudu transformation, its connection with the Laplace transformation ([3], [24]), and its applications are presented e.g. in the papers [1] and [26]. In particular, in the paper [8], solving of fractional differential equation by using the Sumudu transform is analyzed.

In this paper we consider more general fractional differential equation and construct the exact and the approximate solution of equation (1.1), by taking finite sums instead of infinite series of Sumudu transforms. We also analyze the character of obtained solution.

In Section 2, we present some notations and notions considering the Sumudu transformations, Mikusiński operators, and fractional calculus. The convergence of series of Mikusiński operators and Sumudu transforms are considered in order to analyze the character of the obtained solutions of differential and fractional differential equations under the corresponding initial conditions. In Section 3, the Mikusiński operators are compared with corresponding Sumudu transforms in view of determining the solution of a class of $n-$th order differential equations. In Section 4, the exact and the approximate solution of fractional differential equation (1.1) are constructed and analyzed, by using the Sumudu transformation and the Mikusiński calculus. In Section 5, the discussed procedure is explained by an example.

2. Notions and notations

2.1. **Sumudu transformation.** The Sumudu transformation is a linear integral transform defined on the set of functions

$$\mathcal{A} = \{f(t),\ \ \exists M,\ \tau_1,\ \tau_2, > 0\ \ |f(t)| \leq Me^{M|t|\tau_j},\ \ t \in (-1)^j \times (0, \infty),$$

as follows:

$$G(u) = \mathcal{S}[f(t)](u) = \int\limits_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2).$$

Comparing the Laplace transform:

$$F(s) = \mathcal{L}[f(t)](u) = \int\limits_0^\infty f(t)e^{-st}dt, \quad Re(s) > 0,$$

with the Sumudu transform, we can write:

$$G\left(\frac{1}{s}\right) = sF(s), \quad F\left(\frac{1}{s}\right) = uG(u).$$

The Sumudu transform of a series $\sum\limits_{k=0}^{\infty} a_k t^k$ is equal to $\sum\limits_{k=0}^{\infty} k! a_n u^k$ wherefrom it follows that the inverse transform of $\sum\limits_{k=0}^{\infty} a_k u^k$ is equal to $\sum\limits_{k=0}^{\infty} \dfrac{a_n t^k}{k!}$. The last series of Sumudu transforms converge for each $t \in R$.

2.2. **Some elements of the Mikusiński calculus.** The set C_+ of continuous functions with supports in $[0,\infty)$ is a commutative ring without unit element, with the usual addition and multiplication given by the convolution

$$(f * g)(t) \quad = \quad \int_0^t f(\tau)g(t-\tau)\,d\tau, \qquad t \geq 0.$$

By the Titchmarsh theorem, C_+ has no divisors of zero. The quotients of the form

$$\frac{f}{g}, \quad f \in C_+,\ 0 \not\equiv g \in C_+,$$

where the division is considered in the sense of the convolution (see [14]), are elements of the Mikusiński operator field, or just called Mikusiński operators.

The most important operators in the Mikusiński scheme are the integral operator ℓ, the differential operator s, and the identity operator I, and it holds

$$\ell \cdot s = I.$$

Neither s nor I are operators representing continuous functions.

The integral operator ℓ represents the constant function 1 on $[0,\infty)$, and the operator ℓ^α represents a continuous function for $\alpha > 0$, as follows:

$$\ell = \{1\}, \quad \ell^\alpha = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \ \ \alpha \geq 0. \tag{2.1}$$

The series $\displaystyle\sum_{i=0}^{\infty} a_i \ell^{i+1}$ in the field of Mikusiński operators corresponds to the convergent functional series $\displaystyle\sum_{i=0}^{\infty} \frac{a_i t^i}{i!}$ and represents a continuous function, for each $t \in R$.

2.3. **The fractional calculus.** In the recent years the research in fractional calculus is increasing ([18], [9], [17], [10], [11]), because of its wide application in sciences, engineering, economics and other studies ([12], [13]). It concerns the operators of integration and differentiation of arbitrary (fractional) order. There are lot of different definitions of fractional derivatives and fractional integrals and some of them are based on convolutional representations.

