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A GENERALIZED INTEGRAL TRANSFORM ON THE CLASSICAL WIENER SPACE AND ITS APPLICATIONS

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Abstract. In this paper we define an integral transform, that generalizes several previously known integral transforms, and establish its existence and some properties on the classical Wiener space. We then give useful examples and applications in quantum mechanics.

1. Introduction and preliminaries

Let $K \equiv K_0[0,T]$ be the space of complex-valued continuous functions defined on $[0,T]$ which vanish at $t=0$, and $C_0[0,T]$ be the space of real-valued continuous functions defined on $[0,T]$ which also vanish at $t=0$. Let the triple $(C_0[0,T], \mathcal{M}, m)$ denote the classical Wiener space. We denote the Wiener integral of a Wiener integrable functional F by

$$E[F] = \int_{C_0[0,T]} F(x)dm(x).$$

In the unifying paper [20], Lee introduced an integral transform of analytic functionals on abstract Wiener spaces

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(\gamma x + \beta y)dm(x), \quad y \in K. \tag{1.1}$$

The Fourier-Wiener transform, the modified Fourier-Wiener transform, the Fourier-Feynman transform and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$ [1, 2, 3, 6, 7, 8, 9, 13, 17, 18].

In this paper, we introduce a new generalized integral transform which has a kernel in its definition. We then establish the existence and some properties of the integral transform. Furthermore, we give examples to illustrate usefulness of the transform. Finally, we give an application in quantum mechanic. Many of the results and formulas obtained here have some rather different aspects than the results and formulas for integral transforms studied in previous papers [2, 3, 6, 7, 8, 9, 13, 17, 18].

We recall some definitions and notations to understand this paper.

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A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{M}$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.), [16].

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. For a more detailed study of the PWZ stochastic integral, see [1, 2, 3, 7, 8, 9, 14, 17, 18].

2. A new generalized integral transform

In this section we define a new generalized integral transform and give simple examples.

Definition 2.1. *Let γ and β be nonzero complex numbers. Let F and G be functionals defined on K. The generalized integral transform $T^G_{\gamma,\beta}(F)$ of F given G is defined by the formula*

$$T^G_{\gamma,\beta}(F)(y) = \int_{C_0[0,T]} F(x)G(\gamma x + \beta y)dm(x), \quad y \in K, \tag{2.1}$$

if it exists. The functional G is called the kernel of the generalized integral transform.

Remark. *When $F \equiv 1$ on K, $T^G_{\gamma,\beta}(1)(y) = \mathcal{F}_{\gamma,\beta}(G)(y)$ for each $y \in K$. That is to say, it is Lee's integral transform* (1.1) *of G. This means that all formulas and results in previous papers* [2, 3, 6, 7, 8, 9, 13, 17, 18] *are corollaries of our results and formulas in this paper.*

Remark. *Unlike the classical Fourier and Laplace transforms that have kernels, Lee's integral transform* (1.1) *and its special cases on the Wiener space has no kernel. Our generalized integral transform* (2.1) *is the first integral transform studied on the Wiener space that has a kernel.*

We give simple examples to illustrate the usefulness of the kernel with respect to the generalized integral transform and to explain some differences between the generalized integral transform $T^G_{\gamma,\beta}$ and the integral transform $\mathcal{F}_{\gamma,\beta}$.

Example 2.2. *Let $G(x) = \exp\{x(T)\}$, $F_1(x) = x(T), F_2(x) = x^2(T)$, and $F_3(x) = \exp\{x(T)\}$. Then*

$$\begin{aligned}
T^G_{\gamma,\beta}(F_1)(y) &= \exp\{\beta y(T)\} \int_{C_0[0,T]} x(T)\exp\{\gamma x(T)\}dm(x) \\
&= \exp\Big\{\beta y(T) + \frac{\gamma^2 T}{2}\Big\} \frac{1}{\sqrt{2\pi T}} \int_{\mathbb{R}} u \exp\Big\{-\frac{(u-\gamma T)^2}{2T}\Big\}du \\
&= \gamma T \exp\Big\{\beta y(T) + \frac{\gamma^2 T}{2}\Big\},
\end{aligned}$$

$$\begin{aligned}
T^G_{\gamma,\beta}(F_2)(y) &= \exp\{\beta y(T)\} \int_{C_0[0,T]} x^2(T)\exp\{\gamma x(T)\}dm(x) \\
&= \exp\Big\{\beta y(T) + \frac{\gamma^2 T}{2}\Big\} \frac{1}{\sqrt{2\pi T}} \int_{\mathbb{R}} u^2 \exp\Big\{-\frac{(u-\gamma T)^2}{2T}\Big\}du \\
&= (T + \gamma^2 T^2) \exp\Big\{\beta y(T) + \frac{\gamma^2 T}{2}\Big\},
\end{aligned}$$

and

$$\begin{aligned} T^G_{\gamma,\beta}(F_3)(y) &= \exp\{\beta y(T)\}\int_{C_0[0,T]}\exp\{x(T)\}\exp\{\gamma x(T)\}dm(x)\\ &= \exp\{\beta y(T)\}\frac{1}{\sqrt{2\pi T}}\int_{\mathbb{R}}\exp\bigg\{(1+\gamma)u-\frac{u^2}{2T}\bigg\}du\\ &= \exp\bigg\{\beta y(T)+\frac{T(1+\gamma)^2}{2}\bigg\}. \end{aligned}$$

