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SOME NEW PROPERTIES AND APPLICATIONS OF A FRACTIONAL FOURIER TRANSFORM

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Abstract. In this paper, we deal with the fractional Fourier transform in the form introduced a little while ago by the first named author and his coauthors. This transform is closely connected with the Fractional Calculus operators and has been already employed for solving of both the fractional diffusion equation and the fractional Schrödinger equation. In this paper, we continue the investigation of the fractional Fourier transform, and in particular prove some new operational relations for a linear combination of the left- and right-hand sided fractional derivatives. As an application of the obtained results, we provide a schema for solving the fractional differential equations with both left- and right-hand sided fractional derivatives without and with delays and give some examples of realization of our method for several fractional differential equations.

1. Introduction

The prehistory of the fractional Fourier analysis started most probably with the paper [14] by Wiener published as early as 1929. In this paper, Fourier developments of fractional order were introduced and applied for solving of certain integral equations.

Later on, some elements of the fractional Fourier analysis have been employed for solving mathematical problems in different applied areas and especially in signal processing (see e.g. [8] and the references therein), where several definitions of the fractional Fourier transform have been used in different contexts like the voice, images, or signal processing.

Despite of the term "fractional" in the notation, the fractional Fourier transforms that were introduced until recently, have nothing in common with Fractional Calculus, i.e., with the derivatives and integrals of the fractional order. The situation changed with release of the paper [6], where a new definition of the fractional Fourier transform $\mathcal{F}_\alpha$ of order α was suggested and the following operational relation was derived:

$$(\mathcal{F}_\alpha D^\alpha u)(\omega) = (-i\, c_\alpha \omega)(\mathcal{F}_\alpha u)(\omega), \tag{1.1}$$

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where D^α is a suitably defined fractional derivative and c_α is a constant depending on the order α of the fractional derivative.

In contrast to the well-known operational relation (see e.g. [12])

$$(\mathcal{F}D^\alpha_\pm u)(\omega) = (\mp i\omega)^\alpha(\mathcal{F}u)(\omega), \quad \alpha \geq 0, \tag{1.2}$$

where $\mathcal{F}$ is the conventional Fourier transform and $D^\alpha_\pm$ are the Riemann-Liouville fractional derivatives, the operational relation (1.1) avoids potential problems with different branches of the multi-valued complex function $(\mp i\omega)^\alpha$ that might appear while using the operational relation (1.2) and makes it an easy task to deal with certain linear fractional differential equations.

In particular, in [3] the new fractional Fourier transform has been applied for analytical treatment of the Cauchy problem for a partial space- and time-fractional differential equation with the Caputo time-fractional derivative and a linear combination of the spatial left- and right-hand sided Riemann-Liouville fractional derivatives. In [1], the fractional Fourier transform technique was employed to derive a solution of the Cauchy problem for the fractional Schrödinger equation with the quantum Riesz-Feller derivative in the case of a free particle in terms of the Fox H-function.

In this paper, we first derive some new important properties of the fractional Fourier transform including new operational relations. These results are then used to work out a schema for solving the fractional differential equations with both left- and right-hand sides fractional derivatives without and with delays. This kind of fractional differential equations is especially important for the fractional variation calculus, where the necessary optimality conditions of the Euler-Lagrange type are often formulated in form of some fractional differential equations that involve both the left- and the right-hand sided fractional derivatives (see e.g. the very recent book [7] and numerous references therein). To illustrate our method, several examples are provided.

The rest of the paper is organized as follows. In the second section, a definition of the fractional Fourier transform and some known results regarding the fractional Fourier transform that will be used in our paper are provided. In the third section, we mainly deal with the new operational relations for the fractional derivatives that are formulated via the fractional Fourier transform. In the last section, a general scheme for solving of a class of fractional differential equations with the left- and right-hand sided fractional derivatives is introduced. Moreover, the same method is applied for the fractional differential equations with the left- and right-hand sided fractional derivatives with delays, too. Several examples of realization of this schema are provided, too.

2. Fractional Fourier transform

2.1. **Definition and examples.** Before introducing the fractional Fourier transform, let us remind the reader of the definition of the conventional Fourier transform. For a function $f \in S$, S being the space of rapidly decreasing test functions on the real axis $I\!R$, the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are defined as follows:

$$\hat{f}(w) \;=\; (\mathcal{F}f)(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-iwx}\,dx, \quad w \in I\!R, \tag{2.1}$$

$$f(x) \ = \ (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(w)e^{iwx}\,dw, \quad x\in I\!R. \tag{2.2}$$

The operator $\mathcal{F}$ can be defined on the spaces $L_p(I\!R),\ 1\le p\le 2$ and extended to the space $S^{'}$ of tempered distributions ([2], [13]).

For working with the fractional Fourier transform and with fractional differentiation and integration operators some other spaces of functions are usually employed. In this paper, we stay in the framework of the Lizorkin space of functions (see e.g. [4], [5], [9] - [12]). For the sake of completeness, its definition is provided below.

Definition 2.1. *Let us denote by $V(I\!R)$ the set of functions $v\in S$ satisfying the conditions*

$$\frac{d^n v}{dx^n}\Big|_{x=0} = 0,\ \ n=0,1,2,\dots.$$

The Lizorkin space $\Phi(I\!R)$ is defined as the Fourier pre-image of the space $V(I\!R)$ in the space S, i.e.,

$$\Phi(I\!R) = \{\varphi\in S:\ \ \hat{\varphi}\in V(I\!R)\}.$$

According to Definition 2.1, a function $\varphi\in\Phi(I\!R)$ satisfies the orthogonality conditions

$$\int_{-\infty}^{+\infty} x^n\varphi(x)dx = 0,\ \ n=0,1,2,\dots.$$

It is known that the Lizorkin space $\Phi(I\!R)$ is invariant with respect to the fractional integration and differentiation operators (this is not the case for the whole space S of the rapidly decreasing test functions because the fractional integrals and derivatives of the functions from the space S do not always belong to the space S).