In this paper we consider the Riemann-Liouville fractional integral and the Caputo fractional derivative. The Riemann-Liouville fractional integral operator J^α of order $\alpha > 0$ is defined as follows ([19]):

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.2}$$

The Caputo fractional derivative of order α, $n-1 < \alpha < n$ is defined by ([2]):

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \tag{2.3}$$

In the theory of fractional order differential and integral equations the main role plays the Mittag-Leffler function, an entire function in the complex plane, which is a natural extension of the exponential and trigonometric functions:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{2.4}$$

where α ($\operatorname{Re}\alpha > 0$) and β are complex numbers (see e.g. [9], [7], [16]). Usually, α and β are taken positive real numbers.

3. Sumudu transforms and Mikusiński operators

Let us compare the Sumudu transforms with the Mikusiński operators in the frames of their application for solving of integer and fractional order differential equations.

The Mikusiński operator s^n applied to operator x, representing the n-th derivative of an n times differentiable function satisfies the following relation:

$$\{x^{(n)}(t)\} = s^n x - x(0)s^{n-1} - \cdots - x^{(n-1)}(0)I, \tag{3.1}$$

while the Sumudu transform of the n-th derivative of the function $x = x(t)$ has the form:

$$G(x^{(n)}(t))(u) = \frac{x}{u^n} - \frac{x(0)}{u^n} - \cdots - \frac{x^{(n-1)}(0)}{u}. \tag{3.2}$$

The convolution of two functions corresponds to the product of the Mikusiński operators related to these functions, i.e. we have:

$$\{f(t) * g(t)\} = f \cdot g.$$

The Sumudu transform of a convolution is given by:

$$G(f(t) * g(t)) = uf(u)g(u).$$

In this sense, using the relation (2.2), we can see that the Mikusiński operator ℓ^α applied to operator f (representing continuous function) corresponds to the Riemann-Liouville fractional integral operator J^α of order $\alpha > 0$ (applied to a differentiable function f), as follows:

$$\ell^\alpha f = \{J^\alpha f(t)\}, \quad \alpha > 0. \tag{3.3}$$

The Sumudu transform of Riemann-Liouville fractional integral operator J^α of order $\alpha > 0$, defined as a convolution by relation (2.2) is given by:

$$G(J^\alpha)(u) = u^{\alpha+1} f(u). \tag{3.4}$$

The Caputo fractional derivative $D^\alpha f(t)$ of order α, $n-1 < \alpha < n$, applied to a differentiable function x (relation (2.3)) corresponds to the Mikusiński operator s^α applied to operator x as follows:

$$\{D^\alpha f(t)\} = s^\alpha x - \sum_{i=0}^{n-1} x^{(i)}(0) s^{\alpha-i-1}, \quad n-1 < \alpha < n. \tag{3.5}$$

The Sumudu transform of the Caputo fractional derivative defined by (2.3) is:

$$G(D^\alpha x(t))(u) = \frac{x(u)}{u^\alpha} - \sum_{i=0}^{n-1} \frac{x^{(i)}(0)}{u^{\alpha-i}}. \tag{3.6}$$

The Sumudu transform of the Mittag-Leffler function (2.4) has the form

$$G(E_{\alpha,\beta}(z))(u) = \sum_{k=0}^{\infty} \Gamma(\alpha k + \beta) u^k. \tag{3.7}$$

The series of Mikusiński operators

$$\sum_{k=0}^{\infty} \ell^{\alpha k + \beta + 1}, \tag{3.8}$$

where α and β are complex numbers, corresponds to the Mittag-Leffler function

$$t^\beta E_{\alpha,\beta+1}(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k + \beta}}{\Gamma(\alpha k + \beta + 1)}. \tag{3.9}$$

The series (3.8) converges for $\alpha > 0$ and $\beta > 0$ to the continuous function (3.9), $t > 0$.