While, the formulas for the integral transform $\mathcal{F}_{\gamma,\beta}$ for functionals F_1, F_2 and F_3 are given by

$$\mathcal{F}_{\gamma,\beta}(F_1)(y) = \beta y(T),$$
$$\mathcal{F}_{\gamma,\beta}(F_2)(y) = \gamma^2 T + \beta^2 y^2(T),$$

and

$$\mathcal{F}_{\gamma,\beta}(F_3)(y) = \exp\bigg\{\beta y(T)+\frac{\gamma^2 T}{2}\bigg\}.$$

We next describe the class of functionals that we work with in this paper. Let $\{\alpha_1,\cdots,\alpha_n\}$ be an orthonormal set of functions in $L_2[0,T]$ and let $\mathbb{E}_0^{(n)}$ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(x) = f(\langle\alpha_1,x\rangle,\cdots,\langle\alpha_n,x\rangle) \equiv f(\langle\vec{\alpha},x\rangle) \tag{2.2}$$

for some positive integer n, where $f(\lambda_1,\cdots,\lambda_n)$ is an entire function of n complex variables $\lambda_1,\cdots,\lambda_n$ of exponential type; that is to say,

$$|f(\lambda_1,\cdots,\lambda_n)| \le A_F\exp\bigg\{B_F\sum_{j=1}^n|\lambda_j|\bigg\} \tag{2.3}$$

for a positive constant A_F and a nonnegative constant B_F.

Remark. *Note that $\mathbb{E}_0^{(n)}$ is a very rich class of functionals as $\mathbb{E}_0^{(n)}$ contains many unbounded functionals. For $F\in\mathbb{E}_0^{(n)}$, f is a constant function if and only if f is bounded. Furthermore we can show that for appropriate θ and ϕ, the following functionals of the forms*

$$\exp\bigg\{\int_0^T\theta(t,x(t))dt\bigg\},\ \exp\bigg\{\int_0^T\theta(t,x(t))dt\bigg\}\phi(x(T)),\ \exp\bigg\{i\int_0^T x(t)dt\bigg\}$$

are all elements of $\mathbb{E}_0^{(n)}$. These functionals are of interest in functional analysis and quantum mechanics, [17].

We finish this section by stating a well-known integration formula (see [22, formula (29)]) which will be used later in this paper.

Theorem 2.3. *Let $\{\alpha_1,\cdots,\alpha_n\}$ be an orthonormal set of functions in $L_2[0,T]$. Let $f:\mathbb{R}^n\to\mathbb{C}$ be Borel measurable and let*

$$F(x) = f(\langle\vec{\alpha},x\rangle).$$

Then

$$\begin{aligned} \int_{C_0[0,T]}F(x)dm(x) &= \int_{C_0[0,T]}f(\langle\vec{\alpha},x\rangle)dm(x)\\ &= \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}}\int_{\mathbb{R}^n}f(\vec{u})\exp\bigg\{-\sum_{j=1}^n\frac{u_j^2}{2}\bigg\}d\vec{u} \end{aligned} \tag{2.4}$$

in the sense that if either side of (2.4) *exists, then both sides exist and the equality holds.*

3. Existence of the generalized integral transform

In this section, we establish the existence of generalized integral transform $T^G_{\gamma,\beta}(F)$ of $F \in \mathbb{E}_0^{(n)}$ given the kernel G with useful examples. We then obtain some relationships involving the generalized integral transform.

3.1. **The classical kernel** G**.** We first consider the classical kernel G of the form

$$G(x) = g(x(T)), \tag{3.1}$$

where $g : \mathbb{C} \to \mathbb{C}$ is an entire function such that the restriction on $\mathbb{R}$ is an element of $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. This kernel is used to explain various physical phenomena in quantum mechanics.

Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions of bounded variation in $L_2[0, T]$. That is to say

$$(\alpha_i, \alpha_j)_2 = \int_0^T \alpha_i(t)\overline{\alpha_j(t)}dt = \begin{cases} 0 & , \ i \neq j \\ 1 & , \ i = j \end{cases}$$

for all i and j in the set $\{1, 2, \cdots, n\}$. In order to obtain the existence of $T^G_{\gamma,\beta}(F)$ for all functionals $F \in \mathbb{E}_0^{(n)}$ given the kernel G as in equation (3.1), we have to consider the following observation. Let α_0 be the constant function

$$\alpha_0(t) = \frac{1}{\sqrt{T}}$$

for all $t \in [0, T]$. Then four cases arise:

(a) The normalized constant function α_0 is orthogonal to span $\{\alpha_1, \cdots, \alpha_n\}$.
(b) $\alpha_0 \in \{\alpha_1, \cdots, \alpha_n\}$, say $\alpha_0 = \alpha_1$ for convenience.
(c) $\alpha_0 \notin \{\alpha_1, \cdots, \alpha_n\}$ but $\alpha_0 \in \text{span}\{\alpha_1, \cdots, \alpha_n\}$. In this case, one may choose a new orthonormal basis $\{\beta_1 \cdots, \beta_n\}$ for $\text{span}\{\alpha_1, \cdots, \alpha_n\}$ such that $\alpha_0 = \beta_1$. Now by an appropriate change in f it reduces to the case (b).
(d) $\alpha_0 \notin \text{span}\{\alpha_1, \cdots, \alpha_n\}$. In this case one may choose a basis $\{\beta_1 \cdots, \beta_{n+1}\}$ for the $\text{span}\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ such that $\alpha_0 = \beta_1$. Again after making an appropriate change in f, we are back to the case (b) except that the dimension is raised by one, for more details see [10, 15].

These mean that we can consider two cases as follows: either α_0 is orthogonal to $\text{span}\{\alpha_1, \cdots, \alpha_n\}$ or else $\alpha_0 \in \text{span}\{\alpha_1, \cdots, \alpha_n\}$. For each of these two cases we will obtain a formula for the generalized integral transform $T^G_{\gamma,\beta}(F)$ for all functionals $F \in \mathbb{E}_0^{(n)}$ given the kernel G as in equation (3.1) above.

Case 1: α_0 **is orthogonal to** $\text{span}\{\alpha_1, \cdots, \alpha_n\}$**.** That means

$$(\alpha_0, \alpha_i)_2 = \int_0^T \alpha_0(t)\overline{\alpha_i(t)}dt = 0$$

for all $i = 1, 2, \cdots, n$; and we proceed to establish the following theorem.

Theorem 3.1. *Assume that α_0 is orthogonal to* $\operatorname{span}\{\alpha_1, \cdots, \alpha_n\}$ *and let F be given by equation (2.2) and G be given by equation (3.1). Then for all nonzero complex numbers γ and β with $Re(1/\gamma^2) \geq 0$,*

$$T^G_{\gamma,\beta}(F)(y) = \Gamma_1(\langle\alpha_0, y\rangle),$$

where

$$\begin{aligned}\Gamma_1(w) = &\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\bigg\{-\sum_{j=1}^n \frac{u_j^2}{2}\bigg\} d\vec{u} \\ &\times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(v) \exp\bigg\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}w]^2\bigg\} dv.\end{aligned} \tag{3.2}$$

Furthermore, $T^G_{\gamma,\beta} : \mathbb{E}_0^{(n)} \longrightarrow \mathbb{E}_0^{(1)}$ is a linear operator.

Proof. First, let γ and β be nonzero real numbers. Using equations (2.1), (2.2), (2.4) and (3.1), it follows that for $y \in K$

$$\begin{aligned}T^G_{\gamma,\beta}(F)(y) &= \int_{C_0[0,T]} f(\langle\vec{\alpha}, x\rangle) g(\gamma x(T) + \beta y(T)) dm(x) \\ &= \int_{C_0[0,T]} f(\langle\vec{\alpha}, x\rangle) g(\gamma\sqrt{T}\langle\alpha_0, x\rangle + \beta\sqrt{T}\langle\alpha_0, y\rangle) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\bigg\{-\sum_{j=1}^n \frac{u_j^2}{2}\bigg\} d\vec{u} \\ &\qquad \times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(v) \exp\bigg\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\bigg\} dv \\ &\equiv \Gamma_1(\langle\alpha_0, y\rangle).\end{aligned}$$