Following [3] and [6], we now provide a definition of the fractional Fourier transform.

Definition 2.2. *For a function $f\in\Phi(I\!R)$, the fractional Fourier transform $\hat{f}_\alpha$ of a positive order α is defined as follows:*

$$\hat{f}_\alpha(w) \ = \ (\mathcal{F}_\alpha f)(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e_\alpha(w,x)f(x)\,dx, \quad w\in I\!R, \tag{2.3}$$

where

$$e_\alpha(w,x) = \left\{\begin{array}{l} e^{i|w|^{1/\alpha}x},\ \ w\le 0 \\ e^{-i|w|^{1/\alpha}x},\ \ w\ge 0 \end{array}\right. \equiv e^{-isign(w)|w|^{\frac{1}{\alpha}}x}. \tag{2.4}$$

Of course, for $\alpha=1$ the kernel e_α defined by (2.4) coincides with the kernel of the conventional Fourier transform and thus the fractional Fourier transform of the order 1 is just the conventional Fourier transform, i.e., $\mathcal{F}_1\equiv\mathcal{F}$.

For an arbitrary positive α the relation between the fractional and conventional Fourier transforms is given by the following simple formula:

$$\hat{f}_\alpha(w) \ = \ (\mathcal{F}_\alpha f)(w) \equiv (\mathcal{F}f)(w_1) = \hat{f}(w_1), \tag{2.5}$$

where

$$w_1 = sign(w)|w|^{\frac{1}{\alpha}}. \tag{2.6}$$

The formulas (2.5)-(2.6) allow us to use the known properties of the Fourier transform to determine the fractional Fourier transform of the concrete functions. Let us consider some examples.

Example: The fractional Fourier transform of the square pulse

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

can be easily determined as follows:

$$\begin{aligned} \hat{f}_\alpha(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e_\alpha(w,x) f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-isign(w)|w|^{\frac{1}{\alpha}} x}\, dx \\ &= \frac{2}{\sqrt{2\pi}|w|^{1/\alpha}} \sin(|w|^{1/\alpha}). \end{aligned} \tag{2.7}$$

Example: To determine the fractional Fourier transform of the one-sided exponential function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

we calculate the following integral:

$$\begin{aligned} (\mathcal{F}_\alpha f(x))(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e_\alpha(w,x) f(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x(isign(w)|w|^{\frac{1}{\alpha}}+1)}\, dx \\ &= \frac{2}{\sqrt{2\pi}\left(1 + i|w|^{1/\alpha} sign(w)\right)}. \end{aligned} \tag{2.8}$$

Taking into account the formula (2.8) and using integration by parts, we can also compute the fractional Fourier transform of the function $xe^{-x}H(x)$, H being the Heaviside function:

$$\begin{aligned} (\mathcal{F}_\alpha xe^{-x}H(x))(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e_\alpha(w,x) xe^{-x} H(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x(isign(w)|w|^{\frac{1}{\alpha}}+1)} x dx \\ &= \frac{2}{\sqrt{2\pi}\left(1 + i|w|^{1/\alpha} sign(w)\right)^2}. \end{aligned} \tag{2.9}$$

2.2. **Properties of the fractional Fourier transform.** We start with some simple but important rules for calculation of the fractional Fourier transform and then proceed with the new operational relations for the fractional derivatives. For the proofs of the formulas (2.10) - (2.15) we refer the reader to [3].

Let $f,\ g$ belong to the Lisorkin space $\Phi(I\!R)$ and $\alpha > 0,\ w \in I\!R$.

Then the following transformation rules for the fractional Fourier transform are valid:

$$(\mathcal{F}_\alpha f(x-y))(w) = e_\alpha(w,y) \hat{f}_\alpha(w), \quad y \in I\!R, \tag{2.10}$$

$$(\mathcal{F}_\alpha\ f(\lambda x))(w) = \frac{1}{\lambda} (\mathcal{F}_\alpha f) \left(\frac{\omega}{\lambda^\alpha}\right), \quad \lambda > 0, \tag{2.11}$$

$$(\mathcal{F}_\alpha f'(x))(w) = ig_\alpha(w)\hat{f}_\alpha(w),\ g_\alpha(w) = \begin{cases} -|w|^{1/\alpha},\ w \le 0 \\ |w|^{1/\alpha},\ w \ge 0 \end{cases} \equiv sign(w)|w|^{1/\alpha}, \tag{2.12}$$

$$(\mathcal{F}_\alpha f^{(n)}(x))(w) = (ig_\alpha(w))^n \hat{f}_\alpha(w), \quad n \in I\!N, \tag{2.13}$$

$$i\frac{d}{dw}\hat{f}_\alpha(w) = g'_\alpha(w)(\mathcal{F}_\alpha\, xf(x))(w), \tag{2.14}$$

$$\mathcal{F}_\alpha(f * g)(w) = \sqrt{2\pi}\hat{f}_\alpha(w)\hat{g}_\alpha(w), \tag{2.15}$$

$$\begin{aligned}\hat{f}_\alpha(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e_\alpha(w,x)f(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\cos\left(|w|^{1/\alpha}x\right)f(x)dx - \frac{i}{\sqrt{2\pi}}sign(w)\int_{-\infty}^{+\infty}\sin\left(|w|^{1/\alpha}x\right)f(x)dx.\end{aligned} \tag{2.16}$$