3.1. **Linear differential equations.** In this subsection we analyze the procedure of solving linear differential equation of $n-$th order with constant coefficients by using the Mikusinski calculus and Sumudu transforms.

3.1.1. *Mikusiński calculus.* Let us consider the linear differential equation with constant coefficients A_i, $i = 1, 2, \ldots, n-1$, of the form:

$$x^{(n)}(t) - \sum_{i=1}^{n-1} A_i x^{(i)}(t) - A_0 x(t) \quad = \quad f(t), \tag{3.10}$$

with the initial conditions

$$x^{(i)}(0) \quad = \quad x_i, \quad i = 0, 1, \ldots, n-1, \tag{3.11}$$

where f is a continuous function, and x_i are numerical constants. The equation (3.10) with the conditions (3.11) corresponds to the Mikusiński operational equation:

$$s^n x - x \sum_{i=1}^{n-1} A_i s^i - A_0 x = \sum_{i=1}^{n} s^{i-1} x_{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} s^{k-i} + F, \tag{3.12}$$

where x is the operator corresponding to the solution $x(t)$, s is the differential operator, and F is the Mikusiński operator corresponding to the function f. The exact solution of the above equation in the field of Mikusiński operators has the following form:

$$x = \frac{\sum_{i=1}^{n} s^{i-1} x_{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} s^{k-i} + F}{s^n - \sum_{i=0}^{n-1} A_i s^i}.$$

This solution x can be transformed as follows:

$$\begin{aligned} x \quad &= \quad \frac{\sum_{i=1}^{n} \ell^{n-i+1} x_{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} \ell^{n-k+i} + F\ell^n}{I - \sum_{i=0}^{n-1} A_i \ell^{n-i}} =: \frac{B_M}{I - \sum_{i=0}^{n-1} A_i \ell^{n-i}} \\ &= \quad B_M \cdot \left(\sum_{j=0}^{\infty} \left(\sum_{i=0}^{n-1} A_i \ell^{n-i} \right)^j \right), \end{aligned} \tag{3.13}$$

where ℓ is the integral operator ($\ell = s^{-1}$).

The approximate solution of the problem (3.10), (3.11) in the field of Mikusiński operators can be obtained by using finite sum instead of infinite series in the relation (3.13), i.e., it can be written as:

$$x_N = B_M \cdot \left(\sum_{j=0}^{N} \left(\sum_{i=0}^{n-1} A_i \ell^{n-i} \right)^j \right).$$

In the relation (3.13) the operator B_M, defined as:

$$B_M = \sum_{i=1}^{n} \ell^{n-i+1} x_{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} \ell^{n-k+i} + F\ell^n$$

represents the continuous function $\mathcal{B}_M$, and also the operators ℓ^{n-i}, for $i = 1, 2, \dots, n-1$, represent continuous functions, therefore the series $\sum_{j=0}^{\infty} \left(\sum_{i=0}^{n-1} A_i \ell^{n-i} \right)^j$ converges and represents a continuous function. From relation (3.13) it follows that the exact solution of the problem (3.10), (3.11) has the form:

$$x(t) = \mathcal{B}_M * \left(\sum_{j=0}^{\infty} \left(\sum_{i=0}^{n-1} A_i \frac{t^{n-i-1}}{\Gamma(n-i)} \right)^{*j} \right),$$

where $*j$ means $j-$th times applied convolution.

While, the approximate solution of the problem (3.10), (3.11) can be written as:

$$x_N(t) = \mathcal{B}_M * \left(\sum_{j=0}^{N} \left(\sum_{i=0}^{n-1} A_i \frac{t^{n-i-1}}{\Gamma(n-i)} \right)^{*j} \right),$$

where $*j$ means $j-$th times applied convolution.

3.1.2. *Sumudu transformation.* The Sumudu transformation of the equation (3.10) with the conditions (3.11) has the form

$$\frac{x}{u^n} - \sum_{i=0}^{n-1} A_i \frac{x}{u^i} = \sum_{i=1}^{n} \frac{x_{n-i}}{u^i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} \frac{A_{k+1}}{u^{k-i+1}} + F_S, \tag{3.14}$$

where x is the Sumudu transform of the solution and F_S is the Sumudu transform of the function f.