This is also valid for all nonzero complex numbers γ and β with $Re(1/\gamma^2) \geq 0$ because the function $\exp\Big\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}w]^2\Big\}$ is analytic as a function of nonzero complex numbers γ and β with $Re(1/\gamma^2) \geq 0$. Next, the function Γ_1 is an entire function of variable w and by using equation (2.3),

$$\begin{aligned}|\Gamma_1(w)| &\leq \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \|g\|_1 \left(\frac{1}{2\pi|\gamma^2|T}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} |f(\vec{u})| \exp\bigg\{-\sum_{j=1}^n \frac{u_j^2}{2}\bigg\} d\vec{u} \\ &\leq \|g\|_1 \left(\frac{A_F^2}{2\pi|\gamma^2|T}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\bigg\{B_F \sum_{j=1}^n |u_j| - \sum_{j=1}^n \frac{u_j^2}{2}\bigg\} d\vec{u} \\ &= A_1 \exp\{B_1|w|\},\end{aligned}$$

where $B_1 = 0$ and

$$A_1 = \|g\|_1 \left(\frac{A_F^2}{2\pi|\gamma^2|T}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\bigg\{B_F \sum_{j=1}^n |u_j| - \sum_{j=1}^n \frac{u_j^2}{2}\bigg\} d\vec{u} < \infty.$$

This means that $T^G_{\gamma,\beta}(F)$ is an element of $\mathbb{E}_0^{(1)}$ and we complete the proof of Theorem 3.1. □

We next give a table of useful special cases of the generalized integral transform $T^G_{\gamma,\beta}(F)$ by giving two functions f and g as in Theorem 3.1 above.

	$g(u)=\exp\{-\frac{u^2}{2}\}$
$f(\vec{u})=\exp\{\sum_{j=1}^{n} u_j\}$	$\frac{1}{\sqrt{T(1+\gamma^2)}}\exp\left\{\frac{n}{2}-\frac{\beta^2T}{2+2\gamma^2}\langle\alpha_0,y\rangle\right\}$
$f(\vec{u})=\exp\{\sum_{j=1}^{n}-\frac{u_j^2}{2}\}$	$\frac{1}{\sqrt{T}}\exp\left\{\frac{n+\gamma^2}{2}+\beta\sqrt{T}\langle\alpha_0,y\rangle\right\}$

Table 1. Examples 1.

Case 2: $\alpha_0\in\mathrm{span}\{\alpha_1,\cdots,\alpha_n\}$**.** In this case we know that there exists constants $c_1,c_2,\cdots,c_n$, not all zero such that

$$\alpha_0(t)=\sum_{j=1}^{n}c_j\alpha_j(t) \tag{3.3}$$

for all $t\in[0,T]$. We have the following theorem.

Theorem 3.2. *Assume that α_0 is given by equation* (3.3)*. Let F be given by equation* (2.2) *and G be given by equation* (3.1)*. Then for all nonzero complex numbers γ and β,*

$$T^G_{\gamma,\beta}(F)(y)=\Gamma_2(\langle\alpha_0,y\rangle)$$

where

$$\Gamma_2(w)=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}f(\vec{u})g\left(\gamma\sqrt{T}\sum_{j=1}^{n}c_ju_j+\beta\sqrt{T}w\right)\exp\left\{-\sum_{j=1}^{n}\frac{u_j^2}{2}\right\}d\vec{u}.$$

Furthermore, $T^G_{\gamma,\beta}:\mathbb{E}_0^{(n)}\longrightarrow\mathbb{E}_0^{(1)}$ is a linear operator.

Proof. First, let γ and β be nonzero real numbers. Using equations (2.1), (2.2), (2.4) and (3.1), it follows that for $y\in K$

$$\begin{aligned}
&T^G_{\gamma,\beta}(F)(y)\\
&=\int_{C_0[0,T]}f(\langle\vec{\alpha},x\rangle)g(\gamma x(T)+\beta y(T))dm(x)\\
&=\int_{C_0[0,T]}f(\langle\vec{\alpha},x\rangle)g(\gamma\sqrt{T}\sum_{j=1}^{n}c_j\langle\alpha_j,x\rangle+\beta\sqrt{T}\langle\alpha_0,y\rangle)dm(x)\\
&=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}f(\vec{u})g(\gamma\sqrt{T}\sum_{j=1}^{n}c_ju_j+\beta\sqrt{T}\langle\alpha_0,y\rangle)\exp\left\{-\sum_{j=1}^{n}\frac{u_j^2}{2}\right\}d\vec{u}\\
&\equiv\Gamma_2(\langle\alpha_0,y\rangle).
\end{aligned}$$

This is still valid for all nonzero complex numbers γ and β because g is an entire function. Next, the function Γ_2 is an entire function of variable w and by using equation (2.3),

$$\begin{aligned}
|\Gamma_2(w)|&\le\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}|f(\vec{u})||g(\gamma\sqrt{T}\sum_{j=1}^{n}c_ju_j+\beta\sqrt{T}\langle\alpha_0,y\rangle)|\exp\left\{-\sum_{j=1}^{n}\frac{u_j^2}{2}\right\}d\vec{u}\\
&\le\|g\|_\infty A_F\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\exp\left\{B_F\sum_{j=1}^{n}|u_j|-\sum_{j=1}^{n}\frac{u_j^2}{2}\right\}d\vec{u}\\
&=A_2\exp\{B_2|w|\},
\end{aligned}$$

where $B_2 = 0$ and

$$A_2 = \|g\|_\infty A_F \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{B_F \sum_{j=1}^n |u_j| - \sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u} < \infty.$$

This means that $T_{\gamma,\beta}^G(F)$ is an element of $\mathbb{E}_0^{(1)}$ and hence we complete the proof of Theorem 3.2. □

We also give a table of special cases for the generalized integral transform $T_{\gamma,\beta}^G(F)$ by giving two functions f and g as in Theorem 3.2. We will only state for the case $n = 2$ to avoid some complexity.