We also mention an important formula for the Fractional inverse transform that was derived in [3]:

$$(\mathcal{F}_\alpha^{-1}f)(x) = \frac{1}{\sqrt{2\pi}\alpha}\int_{-\infty}^{+\infty} e^{isign(w)|w|^{1/\alpha}x}|w|^{\frac{1}{\alpha}-1}f(w)\,dw, \quad \alpha > 0,\ x \in I\!R. \tag{2.17}$$

For $f \in \Phi(I\!R)$, the relation

$$\mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha f \;=\; f$$

holds true.

As an illustration of possible applications of the transformation rules mentioned above, let us determine the fractional Fourier transform of the Gaussian function

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}. \tag{2.18}$$

Evidently, it satisfies the differential equation

$$\frac{d}{dx}\phi(x) = -x\phi(x). \tag{2.19}$$

Taking into account the transformation rules (2.12) and (2.14), we get, respectively,

$$\mathcal{F}_\alpha(\phi'(x)) = ig_\alpha(w)\hat{\phi}_\alpha(w),$$

and

$$\mathcal{F}_\alpha(-x\phi(x)) = -i\frac{1}{g'_\alpha(w)}\frac{d}{dw}\hat{\phi}_\alpha(w).$$

Application of the fractional Fourier transform to the equation (2.19) leads then to the ordinary differential equation

$$\frac{\hat{\phi'}_\alpha(w)}{\hat{\phi}_\alpha(w)} = -\frac{1}{\alpha}w^{\frac{2}{\alpha}-1} \tag{2.20}$$

with the solution given by

$$\hat{\phi}_\alpha(w) = \hat{\phi}_\alpha(0)e^{-\frac{1}{2}w^{\frac{2}{\alpha}}},\ \hat{\phi}_\alpha(0) = \frac{1}{\sqrt{2\pi}}. \tag{2.21}$$

Thus the fractional Fourier transform $\hat{\phi}_\alpha$ of the Gaussian distribution function ϕ can be written in the following form:

$$\hat{\phi}_\alpha(w) = \phi(w^{1/\alpha}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}w^{\frac{2}{\alpha}}}.$$

2.3. **Operational relations for the fractional derivatives.** The rest of this section is devoted to presentation of some new operational relations that will be derived for a general fractional differential operator which contains both the left- and the right-hand sided fractional derivatives:

$$(D^{\beta}_{a,b}u)(x) = a(D^{\beta}_{+}u)(x)+b(D^{\beta}_{-}u)(x),\ n-1 < \beta \leq n,\ n = [\beta]+1,\ a, b \in \mathbb{R}, \tag{2.22}$$

where D^{β}_{+} and D^{β}_{-} are the Riemann-Liouville fractional derivatives defined on the real axis, i.e.,

$$(D^{\beta}_{+}u)(x) = \left(\frac{d}{dx}\right)^{n} (I^{n-\beta}_{+}u)(x), \tag{2.23}$$

and

$$(D^{\beta}_{-}u)(x) = \left(-\frac{d}{dx}\right)^{n} (I^{n-\beta}_{-}u)(x), \tag{2.24}$$

I^{β}_{+} and I^{β}_{-} being the Riemann-Liouville fractional integral operators

$$(I^{n-\beta}_{+}u)(x) = \frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x} (x-t)^{n-\beta-1}u(t)dt, \tag{2.25}$$

$$(I^{n-\beta}_{-}u)(x) = \frac{1}{\Gamma(n-\beta)}\int_{x}^{+\infty} (t-x)^{n-\beta-1}u(t)dt. \tag{2.26}$$

Let us note here that in [3] and [6] a particular case of the fractional derivative (2.22) has been considered, namely, a one-parametric family of operators of the type (2.22) with $b = a - 1$.

In the proofs of our results, we use the formula of integration by parts for the Riemann-Liouville derivatives

$$\int_{-\infty}^{+\infty} v(x)(D^{\alpha}_{+}u)(x)dx = \int_{-\infty}^{+\infty} (D^{\alpha}_{-}v)(x)u(x)dx,\ \alpha > 0 \tag{2.27}$$

that holds true, e.g., for any $u, v \in \Phi(\mathbb{R})$ (see e.g. [6] or [12])). We also need some auxiliary formulas that are provided in two lemmas below.

Lemma 2.3. *Let $w \in \mathbb{R}\backslash\{0\}$ and $0 < \gamma < 1$. Then*

$$(I^{\gamma}_{+}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x}|w|^{-\frac{\gamma}{\alpha}}\left(\cos\left(\frac{\gamma\pi}{2}\right) + isign(w)\sin\left(\frac{\gamma\pi}{2}\right)\right) \tag{2.28}$$

and

$$(D^{\gamma}_{+}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x}|w|^{\frac{\gamma}{\alpha}}\left(\cos\left(\frac{\gamma\pi}{2}\right) - isign(w)\sin\left(\frac{\gamma\pi}{2}\right)\right). \tag{2.29}$$