The solution of equation (3.14) has the form

$$\begin{aligned} x &= \frac{\sum_{i=1}^{n} \frac{x_{n-i}}{u^i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} \frac{A_{k+1}}{u^{k-i+1}} + F_S}{\frac{1}{u^n} - \sum_{i=0}^{n-1} \frac{A_i}{u^i}} \\ &= \frac{\sum_{i=1}^{n} x_{n-i} u^{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} u^{n-i+k-1} + F_S u^n}{1 - \sum_{i=0}^{n-1} A_i u^{n-i}}, \end{aligned}$$

or

$$x = \left(\sum_{i=1}^{n} x_{n-i} u^{n-i} - \sum_{i=0}^{n-2} x_i \sum_{k=i}^{n-2} A_{k+1} u^{n-i+k-1} + F_S u^n \right) \sum_{j=0}^{\infty} \left(\sum_{i=0}^{n-1} A_i u^{n-i} \right)^j. \tag{3.15}$$

It is obvious that the infinite series in (3.15) converges to the continuous function.

4. The solution of fractional differential equations

In this section we determine the exact and the approximate solution of the fractional orders (multi-term) differential equation

$$\mathcal{D}_t^{\beta_n} x(t) + \sum_{i=1}^{n-1} A_i \mathcal{D}_t^{\beta_i} x(t) + A_0 x(t) = f(t), \quad t > 0, \tag{4.1}$$

where the fractional derivatives are considered in the Caputo sense, β_i, $i=1,2,\ldots,n$, are rational numbers, such that $i-1 \leq \beta_i \leq i$, and A_i, $i=1,2,\ldots,n-1$, are real numbers. We take the same initial conditions as in (3.11):

$$x^{(i)}(0) = x_i, \quad i = 0,1,\ldots,n-1.$$

The problem (4.1), (3.11) corresponds to the Mikusiński operational equation:

$$s^{\beta_n}x + \sum_{i=1}^{n-1} A_i s^{\beta_i} x + A_0 x = \sum_{k=1}^{n} s^{\beta_n - k} x_{k-1} + \sum_{i=1}^{n-1}\sum_{k=1}^{i} A_i s^{\beta_i - k} x_{i-k} + F, \tag{4.2}$$

where $i-1 \leq \beta_i \leq i$, x is the operator corresponding to the solution $x(t)$, s is the differential operator and F is the Mikusiński operator corresponding to the function f.

The exact solution of the above equation has the form:

$$x = \frac{\sum\limits_{k=1}^{n} s^{\beta_n - k} x_{k-1} + \sum\limits_{i=1}^{n-1}\sum\limits_{k=1}^{i} A_i s^{\beta_i - k} x_{i-k} + F}{s^{\beta_n} + \sum\limits_{i=1}^{n-1} A_i s^{\beta_i} + A_0}.$$

It can be be transformed as follows:

$$\begin{aligned} x &= \frac{\sum\limits_{k=1}^{n} s^{\beta_n - k} x_{k-1} + \sum\limits_{i=1}^{n-1}\sum\limits_{k=1}^{i} A_i \ell^{\beta_n - \beta_i + k} x_{i-k} + F\ell^{\beta_n}}{I + \sum\limits_{i=1}^{n-1} A_i \ell^{\beta_n - \beta_i} + A_0 \ell^{\beta_n}} \\ &=: \frac{C_M}{I + \sum\limits_{i=1}^{n-1} A_i \ell^{\beta_n - \beta_i} + A_0 \ell^{\beta_n}} = C_M \cdot \left(\sum_{j=0}^{\infty} (-1)^j \left(\sum_{i=1}^{n-1} A_i \ell^{\beta_n - \beta_i} + A_0 \ell^{\beta_n} \right)^j \right). \end{aligned} \tag{4.3}$$