	$g(u) = \exp\{-\frac{u^2}{2}\}$
$f(\vec{u}) = e^{-\frac{u_1^2+u_2^2}{2}}$	$\frac{1}{\sqrt{4+2\gamma^2 T \sum_{j=1}^2 c_j^2}} \exp\left\{-\frac{\beta^2 T\langle\alpha_0,y\rangle^2}{2+\gamma^2 T \sum_{j=1}^2 c_j^2}\right\}$
$f(\vec{u}) = e^{u_1+u_2}$	$\frac{1}{\sqrt{1+\gamma^2 T \sum_{j=1}^2 c_j^2}} \exp\left\{\frac{2+\gamma^2 T \sum_{j=1}^2 (c_1^2-c_2^2) - \beta T\langle\alpha_0,y\rangle(2\gamma \sum_{j=1}^2 c_j + \beta\langle\alpha_0,y\rangle)}{2+2\gamma^2 T \sum_{j=1}^2 c_j^2}\right\}$

TABLE 2. Examples 2.

3.2. **The kernel G in the class $\mathbb{E}_0^{(n)}$.** We now consider the kernel $G \in \mathbb{E}_0^{(n)}$.

Theorem 3.3. *Let F be given by equation (2.2) and let G be an element of $\mathbb{E}_0^{(n)}$ which has the form*

$$G(x) = g(\langle\vec{\alpha}, x\rangle),$$

where g is a function which satisfies the conditions of the class $\mathbb{E}_0^{(n)}$. Then for all nonzero complex numbers γ and β,

$$T_{\gamma,\beta}^G(F)(y) = \Gamma_3(\langle\vec{\alpha}, y\rangle),$$

where

$$\Gamma_3(\vec{w}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) g(\gamma\vec{u} + \beta\vec{w}) \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}.$$

Proof. Let γ and β be nonzero complex numbers. Using equations (2.1), (2.2) and (2.4), it follows that for $y \in K$,

$$\begin{aligned} T_{\gamma,\beta}^G(F)(y) &= \int_{C_0[0,T]} f(\langle\vec{\alpha}, x\rangle) g(\gamma\langle\vec{\alpha}, x\rangle + \beta\langle\vec{\alpha}, y\rangle) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) g(\gamma\vec{u} + \beta\langle\vec{\alpha}, y\rangle) \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u} \\ &\equiv \Gamma_3(\langle\vec{\alpha}, y\rangle). \end{aligned}$$

In fact, the function Γ_3 is an entire function and by using equation (2.3),

$$|\Gamma_3(\vec{w})| \le \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |f(\vec{u})||g(\gamma\vec{u}+\beta\vec{w})| \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}$$

$$\le A_F A_G \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{B_F \sum_{j=1}^n |u_j| + B_G|\gamma| \sum_{j=1}^n |u_j| + B_G|\beta| \sum_{j=1}^n |w_j| - \sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}$$

$$= A_3 \exp\{B_3 \sum_{j=1}^n |w_j|\},$$

where $B_3 = B_G|\beta|$ and

$$A_3 = A_F A_G \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{B_F \sum_{j=1}^n |u_j| + B_G|\gamma| \sum_{j=1}^n |u_j| - \sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u} < \infty.$$

This means that $T_{\gamma,\beta}^G(F)$ is an element of $\mathbb{E}_0^{(n)}$ and hence we complete the proof of Theorem 3.3. □

4. Some relationships

In this section, we establish some relationships with respect to the generalized integral transform for Case 1: α_0 is orthogonal to span$\{\alpha_1, \cdots, \alpha_n\}$. Before doing so, we need the following lemma.

Lemma 4.1. *Let F, G, γ and β be as in Theorem 3.1 and let $\mathcal{F}_{\gamma,\beta}$ is the integral operator defined by equation* (1.1). *Then*

$$T_{\gamma,\beta}^G(F)(y) = E[F]\mathcal{F}_{\gamma,\beta}(G)(y) \tag{4.1}$$

for $y \in K$. In particular, if $E[F] = 1$, then $T_{\gamma,\beta}^G(F)(y) = \mathcal{F}_{\gamma,\beta}(G)(y)$ for all $y \in K$.