Proof. To prove the formula (2.28), the substitution $t = x - \tau$ is made in the Riemann-Liouville fractional integral I^{γ}_{+} and we get the result after some elementary transformations:

$$\begin{aligned}\left(I^{\gamma}_{+}e^{-isign(w)|w|^{\frac{1}{\alpha}}t}\right)(x) &= \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{x} (x-t)^{\gamma-1}e^{-isign(w)|w|^{\frac{1}{\alpha}}t}dt \\ &= e^{-isign(w)|w|^{\frac{1}{\alpha}}x}\int_{0}^{+\infty} \tau^{\gamma-1}e^{isign(w)|w|^{\frac{1}{\alpha}}\tau}d\tau\end{aligned}$$

$$= e^{-isign(w)|w|^{\frac{1}{\alpha}}x} \left[\frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \tau^{\gamma-1} \cos(|w|^{1/\alpha}\tau)d\tau \right.$$
$$\left. +isign(w) \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \tau^{\gamma-1} \sin(|w|^{1/\alpha}\tau)d\tau \right]$$
$$= e^{-isign(w)|w|^{\frac{1}{\alpha}}x} |w|^{-\frac{\gamma}{\alpha}} \left(\cos\left(\frac{\gamma\pi}{2}\right) + isign(w) \sin\left(\frac{\gamma\pi}{2}\right)\right).$$

The formula (2.29) directly follows from (2.28):

$$\left(D_+^\gamma e^{-isign(w)|w|^{\frac{1}{\alpha}}t}\right)(x) = \frac{d}{dx}\left(I_+^{1-\gamma} e^{-isign(w)|w|^{\frac{1}{\alpha}}t}\right)(x)$$
$$= \frac{d}{dx}\left[e^{-isign(w)|w|^{\frac{1}{\alpha}}x} |w|^{-\frac{1-\gamma}{\alpha}} \left(\cos\left(\frac{(1-\gamma)\pi}{2}\right) \right.\right.$$
$$\left.\left. + isign(w) \sin\left(\frac{(1-\gamma)\pi}{2}\right)\right)\right]$$
$$= e^{-isign(w)|w|^{\frac{1}{\alpha}}x} |w|^{\frac{\gamma}{\alpha}} \left(\cos\left(\frac{\gamma\pi}{2}\right) - isign(w) \sin\left(\frac{\gamma\pi}{2}\right)\right).$$

□

Proceeding in the similar way, we prove the formulas given in the next lemma.

Lemma 2.4. *Let* $w \in \mathbb{R}\backslash\{0\}$ *and* $0 < \gamma < 1$. *Then*

$$(I_-^\gamma e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x} |w|^{-\frac{\gamma}{\alpha}} \left(\cos\left(\frac{\gamma\pi}{2}\right) - isign(w) \sin\left(\frac{\gamma\pi}{2}\right)\right) \tag{2.30}$$

and

$$(D_-^\gamma e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x} |w|^{\frac{\gamma}{\alpha}} \left(\cos\left(\frac{\gamma\pi}{2}\right) + isign(w) \sin\left(\frac{\gamma\pi}{2}\right)\right). \tag{2.31}$$

Now we formulate our main result regarding a new operational relation for the general differential operator $D_{a,b}^\gamma$ defined by (2.22).

Theorem 2.5. *Let* $0 < \gamma \le 1$ *and* $u \in \Phi(\mathbb{R})$. *Then the following operational relation holds true for any values of the parameters* a, $b \in \mathbb{R}$*:*

$$(\mathcal{F}_\alpha D_{a,b}^\gamma u)(w) = c_\gamma(w)|w|^{\frac{\gamma}{\alpha}}(\mathcal{F}_\alpha u)(w), \tag{2.32}$$

where $c_\gamma(w)$ *is defined by*

$$c_\gamma = (a+b)\cos\left(\frac{\gamma\pi}{2}\right) + isign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right).$$

Proof. Using arguments similar to those employed in [6], we first get the relation

$$(\mathcal{F}_\alpha D_{a,b}^\gamma u)(0) = 0$$

for any function u that belong to the Lizorkin space $\Phi(\mathbb{R})$. For $\omega \neq 0$, the formulas (2.27), (2.29), and (2.31) are applied and we have the following chain of equalities:

$$
\begin{aligned}
&(\mathcal{F}_\alpha D^\gamma_{a,b}u)(w) = \int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}(D^\gamma_{a,b}u)(x)dx \\
&= a\int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}(D^\gamma_{+}u)(x) + b\int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}(D^\gamma_{-}u)(x) \\
&= a\int_{-\infty}^{+\infty} (D^\gamma_{-}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x)u(x)dx + b\int_{-\infty}^{+\infty} (D^\gamma_{+}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x)u(x)dx \\
&= a|w|^{\frac{\gamma}{\alpha}}\left(\cos\left(\frac{\gamma\pi}{2}\right) + isign(w)\sin\left(\frac{\gamma\pi}{2}\right)\right)\int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}u(x)dx \\
&\qquad + b|w|^{\frac{\gamma}{\alpha}}\left(\cos\left(\frac{\gamma\pi}{2}\right) - isign(w)\sin\left(\frac{\gamma\pi}{2}\right)\right)\int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}u(x)dx \\
&= |w|^{\frac{\gamma}{\alpha}}(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + isign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)(\mathcal{F}_\alpha u)(w),
\end{aligned}
$$

that prove the formula (2.32). □

The formula (2.32) can be applied several times and we thus get a more general operational relation which is given by the formula (2.33) below.