Since $\beta_n > 0$, and $\beta_n - \beta_i > 0$, for $i = 1,2,\ldots,n-1$, the series of operators converge and represent the continuous Mittag-Leffler function (3.8). In this sense the solution of the problem (4.1), (3.11) can be written as:

$$C_M(t) * \left(\sum_{j=0}^{\infty} (-1)^j \left(\sum_{i=1}^{n-1} A_i \frac{t^{\beta_n - \beta_i - 1}}{\Gamma(\beta_n - \beta_i)} + A_0 \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)} \right)^{*j} \right).$$

Analogously, the approximate solution of the problem (4.1), (3.11) can be written as:

$$x_N(t) = C_M(t) * \left(\sum_{j=0}^{N} (-1)^j \left(\sum_{i=1}^{n-1} A_i \frac{t^{\beta_n - \beta_i - 1}}{\Gamma(\beta_n - \beta_i)} + A_0 \frac{t^{\beta_n - 1}}{\Gamma(\beta_n)} \right)^{*j} \right),$$

where $*j$ mean j times applied convolution.

Applying the Sumudu transform the problem (4.1), (3.11) we get the equation

$$\frac{x}{u^{\beta_n}} + \sum_{i=0}^{n-1} A_i \frac{x}{u^{\beta_i}} + A_0 x = \sum_{k=0}^{n-1} \frac{x_{n-i}}{u^{\beta_n - k}} + \sum_{i=1}^{n-1}\sum_{k=0}^{i-1} A_i \frac{x_{i-k}}{u^{\beta_i - k}} + F_S, \tag{4.4}$$

which exact solution is:

$$
\begin{aligned}
x &= \frac{\sum_{k=0}^{n-1} \frac{x_{n-i}}{u^{\beta_n - k}} + \sum_{i=1}^{n-1}\sum_{k=0}^{i-1} A_i \frac{x_{i-k}}{u^{\beta_i - k}} + F_S}{\frac{1}{u^{\beta_n}} + \sum_{i=1}^{n-1} \frac{A_i}{u^{\beta_i}} + A_0} \\
&= \frac{\sum_{k=0}^{n-1} u^k x_{n-i} + \sum_{i=1}^{n-1}\sum_{k=0}^{i-1} A_i u^{\beta_n - \beta_i + k} x_{i-k} + F_S u^{\beta_n}}{1 + \sum_{i=1}^{n-1} A_i u^{\beta_n - \beta_i} + A_0 u^{\beta_n}} \\
&=: \frac{D_S}{1 + \sum_{i=1}^{n-1} A_i u^{\beta_n - \beta_i} + A_0 u^{\beta_n}} \\
&= D_S \sum_{j=0}^{\infty} (-1)^j \left(\sum_{i=1}^{n-1} A_i u^{\beta_n - \beta_i} + A_0 u^{\beta_n} \right)^j ,
\end{aligned}
$$

where

$$D_S = \sum_{k=0}^{n-1} u^k x_{n-i} + \sum_{i=1}^{n-1}\sum_{k=0}^{i-1} A_i u^{\beta_n - \beta_i + k} x_{i-k} + F_S u^{\beta_n}.$$

The approximate solution of equation (4.4) can be considered in the form

$$x_N = D_S \sum_{j=1}^{N} (-1)^j \left(\sum_{i=1}^{n-1} A_i u^{\beta_n - \beta_i} + A_0 u^{\beta_n} \right)^j .$$

5. An Example

Let us consider the fractional differential equation, with $1 < \beta_2 < 2,\ 0 < \beta_1 < 1$, and the coefficient A:

$$x_t^{\beta_2}(t) + A x_t^{\beta_1}(t) = 0 \tag{5.1}$$

with the initial conditions

$$x(0) = x_0, \qquad x'(0) = x_1. \tag{5.2}$$

The problem (5.1), (5.2) corresponds to the operational equation:

$$s^{\beta_2} x + A s^{\beta_1} x = s^{\beta_2 - 1} x_0 + s^{\beta_2 - 2} x_1 + s^{\beta_1 - 1} A x_0. \tag{5.3}$$

