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INEQUALITIES FOR THE PARTIAL SUMS OF SOME MITTAG-LEFFLER TYPE SERIES

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Abstract. In this paper we consider series defined by means of the generalized (3-index) Mittag-Leffler functions. Their partial sums subsequences as well as integrals of functions, connected with these subsequences, are estimated.

1. Introduction

Let $E^{\gamma}_{\alpha,\,n}(z)$ denote the generalized Mittag-Leffler (M-L) function with 3 indices: $\alpha > 0$, non-negative integer n and $-1 \le \gamma \le 1$, arising as a natural generalization of the Mittag-Leffler functions E_α and $E_{\alpha,n}$ ([6], Section 18.1) and introduced by Prabhakar [22]. It is defined in the whole complex plane $\mathbb{C}$ by the power series

$$E^{\gamma}_{\alpha,\,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!}, \quad z \in \mathbb{C}, \tag{1.1}$$

where $(\gamma)_k$ is the Pochhammer symbol ([6], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1).$$

Bearing in mind that the value of $1/\Gamma(\alpha k + n)$ is equal to 0 for $n = 0$ and $k = 0$, for convenience we denote

$$\widetilde{E}^{\gamma}_{\alpha,\,n}(z) = \frac{\Gamma(\alpha p + n)}{(\gamma)_p}\, z^{n-p}\, E^{\gamma}_{\alpha,\,n}(z), \ \ \gamma \neq 0, \ \ n \in \mathbb{N}_0, \tag{1.2}$$

with $p = 1$ for $n = 0$, and $p = 0$ for $n \in \mathbb{N}$. We consider series in these special functions of the kind:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,\,n}(z), \tag{1.3}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$. For $\gamma = 0$ the function in (1.1) reduces to the constants $E^0_{\alpha,\,0} = 0$ and $E^0_{\alpha,\,n} = 1/\Gamma(n)$ for $n \in \mathbb{N}$, and then the notations in (1.2) yield $\widetilde{E}^0_{\alpha,\,0}(z) = 1$ (just for completeness) and $\widetilde{E}^0_{\alpha,\,n}(z) = z^n$. We do not

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deal with this trivial case, because then the series (1.3) is simply a classical power series.

In a number of recently published papers (see e.g. [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]) and also in the book [21], the author has provided various results related to the convergence of such type of series as well as to series in other special functions in a sense of [7], [9]. For example, the Bessel functions and their generalizations, including the hyper-Bessel functions, introduced by Delerue in 1953, belong to this category of special functions. These interesting class of multi-index functions is closely related to the hyper-Bessel differential operators of arbitrary integer order $m > 1$, introduced by Dimovski [3]. The obtained results, discussed above, refer to the domains where these series converge and where they do not converge, and moreover, where the convergence is uniform and where it is not. Finding of such kind of results is motivated by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions and their generalizations (as for example, in Luchko and Gorenflo [10], Kiryakova [8], Ding and Nieto [4], Ashurov, Cabada and Turmetov [1]). The functions (1.2) and series in them have recently been used to represent solutions of the generalized Langevin equation by Sandev, Tomovski and Dubbeldam [26], as well as in the eigenfunction expansion of the solution of two-term time-fractional equations by Bazhlekova and Dimovski [2].

Other studies, connected to various anomalous diffusion and relaxation processes, generalized diffusion and Fokker-Planck-Smoluchowski equations with the corresponding memory kernels, can be seen in the survey paper by Sandev, Chechkin, Kantz and Metzler [23]. For various distributed-order and generalized distributed-order diffusion equations, involving the Prabhakar functions, see the papers by Sandev, Chechkin, Korabel, Kantz, Sokolov and Metzler [24] and by Sandev, Tomovski, and Crnkovic [25].

In what follows we use the notations $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with radius R and for its boundary, i.e.