Corollary 2.6. *Let $0 < \gamma \le 1$, $u \in \Phi(\mathbb{R})$, and $n \in \mathbb{N}$. Then the following operational relation holds true for any values of the parameters a, $b \in \mathbb{R}$:*

$$(\mathcal{F}_\alpha D^{\gamma n}_{a,b}u)(w) = c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}}(\mathcal{F}_\alpha u)(w), \tag{2.33}$$

where $c_{\gamma,n}(w)$ is given by

$$
\begin{aligned}
c_{\gamma,n}(w) = (isign(w))^n(a+(-1)^n b)\cos\left(\frac{(1-\gamma)n\pi}{2}\right)& \\
+\ isign(w)(-a+(-1)^n b)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)&.
\end{aligned} \tag{2.34}
$$

Proof. First, we apply the formulas (2.29) and (2.31) and get

$$
\begin{aligned}
&(D^{\gamma n}_{+}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x} \\
&\times\ |w|^{\frac{\gamma n}{\alpha}}(-isign(w))^n\left(\cos\left(\frac{(1-\gamma)n\pi}{2}\right) + isign(w)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)\right),
\end{aligned} \tag{2.35}
$$

and

$$
\begin{aligned}
&(D^{\gamma n}_{-}e^{-isign(w)|w|^{\frac{1}{\alpha}}t})(x) = e^{-isign(w)|w|^{\frac{1}{\alpha}}x} \\
&\times\ |w|^{\frac{\gamma n}{\alpha}}(isign(w))^n\left(\cos\left(\frac{(1-\gamma)n\pi}{2}\right) - isign(w)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)\right).
\end{aligned} \tag{2.36}
$$

Then formulas (2.27), (2.35), and (2.36) lead to the following chain of equalities

$$
\begin{aligned}
(\mathcal{F}_\alpha D^{\gamma n}_{a,b}u)(w) &= \int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}(D^{\gamma n}_{a,b}u)(x)dx \\
&= a|w|^{\frac{\gamma n}{\alpha}}(isign(w))^n\left(\cos\left(\frac{(1-\gamma)n\pi}{2}\right) - isign(w)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)\right) \\
&\quad \times \int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}u(x)dx \\
&+ b|w|^{\frac{\gamma n}{\alpha}}(-isign(w))^n\left(\cos\left(\frac{(1-\gamma)n\pi}{2}\right) + isign(w)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)\right) \\
&\quad \times \int_{-\infty}^{+\infty} e^{-isign(w)|w|^{\frac{1}{\alpha}}x}u(x)dx \\
&= |w|^{\frac{\gamma n}{\alpha}}(isign(w))^n(a+(-1)^n b)\cos\left(\frac{(1-\gamma)n\pi}{2}\right) \\
&\quad + isign(w)(-a+(-1)^n b)\sin\left(\frac{(1-\gamma)n\pi}{2}\right)(\mathcal{F}_\alpha u)(w),
\end{aligned}
$$

and this proves the operational relation (2.33). □

3. Fractional differential equations with the left- and right-hand sided fractional derivatives

As already mentioned in the introduction, fractional differential equations with both left- and right-hand sides fractional derivatives appear in a natural way while dealing with the problems of the fractional variation calculus.

In this section, a general schema for solving a particular class of such equations is introduced and illustrated by several examples. The main idea of the method is to employ the operational relations (2.32) and (2.33) along with other properties of the fractional Fourier transform to reduce the fractional differential equations to some algebraic equations in the frequency domain. Solving these equation leads to the explicit formulas for the fractional Fourier transforms of the solutions we are looking for. Then the inverse fractional Fourier transform is applied to determine the solutions in the time domain.

Let us consider the following class of the fractional differential equations:

$$\sum_{n=0}^{N} \alpha_n D^{\gamma n}_{a,b} y(x) = g(x),$$

where $g \in \Phi(\mathbb{R})$ and the solution is looked in $\Phi(\mathbb{R})$, too. By applying the operational relation (2.33) to this equation we get the algebraic equation

$$\sum_{n=0}^{N} \alpha_n c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}}(\mathcal{F}_\alpha y)(w) = (\mathcal{F}_\alpha g)(w)$$

with the solution given by

$$(\mathcal{F}_\alpha y)(w) = \frac{(\mathcal{F}_\alpha g)(w)}{\sum_{n=0}^{N} \alpha_n c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}}}.$$

Finally, the inverse fractional Fourier transform of the last formula leads to the solution in the form

$$y(x) = \left(\mathcal{F}_\alpha^{-1} \frac{(\mathcal{F}_\alpha g)(w)}{\sum_{n=0}^{N} \alpha_n c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}}} \right)(x)$$

that can be rewritten as the following convolution:

$$y(x) = (g \, * \, h_{\gamma,n})\,(x), \;\; h_{\gamma,n}(x) = \left(\mathcal{F}_\alpha^{-1} \frac{1}{\sum_{n=0}^{N} \alpha_n c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}}} \right)(x).$$

In the general case, a simplification of the formula for the inverse fractional Fourier transform at the right-hand side of the solution formula is not possible, but in some concrete cases we were able to get the closed form solutions that are presented below. For calculation of the inverse fractional Fourier transforms in the examples given below the CAS Mathematica has been employed.