The solution of this operational equation is

$$
\begin{aligned}
x &= \frac{s^{\beta_2 - 1} x_0 + s^{\beta_2 - 2} x_1 + s^{\beta_1 - 1} A x_0}{s^{\beta_2} + A s^{\beta_1}} = \frac{\ell x_0 + \ell^2 x_1 + \ell^{\beta_2 - \beta_1 + 1} A x_0}{I + A \ell^{\beta_2 - \beta_1}} \\
&= \left(\ell x_0 + \ell^2 x_1 + \ell^{\beta_2 - \beta_1 + 1} A x_0 \right) \sum_{j=0}^{\infty} (-1)^j \left(A \ell^{\beta_2 - \beta_1} \right)^j .
\end{aligned}
$$

The Sumudu transform of the problem (5.1), (5.2) is:

$$\frac{x}{u^{\beta_2}} + A \frac{x}{u^{\beta_1}} x = \frac{x_0}{u^{\beta_2}} + \frac{x_1}{u^{\beta_2 - 1}} + \frac{A x_0}{u^{\beta_1}}. \tag{5.4}$$

The exact solution of the above equation (5.4) is:

$$\begin{aligned} x &= \frac{\frac{x_0}{u^{\beta_2}} + \frac{x_1}{u^{\beta_2-1}} + \frac{Ax_0}{u^{\beta_1}}}{\frac{1}{u^{\beta_2}} + A\frac{1}{u^{\beta_1}}} = \frac{x_0 + x_1 u + Ax_0 u^{\beta_2-\beta_1}}{1 + Au^{\beta_2-\beta_1}} \\ &= \left(x_0 + x_1 u + Ax_0 u^{\beta_2-\beta_1}\right) \sum_{j=0}^{\infty} (-1)^j \left(Au^{\beta_2-\beta_1}\right)^j . \end{aligned}$$

Applying the inverse Sumudu transform we obtain the solution of the problem (5.1), (5.2), as follows:

$$\begin{aligned} x(t) &= x_0 \sum_{j=0}^{\infty} (-1)^j \frac{A^j t^{j(\beta_2-\beta_1)}}{\Gamma(j(\beta_2-\beta_1)+1)} + x_1 \sum_{j=0}^{\infty} (-1)^j \frac{A^j t^{j(\beta_2-\beta_1)+1}}{\Gamma(j(\beta_2-\beta_1)+2)} \\ &\quad + x_0 \sum_{j=0}^{\infty} (-1)^j \frac{A^j t^{(j+1)(\beta_2-\beta_1)}}{\Gamma((j+1)(\beta_2-\beta_1)+1)} \\ &= x_0\, E_{\beta 2-\beta_1,1}(-At^{\beta_2-\beta_1}) + x_1 t\, E_{\beta 2-\beta_1,2}(-At^{\beta_2-\beta_1}) \\ &\quad + x_0 t^{\beta_2-\beta_1}\, E_{\beta 2-\beta_1,\beta_2-\beta_1+1}(-At^{\beta_2-\beta_1}). \end{aligned}$$

Let us note that the solution of the considered problem consists of three Mittag-Leffler functions, all of them with converging series for $t \in R$, because $\beta_2 > \beta_1$.

The approximate solution of the problem (5.1), (5.2) has the form:

$$\begin{aligned} x_N(t) &= x_0 \sum_{j=0}^{N} (-1)^j \frac{A^j t^{(j)(\beta_2-\beta_1)}}{\Gamma((j)(\beta_2-\beta_1)+1)} + x_1 \sum_{j=0}^{N} (-1)^j \frac{A^j t^{j(\beta_2-\beta_1)+1}}{\Gamma(j(\beta_2-\beta_1)+2)} \\ &\quad + x_0 \sum_{j=0}^{N} (-1)^j \frac{A^j t^{(j+1)(\beta_2-\beta_1)}}{\Gamma((j+1)(\beta_2-\beta_1)+1)}. \end{aligned}$$

References

[1] F. B. M. Belgacem, A. A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, Journal of Applied Mathematics and Stochastic Analysis **2006** (2006) 1–23.