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\},$$

and $\widetilde{D}(0;R)$ for the circular domain

$$\widetilde{D}(0;R) = \{z : |z| > R,\ z \in \mathbb{C}\}.$$

We only recall that the series (1.3) is absolutely convergent in the open disk $D(0;R)$ and divergent in the circular domain $\widetilde{D}(0;R)$ with radius R (see [15] and also [21, p. 174, (9.32)]), given by

$$R = \left[\limsup_{n\to\infty} (|a_n|)^{1/n}\right]^{-1}. \tag{1.4}$$

In other words, the domain of convergence of the series (1.3) is the open disk $D(0;R)$ with a radius of convergence (1.4). Moreover, if this series converges at the point $z_0 \neq 0$, then it is absolutely convergent in the disk $D(0;|z_0|)$. Inside the disk $D(0;R)$, i.e. on each closed disk $|z| \le r$ with $r < R$, the convergence is uniform. The very disk of convergence is not obligatorily a domain of uniform convergence and the series may even be divergent on its boundary. Thus, it is possible a given series of the type (1.3) with a finite radius of convergence $0 < R < \infty$ to be convergent or divergent at some points of the boundary $C(0;R)$. These points could be regular or singular for its sum f, but the series diverges outside the domain of convergence.

However, sometimes it is possible to exist a subsequence of its partial sums which converges in a neighborhood of a regular point to the sum. The elements of such type of subsequence, related to the series (1.3), and integrals connected with them, are estimated modulo in this paper. We are interested in these estimations in order to study the series' behaviour as completely as is possible.

2. Auxiliary statements

Since expressions of the forms $\frac{\Gamma(\alpha)}{\Gamma(m\alpha)}$ and $\frac{\Gamma(n)}{\Gamma(m\alpha+n)}$ $(n = 1, 2, \dots)$ are involved in the sums under consideration, we first need to estimate the modules of these Γ-quotients.

Lemma 2.1. *Let $\alpha \geq 2$ and $m \geq 2$, then the following inequalities hold true*

$$\frac{\Gamma(\alpha)}{\Gamma(m\alpha)} \leq \frac{1}{(m+1)!}. \tag{2.1}$$

Proof. First, for $m = 2$ we have

$$\Gamma(2\alpha) = (2\alpha-2)(2\alpha-1)\Gamma(2\alpha-2)$$

and since $(2\alpha - 2) \geq \alpha \geq 2$,

$$\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha-2)} \frac{1}{(2\alpha-2)(2\alpha-1)} \leq \frac{1}{(2\alpha-2)(2\alpha-1)} \leq \frac{1}{3!}. \tag{2.2}$$

Then by induction, supposing that

$$\frac{\Gamma(\alpha)}{\Gamma(m\alpha)} \leq \frac{1}{(m+1)!},$$

and in view of $m\alpha + \alpha - 1 \geq m + m + \alpha - 1 \geq m + 2$, the conclusion

$$\begin{aligned} \frac{\Gamma(\alpha)}{\Gamma((m+1)\alpha)} &= \frac{\Gamma(\alpha)}{(m\alpha+\alpha-1)\Gamma(m\alpha+\alpha-1)} \\ &= \frac{\Gamma(\alpha)}{\Gamma(m\alpha)} \frac{\Gamma(m\alpha)}{\Gamma(m\alpha+\alpha-1)} \frac{1}{(m\alpha+\alpha-1)} \\ &\leq \frac{1}{(m+1)!} \frac{1}{(m\alpha+\alpha-1)} \leq \frac{1}{(m+2)!} \end{aligned} \tag{2.3}$$

can be made, which means that inequality (2.1) is satisfied for each integer value of $m \geq 2$. □

Lemma 2.2. *Let $\alpha \geq 2$ and $m \in \mathbb{N}$, then the following inequalities hold true*

$$\frac{\Gamma(n)}{\Gamma(m\alpha+n)} \leq \frac{1}{(m+n-1)_{m+1}}, \tag{2.4}$$

for all values of $n \in \mathbb{N}$.

Proof. Note that $m + n - 1 = m$ if $n = 1$, and checking the result (2.4) in this case goes separately for $m = 1$ and $m \geq 2$.