Example: Let us consider the following fractional differential equation

$$D_{a,b}^{\gamma n} y(x) = g(x). \tag{3.1}$$

Following our general method, we first determine the Fourier transform of the solution:

$$\begin{aligned}
&(\mathcal{F}_\alpha D_{a,b}^{\gamma n} y(x))(w) = (\mathcal{F}_\alpha g(x))(w), \\
&\Leftrightarrow c_{\gamma,n}(w)|w|^{\frac{\gamma n}{\alpha}} (\mathcal{F}_\alpha y)(w) = (\mathcal{F}_\alpha g(x))(w), \\
&\Leftrightarrow (\mathcal{F}_\alpha y)(w) = \frac{1}{c_{\gamma,n}(w)} |w|^{-\frac{\gamma n}{\alpha}} (\mathcal{F}_\alpha g(x))(w).
\end{aligned}$$

Using the inverse fractional Fourier transform, Fourier transform convolution, and then evaluating the inverse Fourier transform for a particular case of the function $h_{\gamma,n}$, we get a closed form solution:

$$\begin{aligned}
&y(x) = \mathcal{F}_\alpha^{-1} \left(\frac{1}{c_{\gamma,n}(w)} e^{i sign(w)|w|^{\frac{1}{\alpha}} x} |w|^{-\frac{\gamma n}{\alpha}} \right) * g(x) \\
&\Leftrightarrow y(x) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} \frac{1}{c_{\gamma,n}(w)} e^{i sign(w)|w|^{\frac{1}{\alpha}} x} |w|^{\frac{1-\gamma n}{\alpha} - 1} dw \right) * g(x) \\
&\Leftrightarrow y(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{c_{\gamma,n}^2} \int_{-\infty}^{0} e^{-i(-w)^{\frac{1}{\alpha}} x} (-w)^{\frac{1-\gamma n}{\alpha} - 1} dw \right. \\
&\qquad\qquad \left. + \frac{1}{c_{\gamma,n}^1} \int_{0}^{+\infty} e^{i w^{\frac{1}{\alpha}} x} w^{\frac{1-\gamma n}{\alpha} - 1} dw \right) * g(x) \\
&\Leftrightarrow y(x) = \frac{\alpha}{\sqrt{2\pi}} \left(\frac{1}{c_{\gamma,n}^2} \int_{0}^{+\infty} e^{-izx} z^{-\gamma n} dz + \frac{1}{c_{\gamma,n}^1} \int_{0}^{+\infty} e^{izx} z^{-\gamma n} dz \right) * g(x) \\
&\Leftrightarrow y(x) = \frac{\alpha}{\sqrt{2\pi}} sign(x)|x|^{\gamma n - 1} \Gamma(1-\gamma n) \left[\cos\left(\frac{\gamma n \pi}{2}\right) \left(\frac{-i}{c_{\gamma,n}^2} + \frac{i}{c_{\gamma,n}^1} \right) \right. \\
&\qquad\qquad \left. - \, i sign(x) \sin\left(\frac{\gamma n \pi}{2}\right) \left(\frac{i}{c_{\gamma,n}^2} + \frac{i}{c_{\gamma,n}^1} \right) \right] \; * g(x) \\
&\Leftrightarrow y(x) = \frac{\alpha}{\sqrt{2\pi}} \Gamma(1-\gamma n) \int_{-\infty}^{\infty} \left[C_1 \cos\left(\frac{\gamma n \pi}{2}\right) - \; i \, C_2 \; sign(x-\tau) \sin\left(\frac{\gamma n \pi}{2}\right) \right] \\
&\qquad\qquad \times \; sign(x-\tau)|x-\tau|^{\gamma n - 1} g(\tau) d\tau
\end{aligned}$$

with $\gamma n < 1$, where:

$$c^2_{\gamma,n} = (-i)^n(a+(-1)^n b)\cos\left(\frac{(1-\gamma)n\pi}{2}\right) - i(-a+(-1)^n b)\sin\left(\frac{(1-\gamma)n\pi}{2}\right),$$
$$c^1_{\gamma,n} = (i)^n(a+(-1)^n b)\cos\left(\frac{(1-\gamma)n\pi}{2}\right) + i(-a+(-1)^n b)\sin\left(\frac{(1-\gamma)n\pi}{2}\right),$$
$$C_1 = \frac{-i}{c^2_{\gamma,n}} + \frac{i}{c^1_{\gamma,n}} \quad \text{and} \quad C_2 = \frac{i}{c^2_{\gamma,n}} + \frac{i}{c^1_{\gamma,n}}.$$

Our method also works for equations that contain both ordinary derivatives and combinations of the left- and right-hand sided fractional derivatives. In this case, one has to apply the operational relation (2.13) along with the operational relations (2.32) and (2.33).

Example: In this example, we deal with the following fractional differential equation that also contains a first order derivative:

$$y'(x) + D^{\frac{1}{2}}_{a,-a}y(x) = g(x). \tag{3.2}$$

By applying the fractional Fourier transform to both sides of the equation (3.2) and employing the operational relations (2.12) and (2.32), we get

$$\begin{aligned}
&(\mathcal{F}_\alpha y')(w) + (\mathcal{F}_\alpha D^{\frac{1}{2}}_{a,-a}y)(w) = (\mathcal{F}_\alpha g)\\
&\Leftrightarrow (\mathcal{F}_\alpha y)(w) = \frac{1}{isign(w)|w|^{1/\alpha} + isign(w)2a\sin\left(\frac{\pi}{4}\right)|w|^{\frac{1}{2\alpha}}}\ (\mathcal{F}_\alpha g)\\
&\Leftrightarrow (\mathcal{F}_\alpha y)(w) = \frac{1}{i}\frac{sign(w)}{|w|^{1/\alpha} + a\sqrt{2}|w|^{\frac{1}{2\alpha}}}\ (\mathcal{F}_\alpha g).
\end{aligned}$$