[2] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent - II*, Geophys. J. Royal Astronom. Soc. **13**, 5 (1967) 529–539 (Reprinted in: Fract. Calc. Appl. Anal. **11**, 1 (2008) 3–14).

[3] L. Debnath, *Integral Transforms and Their Applications*, CRC Press, Florida (1995).

[4] I. H. Dimovski, *Convolution Calculus*, Kluwer, Dordrecht (1990).

[5] I. H. Dimovski, K. Skornik, *Operational calculi for the Euler operator*, Fract. Calc. Appl. Anal. **9**, 2 (2006) 89–100.

[6] I. H. Dimovski, *Nonclassical convolutions and their uses*, Fract. Calc. Appl. Anal. **17**, 4 (2014) 936–944.

[7] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg (2014).

[8] Q. D. Katatbet, F. B. M., Belgacem, *Applications of the Sumudu transform to fractional differential equations*, Nonlinear Studies **18**, 1 (2011) 99–112.

[9] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman Sci. and Technical, Harlow and J. Wiley, N. York (1994)

[10] V. Kiryakova, *Fractional order differential and integral equations with Erdélyi-Kober operators: Explicit solutions by means of the transmutation method*, American Institute of Physics Conf. Proc. **1410** (2011) 247–258.

[11] V. Kiryakova, *Some operational tools for solving fractional and higher integer order differential equations: A survey on their mutual relations*, American Institute of Physics Conf. Proc. **1497** (2012) 273–289.

[12] F. Mainardi, Yu. Luchko, G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. **4**, 2 (2001) 153–192.

[13] F. Mainardi, G. Pagnini, *The Wright functions as the solutions of time-fractional diffusion equation*, Applied Math. and Comput. **141**, 1 (2003) 51–62.

[14] J. Mikusiński, *Operational Calculus*, Vol. **I**, PWN – Polish Scientific Publishers, Warszawa & Pergamon Press, Oxford (1983).

[15] J. Mikusiński, T. Boehme, *Operational Calculus*, Vol. **II**, PWN – Polish Scientific Publishers, Warszawa & Pergamon Press, Oxford (1987).

[16] J. Paneva–Konovska, *From Bessel to Multi–Index Mittag–Leffler Functions. Enumerable Families, Series in Them and Convergence*, World Scientific, London (2016).

[17] I. Podlubny, *Fractional Differential Equations*, Acad. Press, San Diego (1999).

[18] B. Ross, *A brief history and exposition of fundamental theory of fractional calculus*, in: "Fractional Calculus and Its Applications" (Proc. 1st Internat. Conf. held in New Haven, 1974; Ed. B. Ross), Lecture Notes in Math. **457**, Springer-Verlag, N. York (1975) 1–37.

[19] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Sci. Publ., Switzerland (1993).

[20] B. Stanković, *Operator linear differential equations of order m*, Journal of Differential Equations **5**, 1 (1969) 1–11.

[21] Dj. Takači, A. Takači, *On the approximate solution of mathematical model of a viscoelastic bar*, Nonlinear Analysis **67** (2007) 1560–1569.

[22] Dj. Takači, A. Takači, M. Štrboja, *On the character of operational solutions of the time-fractional diffusion equation*, Nonlinear Analysis **72**, 5 (2010) 2367–2374.

[23] D. Takači, A. Takači, A. Takači, *On the operational solutions of fuzzy fractional differential equations*, Fract. Calc. Appl. Anal. **17**, 4 (2014) 1100–1113.

[24] W. T. Thomson, *Laplace Transformation Theory and Engineering Applications*, Prentice-Hall Engineering Design Ser., Prentice-Hall Inc., New York (1950).

[25] G. K. Watugala, *Sumudu transform: a new integral transform to solve differential equation and control engineering problems*, International J. of Mathematical Education in Science and Technology **24**, 1 (1993) 35–43.

[26] G. K. Watugala, *Sumudu transform – a new integral transform to solve differential equations and control engineering problems*, Mathematical Engineering in Industry **6**, 4 (1998) 319–329.

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