Proof. For $F \in \mathbb{E}_0^{(n)}$, we note that

$$E[F] = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}.$$

In fact,

$$|E[F]| \le \left(\frac{A_F^2}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{\sum_{j=1}^n (B_F|u_j| - \frac{u_j^2}{2})\right\} d\vec{u} < \infty.$$

Also, we note that for $G(x) = g(x(T))$ and for all nonzero real numbers γ and β,

$$\mathcal{F}_{\gamma,\beta}(G)(y) = \int_{C_0[0,T]} g(\gamma\sqrt{T}\langle\alpha_0, x\rangle + \beta\sqrt{T}\langle\alpha_0, y\rangle) dm(x)$$

$$= \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(v) \exp\left\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\right\} dv.$$

This is valid for all nonzero complex numbers γ and β. From these facts, equation (3.2) and a similar method in Theorem 3.1, we obtain equation (4.1) as desired. □

In our next theorem, we establish the formula for composition of generalized integral operators.

Theorem 4.2. *Let F and G be as in Theorem 3.1. Then for all nonzero complex numbers $\gamma_1,\cdots,\gamma_n,\beta_1,\cdots,\beta_n$, with $Re(1/\gamma_j^2)\geq 0$, $j=1,2,\cdots,n$,*

$$\begin{aligned} &T^G_{\gamma_n,\beta_n}(\cdots(T^G_{\gamma_1,\beta_1}(F))\cdots)(y)\\ &=E[F]E[\mathcal{F}_{\gamma_1,\beta_1}(G)]\cdots E[\mathcal{F}_{\gamma_{n-1},\beta_{n-1}}(G)]\mathcal{F}_{\gamma_n,\beta_n}(G)(y)\end{aligned} \tag{4.2}$$

for $y\in K$.

Proof. Applying equation (4.1) twice, we have

$$\begin{aligned} T^G_{\gamma_2,\beta_2}(T^G_{\gamma_1,\beta_1}(F))(y) &= E[T^G_{\gamma_1,\beta_1}(F)]\mathcal{F}_{\gamma_2,\beta_2}(G)(y)\\ &= E[F]E[\mathcal{F}_{\gamma_1,\beta_1}(G)]\mathcal{F}_{\gamma_2,\beta_2}(G)(y),\end{aligned}$$

which establishes equation (4.2) for $n=2$. The general case $n\geq 2$ is deduced by mathematical induction. □

5. Applications

In this section, we present some applications involving quantum mechanics. The solution of a diffusion equation can be expressed as the limit of generalized integral transform. In order to do this we need to first list some results and formulas used in [4, 5]. We will only do this for Case 1; i.e., when θ_0 is orthogonal to $\text{span}\{\theta_1,\cdots,\theta_n\}$. Case 2; when $\theta_0\in\text{span}\{\theta_1,\cdots,\theta_n\}$ and the case $G\in\mathbb{E}_0^{(n)}$ follow in a similar way.

Let

$$C_0'[0,T]=\Big\{w\in C_0[0,T]: w(t)=\int_0^t z(s)ds,\ z\in L_2[0,T]\Big\}.$$

Then $C_0'[0,T]$ is a separable infinite dimensional real Hilbert space with inner product

$$(w_1,w_2)_{C_0'}=\int_0^T w_1'(t)w_2'(t)dt=\int_0^T z_1(t)z_2(t)dt.$$

It is well-known [12, 19] that $(C_0'[0,T],C_0[0,T],m)$ is an abstract Wiener space. Next, let $S: C_0'[0,T]\to C_0'[0,T]$ be the linear operator defined by

$$Sw(t)=\int_0^t w(s)ds.$$

Then the adjoint operator S^* of S is given by

$$S^*w(t)=w(T)t-\int_0^t w(s)ds=\int_0^t (w(T)-w(s))ds$$

and the linear operator $A=S^*S$ is given by

$$Aw(t)=\int_0^T \min\{s,t\}w(s)ds.$$

Furthermore, we see that A is a self-adjoint operator on $C_0'[0,T]$ and that

$$(w_1,Aw_2)_{C_0'}=(Sw_1,Sw_2)_{C_0'}=\int_0^T w_1(s)w_2(s)ds$$

for all $w_1,w_2\in C_0'[0,T]$. Hence A is a positive definite operator, i.e., $(w,Aw)_{C_0'}\geq 0$ for all $w\in C_0'[0,T]$. One can show that the orthonormal eigenfunctions $\{e_n\}$ of A are given by

$$e_n(t)=\frac{\sqrt{2T}}{(n-\frac12)\pi}\sin\left(\frac{(n-\frac12)\pi}{T}t\right)\equiv\int_0^t\alpha_n(s)ds$$