Using the inverse fractional Fourier transform, convolution properties and then evaluating the integral for the inverse Fourier transform, we obtain a closed form formula for the solution y as follows:

$$\begin{aligned}
y(x) &= \frac{1}{i}\mathcal{F}^{-1}_\alpha\left(\frac{sign(w)}{|w|^{1/\alpha} + a\sqrt{2}|w|^{\frac{1}{2\alpha}}}\right) * g(x)\\
&= \frac{i\alpha}{\sqrt{2\pi}}\left(\int_0^{+\infty} e^{-izx}\frac{1}{z+a\sqrt{2z}}dz - \int_0^{+\infty} e^{izx}\frac{1}{z+a\sqrt{2z}}dz\right) * g(x)\\
&= h_{a,\alpha}(x) * g(x),
\end{aligned}$$

where

$$h_{a,\alpha}(x) = \frac{\alpha}{2\sqrt{2}\pi^3}\ sign(x)\ G^{5,3}_{0,0}\left(a^4x^2 \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4} \end{matrix}\right)$$

for $a \geq 0$ and $G^{m,n}_{p,q}$ denotes the Meijer G-function.

In a similar way, we can solve the following fractional differential equation

$$y' + D^\gamma_{a,b}y = g. \tag{3.3}$$

The solution of (3.3) is presented in terms of the fractional inverse Fourier transform as

$$
\begin{aligned}
y(x) &= \mathcal{F}_\alpha^{-1}\left(\frac{1}{isign(w)|w|^{1/\alpha}+c_\gamma(w)|w|^{\frac{\gamma}{\alpha}}}\right) * g(x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{isign(w)|w|^{\frac{1}{\alpha}}x}\frac{1}{isign(w)|w|^{1/\alpha}+c_\gamma(w)|w|^{\frac{\gamma}{\alpha}}}dw * g(x),
\end{aligned}
$$

where $c_\gamma = (a+b)\cos\left(\frac{\gamma\pi}{2}\right) + isign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)$.

Another important point we focus on in the rest of this section is that the operational relations for the general fractional derivatives which we introduced in the previous section can be combined with the shift property (2.10) of the fractional Fourier transform and thus applied to the fractional differential equations with delay. This combined operational property is as follows:

$$
\begin{aligned}
&(\mathcal{F}_\alpha D^\gamma_{a,b} f(x-y))(w) = e^{-isign(w)|w|^{\frac{1}{\alpha}}y}|w|^{\frac{\gamma}{\alpha}}\\
&\times\left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + i\ sign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right](\mathcal{F}_\alpha f)(w). \qquad (3.4)
\end{aligned}
$$

The solution schema we presented at the beginning of the section can be then applied - with some evident adaption - for solving the fractional differential equations containing both the left- and the right-sided fractional derivatives with delay. We demonstrate this technique in the following examples.

Example: Let us consider a single-term fractional differential equation with delay:

$$
D^\gamma_{a,b} z(x-y) = g(x). \qquad (3.5)
$$

Applying the fractional Fourier transform to (3.5), using the operational relation (3.4) and then evaluating the integral for the inverse Fourier transform, we get the following result:

$$
\begin{aligned}
&(\mathcal{F}_\alpha D^\gamma_{a,b} z(x-y))(w) = (\mathcal{F}_\alpha g(x))(w)\\
&\Leftrightarrow e^{-isign(w)|w|^{\frac{1}{\alpha}}y}|w|^{\frac{\gamma}{\alpha}}\left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right)\right.\\
&\qquad\left. + i\ sign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right](\mathcal{F}_\alpha z)(w) = (\mathcal{F}_\alpha g(x))(w)\\
&\Leftrightarrow z(x) = \mathcal{F}_\alpha^{-1}\left(C_{\gamma,a,b}(w)e^{isign(w)|w|^{\frac{1}{\alpha}}y}|w|^{-\frac{\gamma}{\alpha}}\right) * g(x)\\
&\Leftrightarrow z(x) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{isign(w)|w|^{\frac{1}{\alpha}}x}C_{\gamma,a,b}(w)e^{isign(w)|w|^{\frac{1}{\alpha}}y}|w|^{\frac{1-\gamma}{\alpha}-1}dw\right) * g(x)\\
&\Leftrightarrow z(x) = \frac{1}{\sqrt{2\pi}}\left(C^2_{\gamma,a,b}\int_{-\infty}^{0} e^{-i(-w)^{\frac{1}{\alpha}}(x+y)}(-w)^{\frac{1-\gamma}{\alpha}-1}dw\right.\\
&\qquad\left. + C^1_{\gamma,a,b}\int_0^{+\infty} e^{iw^{\frac{1}{\alpha}}(x+y)}w^{\frac{1-\gamma}{\alpha}-1}dw\right) * g(x)\\
&\Leftrightarrow z(x) = \frac{\alpha}{\sqrt{2\pi}}\left(C^2_{\gamma,a,b}\int_0^{+\infty} e^{-iz(x+y)}z^{-\gamma}dz + C^1_{\gamma,a,b}\int_0^{+\infty} e^{iz(x+y)}z^{-\gamma}dz\right) * g(x)
\end{aligned}
$$