Namely, for $m = 1$,

$$\frac{\Gamma(1)}{\Gamma(\alpha+1)} = \frac{1}{\alpha\Gamma(\alpha)} \leq \frac{1}{\alpha} \leq \frac{1}{1.2} = \frac{1}{(1)_2}, \tag{2.5}$$

and respectively for $m \geq 2$, due to $m\alpha - m \geq 2$,

$$\begin{aligned}\frac{\Gamma(1)}{\Gamma(m\alpha+1)} &= \frac{1}{(m\alpha)\dots(m\alpha-m)\Gamma(m\alpha-m)} \\ &\leq \frac{1}{(2m)\dots(2m-m)} = \frac{1}{(m)_{m+1}}.\end{aligned} \tag{2.6}$$

Thus, for $n = 1$ the inequality (2.4) is verified.

Further, using induction, suppose that

$$\frac{\Gamma(n)}{\Gamma(m\alpha+n)} \leq \frac{1}{(m+n-1)_{m+1}}, \tag{2.7}$$

and then

$$\begin{aligned}\frac{\Gamma(n+1)}{\Gamma(m\alpha+n+1)} &= \frac{\Gamma(n)}{\Gamma(m\alpha+n)}\frac{n}{(m\alpha+n)} \leq \frac{1}{(m+n-1)_{m+1}}\frac{n}{(2m+n)} \\ &= \frac{n}{(m+n-1)}\frac{1}{(m+n)}\cdots\frac{1}{(2m+n-1)}\frac{1}{(2m+n)} \leq \frac{1}{(m+n)_{m+1}}.\end{aligned} \tag{2.8}$$

Now, in view of (2.5)–(2.8), the inequality (2.4) follows for all $n \in \mathbb{N}$. □

Observing that for $-1 \leq \gamma \leq 1$ and $m \in \mathbb{N}$, the inequalities

$$\left|\frac{(\gamma+1)_m}{(m+1)!}\right| \leq 1, \quad \left|\frac{(\gamma)_m}{m!}\right| \leq 1 \tag{2.9}$$

are satisfied, and setting for convenience

$$\widetilde{\Gamma}^{\gamma}_{\alpha,m,0} = \frac{\Gamma(\alpha)}{\Gamma(m\alpha)}\frac{(\gamma+1)_{m-1}}{m!}, \quad \widetilde{\Gamma}^{\gamma}_{\alpha,m,n} = \frac{\Gamma(n)}{\Gamma(m\alpha+n)}\frac{(\gamma)_m}{m!}, \tag{2.10}$$

the following elementary statement can be formulated.

Corollary 2.3. *If $\alpha \geq 2$, $-1 \leq \gamma \leq 1$ and $n \in \mathbb{N}$, then the following inequalities are valid:*

$$\begin{aligned}\left|\widetilde{\Gamma}^{\gamma}_{\alpha,m,0}\right| &\leq \frac{1}{(m+1)!}, \quad \textit{for} \quad m = 2, 3, \dots, \\ \left|\widetilde{\Gamma}^{\gamma}_{\alpha,m,n}\right| &\leq \frac{1}{(m+n-1)_{m+1}}, \quad \textit{for} \quad m \in \mathbb{N}.\end{aligned} \tag{2.11}$$

Proof. The validity of (2.11) automatically follows bringing together (2.1), (2.4), (2.9) and (2.10). □

3. Some useful estimates

Let the domain G be bounded by the circles $\widetilde{\gamma} \subset \widetilde{D}(0;1)$ and $C(0;1)$, i.e. G is their outside, and let $C(0;R) \subset G$. Denoting

$$S_{k_n}(z) = \sum_{m=0}^{k_n} a_m \widetilde{E}^{\gamma}_{\alpha,\,m}(z), \qquad n = 1, 2, \dots, \tag{3.1}$$

and

$$I_m = \frac{1}{2\pi i}\int\limits_{|z|=R} \frac{S_{k_n}(z)}{z^{m+1}}\,dz, \tag{3.2}$$

the following result can be derived.