with corresponding eigenvalues $\{\delta_n\}$ given by

$$\delta_n = \left(\frac{T}{(n-\frac{1}{2})\pi}\right)^2. \tag{5.1}$$

Furthermore, it can be shown that $\{e_n\}$ is a basis of $C_0'[0,T]$ and so $\{\alpha_n\}$ is a basis of real $L_2[0,T]$, and that A is a trace class operator and so S is a Hilbert-Schmidt operator on $C_0'[0,T]$. In fact, the trace of A is given by $TrA = \frac{1}{2}T^2 = \int_0^T tdt$. In this case,

$$\int_0^T x^2(s)ds = \lim_{n\to\infty}\sum_{j=1}^n \delta_j\langle\alpha_j, x\rangle^2$$

for a.e. $x \in C_0[0,T]$. Let

$$f_n(\vec{w}) = \exp\left\{-\sum_{j=1}^n \delta_j w_j^2\right\}.$$

Then f_n is an entire function and $|f_n(\vec{w})| \le A_n \exp\{B_n \sum_{j=1}^n |w_j|\}$ where $A_n = \|f_n\|_1$ and $B_n = 0$ for all $n = 1, 2, \cdots$. Hence $F_n(x) = f_n(\langle\vec{\alpha}, x\rangle)$ in $\mathbb{E}_0^{(n)}$ for all $n = 1, 2, \cdots$. Also, for each $n = 1, \cdots$, let δ_n be as in equation (5.1). Then we obtain that

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \exp\left\{-\sum_{j=1}^n \delta_j\langle\alpha_j, x\rangle^2\right\} = \exp\left\{-\int_0^T x^2(s)ds\right\}$$

for a.e. $x \in C_0[0,T]$.

It is a well-known fact that the Wiener integral of the functionals

$$\exp\left\{-\int_0^T V(x(t))dt\right\}\psi(x(T)) \tag{5.2}$$

gives the solution to the diffusion equation

$$\frac{\partial}{\partial t}\varphi(u,t) = \frac{1}{2}\Delta\varphi(u,t) - V(u)\varphi(u,t) \tag{5.3}$$

with initial condition $\varphi(u,0) = \psi(u)$, where Δ is the Laplacian and V is an appropriate potential function. Our new integral transform $T_{\gamma,\beta}^G(F)$ represents the functional (5.2) as follows, when the potential function $V(u) = u^2$,

$$\begin{aligned}
T_{\gamma,\beta}^G(F)(y) &= \int_{C_0[0,T]} F(x)G(\gamma x + \beta y)dm(x) \\
&= \int_{C_0[0,T]} \exp\left\{-\int_0^T x^2(s)ds\right\} g(\gamma x(T) + \beta y(T))dm(x) \\
&= \int_{C_0[0,T]} \lim_{n\to\infty} \exp\left\{-\sum_{j=1}^n \delta_j\langle\alpha_j, x\rangle^2\right\} g(\gamma x(T) + \beta y(T))dm(x) \\
&= \lim_{n\to\infty}\int_{C_0[0,T]} \exp\left\{-\sum_{j=1}^n \delta_j\langle\alpha_j, x\rangle^2\right\} g(\gamma x(T) + \beta y(T))dm(x) \\
&= \lim_{n\to\infty} T_{\gamma,\beta}^G(F_n)(y),
\end{aligned}$$

where $F_n(x) = f_n(\langle\vec{\alpha}, x\rangle)$ in $\mathbb{E}_0^{(n)}$ for all $n = 1, 2, \cdots$. This tells us that the limit of the generalized integral transforms can be used to obtain the solution of diffusion equation (5.3). In fact, if $\gamma = 1$ and $y = 0$, then it gives the solution of the diffusion equation (5.3).

Now we shall explain this for the following cases:

• When $g(v) = \exp\{-\frac{v^2}{2}\}$, then $G(x) = g(x(T)) = \exp\{-\frac{x^2(T)}{2}\}$ and g is an element of $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

Let

$$I_n \equiv T^G_{\gamma,\beta}(F_n)(y) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f_n(\vec{u}) \exp\left\{\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}$$
$$\times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(v) \exp\left\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\right\} dv$$

for each $n = 1, 2, \cdots$. We have

$$I_n = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\{-\sum_{j=1} \delta_j u_j^2\} \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}$$
$$\times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{v^2}{2}\right\} \exp\left\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\right\} dv$$
$$= \left(\prod_{j=1}^n \frac{1}{2\delta_j + 1}\right)^{\frac{1}{2}} \frac{1}{\sqrt{T(1+\gamma^2)}} \exp\left\{-\beta^2 T \frac{\gamma^2(1+\gamma^2) - 1}{\gamma^2(1+\gamma^2)} \langle\alpha_0, y\rangle^2\right\}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left(\prod_{j=1}^n \frac{1}{2\delta_j + 1}\right)^{\frac{1}{2}} = \lim_{n\to\infty} \left(\prod_{j=1}^n \frac{(j - \frac{1}{2})^2\pi^2}{2T^2 + (j - \frac{1}{2})^2\pi^2}\right)^{\frac{1}{2}} = \sqrt{\operatorname{sech}(\sqrt{2}T)}$$