Theorem 3.1. *If the series* (1.1) *has a radius of convergence 1 and the subsequence of its partial sums* (3.1) *converges uniformly on the curve* $\widetilde{\gamma}$, *then a positive number* η *exists, such that the following inequalities hold true*

$$|S_{k_n}(z)| \leq \left(Re^{-\eta}\right)^{k_n} \tag{3.3}$$

for $z \in C(0;R)$ *and*

$$|I_m| \leq \frac{\left(Re^{-\eta}\right)^{k_n}}{R^m}, \quad \textit{for} \quad m \leq k_n. \tag{3.4}$$

Proof. Consider the function

$$u_n(z) = \frac{1}{k_n}\log\left|\frac{S_{k_n}(z)}{z^{k_n}}\right| = \frac{1}{k_n}\log|S_{k_n}(z)| - \log|z|, \quad n = 1, 2, \ldots, \tag{3.5}$$

which is subharmonic in the set, considered further.

Obviously, there exists a constant N_1 such that the inequality $u_n(z) < -q$ holds for $z \in \widetilde{\gamma}$ and $n > N_1$, with q not depending on z and n, and $q > 0$.

Now, letting $\varepsilon > 0$, we set $\delta = e^{-\varepsilon/2}$, whence $\delta < 1$. Since the maximum of the module $\left|\frac{S_{k_n}(z)}{z^{k_n}}\right|$ in the closure $[\widetilde{D}(0;\delta)]$ of the set $\widetilde{D}(0;\delta)$ is attained on the circle $C(0;\delta)$, then setting

$$m_{\delta,n} = \max_{|z|=\delta}|S_{k_n}(z)|,$$

the following inequality holds true in the set $|z| \geq \delta$

$$u_n(z) \leq \frac{1}{k_n}\log m_{\delta,n} - \log\delta = \frac{1}{k_n}\log m_{\delta,n} + \frac{\varepsilon}{2},$$

and therefore on the set $|z| \geq 1$ as well. Since $\lim\limits_{n\to\infty}\frac{1}{k_n}\log m_{\delta,n} = 0$, there exists a number N_2 such that for all the values of $n > N_2$, for $|z| > \delta$ and also in particular for $|z| \geq 1$, the inequality $u_n(z) \leq \varepsilon$ holds. Along with both circles $\widetilde{\gamma}$ and $C(0;1)$, we also consider the circle $C(0;R)$ with a radius $R > 1$ and lying in the domain G, bounded by the curves $\widetilde{\gamma}$ and $C(0;1)$. Now, letting $n > N = N(\varepsilon) = \max(N_1, N_2)$, we can write $u_n(z) < -q$ for $z \in \widetilde{\gamma}$ and $u_n(z) \leq \varepsilon$ for $|z| = 1$. As a result, the following estimation is obtained $u_n(z) \leq -\lambda_1 q + \lambda_2\varepsilon$ for $z \in C(0;R)$ and $n > N$, with $\lambda_{1,2} > 0$ and $\lambda_{1,2}$ only depending on R (in fact, λ_1 is the minimum of a harmonic measure on the circle $C(0;R)$ [5, Ch. VIII, §4, the proof of Theorem 2]). Taking ε such that $-\lambda_1 q + \lambda_2\varepsilon = -\eta < 0$ we conclude that $u_n(z) \leq -\eta$ for $n > N$ and $|z| = R$, which produces the evaluation

$$|S_{k_n}(z)| \leq \left(Re^{-\eta}\right)^{k_n}.$$

Now, using the latest inequality for $|S_{k_n}(z)|$, the following estimation is obtained for the modules $|I_m|$ of the integral (3.2)

$$|I_m| \leq \frac{1}{2\pi}\frac{1}{R^{m+1}}\left(Re^{-\eta}\right)^{k_n} 2\pi R = \frac{\left(Re^{-\eta}\right)^{k_n}}{R^m}, \quad \text{for} \quad m \leq k_n,$$

that is the desired inequality. □

In conclusion, note that if $\gamma = 1$, the functions $\widetilde{E}^{\gamma}_{\alpha,n}$ reduce to the corresponding functions $\widetilde{E}_{\alpha,n}$, and then the results obtained here refer to the series in the 2-parametric M-L functions and its partial sums subsequence as well.

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