and hence we can conclude that

$$\lim_{n\to\infty} T^G_{\gamma,\beta}(F_n)(y) = \sqrt{\frac{\operatorname{sech}(\sqrt{2}T)}{T(1+\gamma^2)}} \exp\left\{-\beta^2 T \frac{\gamma^2(1+\gamma^2) - 1}{\gamma^2(1+\gamma^2)} \langle\alpha_0, y\rangle^2\right\}$$

gives us a solution for the diffusion equation (5.3).

• When $g(v) = \begin{cases} R, & \text{if } |v| \le L \\ 0, & \text{if } |v| > L \end{cases}$, where $R > 0$, then g is an element of $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. In this case, equation (5.3) is called the diffusion equation for a pulse wave packet with constant amplitude [21, 23].

In order to explain this case we first need to discuss the "Gauss error function" which is a non-elementary special function of sigmoid shape which occurs naturally in probability, statistics, and partial differential equations describing diffusion [11, 24]. It is defined for $x \ge 0$ by the formula

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The complementary error function, denoted by $Erfc(x)$, is defined as

$$Erfc(x) = 1 - Erf(x).$$

The imaginary error function, denoted by $Erfi$, is defined as

$$Erfi(x) = iErf(ix) = \frac{2}{\sqrt{\pi}} e^{x^2} D(x),$$

where $D(x)$ is the Dawson function. When the error function is evaluated for arbitrary complex arguments z, the resulting complex error function is usually referred to in scaled form as the Faddeeva function

$$w(z) = e^{-z^2} Erfc(iz).$$

Now, we will give the solution of the diffusion equation (5.3) for a pulse wave packet with constant amplitude as follows:

Let

$$J_n \equiv T^G_{\gamma,\beta}(F_n)(y) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f_n(\vec{u}) \exp\left\{\sum_{j=1}^{n} \frac{u_j^2}{2}\right\} d\vec{u}$$
$$\times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{\mathbb{R}} g(v) \exp\left\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\right\} dv$$

for each $n = 1, 2, \cdots$. Then we have

$$J_n = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \exp\{-\sum_{j=1} \delta_j u_j^2\} \exp\left\{-\sum_{j=1}^{n} \frac{u_j^2}{2}\right\} d\vec{u}$$
$$\times \left(\frac{1}{2\pi\gamma^2 T}\right)^{\frac{1}{2}} \int_{-L}^{L} R \exp\left\{-\frac{1}{2\gamma^2}[v - \beta\sqrt{T}\langle\alpha_0, y\rangle]^2\right\} dv$$
$$= \left(\prod_{j=1}^{n} \frac{1}{2\delta_j + 1}\right)^{\frac{1}{2}} \left(\frac{R^2}{4T}\right)^{\frac{1}{2}} \left[Erf\left(\frac{L - \beta\sqrt{T}\langle\alpha_0, y\rangle}{2\gamma^2}\right) + Erf\left(\frac{L + \beta\sqrt{T}\langle\alpha_0, y\rangle}{2\gamma^2}\right)\right].$$

Taking the limit as $n \to \infty$ again as in the proceeding of the first case, we have

$$\lim_{n\to\infty} \left(\prod_{j=1}^{n} \frac{1}{2\delta_j + 1}\right)^{\frac{1}{2}} = \sqrt{\operatorname{sech}(\sqrt{2}T)}$$

and hence we can conclude that

$$\lim_{n\to\infty} T^G_{\gamma,\beta}(F_n)(y) = \sqrt{\frac{R^2 \operatorname{sech}(\sqrt{2}T)}{4T}}$$
$$\times \left[Erf\left(\frac{L - \beta\sqrt{T}\langle\alpha_0, y\rangle}{2\gamma^2}\right) + Erf\left(\frac{L + \beta\sqrt{T}\langle\alpha_0, y\rangle}{2\gamma^2}\right)\right]$$

gives us the solution for the diffusion equation (5.3).

Remark. *In applications we meet also initial functions*

$$g_1(v) = \frac{m\omega^2}{8c^2}(v^2 - c^2)^2$$

and

$$g_2(v) = -\frac{\alpha^2\omega}{2k} \frac{s(s+1)}{\cosh^2(\alpha v)}.$$

In the case of g_1, we say that the diffusion equation (5.3) *has the double-well initial function. Also in the case of g_2, we say that the diffusion equation* (5.3) *has the Pöschl-Teller initial function.*

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