$$\Leftrightarrow z(x) = \frac{\alpha}{\sqrt{2\pi}} \, sign(x+y)|x+y|^{\gamma-1}\Gamma(1-\gamma) \left[\cos\left(\frac{\gamma\pi}{2}\right)(-iC^2_{\gamma,a,b} + iC^1_{\gamma,a,b}) \right.$$
$$\left. - i \ sign(x+y) \ \sin\left(\frac{\gamma\pi}{2}\right)(iC^2_{\gamma,a,b} + iC^1_{\gamma,a,b})\right] \ * g(x)$$
$$\Leftrightarrow z(x) = \frac{\alpha}{\sqrt{2\pi}}\Gamma(1-\gamma)\int_{-\infty}^{\infty}\left[C_3\cos\left(\frac{\gamma\pi}{2}\right) - i \ C_4 \ sign(x-\tau+y)\sin\left(\frac{\gamma\pi}{2}\right)\right]$$
$$\times \ sign(x-\tau+y) \ |x-\tau+y|^{\gamma-1}g(\tau)d\tau,$$

with $\gamma < 1$, where:

$$C_{\gamma,a,b}(w) = \frac{1}{\left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + i \ sign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right]},$$
$$C^1_{\gamma,a,b} = \frac{1}{\left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + i(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right]},$$
$$C^2_{\gamma,a,b} = \frac{1}{\left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) - i(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right]},$$
$$C_3 = -iC^2_{\gamma,a,b} + iC^1_{\gamma,a,b} \text{ and } C_4 = iC^2_{\gamma,a,b} + iC^1_{\gamma,a,b}.$$

Example: In this example, we consider a fractional differential equation with two fractional derivatives of the same order but with different delays:

$$D^\gamma_{a,b}z(x-y) + D^\gamma_{a,b}z(x-t) = g(x). \tag{3.6}$$

Following our schema, the fractional Fourier transform of the solution is first determined:

$$(\mathcal{F}_\alpha D^\gamma_{a,b}z(x-y))(w) + (\mathcal{F}_\alpha D^\gamma_{a,b}y(x-t))(w) = (\mathcal{F}_\alpha g(x))(w),$$
$$c_{\gamma,a,b}(w)e^{-isign(w)|w|^{\frac{1}{\alpha}}y}|w|^{\frac{\gamma}{\alpha}}(\mathcal{F}_\alpha z)(w)$$
$$+ c_{\gamma,a,b}(w)e^{-isign(w)|w|^{\frac{1}{\alpha}}t}|w|^{\frac{\gamma}{\alpha}}(\mathcal{F}_\alpha z)(w) = (\mathcal{F}_\alpha g(x))(w),$$
$$(\mathcal{F}_\alpha z)(w) = \frac{e^{isign(w)|w|^{\frac{1}{\alpha}}(y+t)}}{c_{\gamma,a,b}(w) \ |w|^{\frac{\gamma}{\alpha}}} \ (\mathcal{F}_\alpha g(x))(w),$$

with $\gamma < 1$, where:

$$c_{\gamma,a,b}(w) = \left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + i \ sign(w)(a-b)\sin\left(\frac{\gamma\pi}{2}\right)\right].$$

Using the inverse fractional Fourier transform, convolution property and calculating the integral for the inverse Fourier transform, the following closed form solution can be obtained:

$$z(x) = \mathcal{F}^{-1}_\alpha\left(\frac{e^{isign(w)|w|^{\frac{1}{\alpha}}(y+t)}}{c_{\gamma,a,b}(w) \ |w|^{\frac{\gamma}{\alpha}}}\right) * g(x)$$
$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty}\frac{1}{c_{\gamma,a,b}(w)}e^{isign(w)|w|^{\frac{1}{\alpha}}(x+y+t)}|w|^{\frac{1-\gamma}{\alpha}-1}dw\right) * g(x)$$

$$= \frac{\alpha}{\sqrt{2\pi}} \left(\frac{1}{c^2_{\gamma,a,b}} \int_0^{+\infty} e^{-iz(x+y+t)} z^{-\gamma} dz + \frac{1}{c^1_{\gamma,a,b}} \int_0^{+\infty} e^{iz(x+y+t)} z^{-\gamma} dz \right) * g(x)$$

$$= \frac{\alpha}{\sqrt{2\pi}} \Gamma(1-\gamma) \int_{-\infty}^{\infty} \left[C_5 \cos\left(\frac{\gamma\pi}{2}\right) - i\, C_6\ sign(x-\tau+y+t) \sin\left(\frac{\gamma\pi}{2}\right) \right]$$
$$\times\ sign(x-\tau+y+t)\ |x-\tau+y+t|^{\gamma-1} g(\tau) d\tau,$$

where:

$$c^1_{\gamma,a,b} = \left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) + i\ (a-b)\sin\left(\frac{\gamma\pi}{2}\right) \right],$$
$$c^2_{\gamma,a,b} = \left[(a+b)\cos\left(\frac{\gamma\pi}{2}\right) - i\ (a-b)\sin\left(\frac{\gamma\pi}{2}\right) \right],$$
$$C_5 = \left(\frac{-i}{c^2_{\gamma,a,b}} + \frac{i}{c^1_{\gamma,a,b}} \right) \quad \text{and} \quad C_6 = \left(\frac{i}{c^2_{\gamma,a,b}} + \frac{i}{c^1_{\gamma,a,b}} \right).$$

Example: The last example is a fractional differential equation with two different fractional derivatives but with the same delay:

$$D^\eta_{a,b} z(x-y) + D^\gamma_{a,b} z(x-y) = g(x). \tag{3.7}$$

Our solution schema leads to the fractional Fourier transform of the solution in the following form:

$$(\mathcal{F}_\alpha D^\eta_{a,b} z(x-y))(w) + (\mathcal{F}_\alpha D^\gamma_{a,b} z(x-y))(w) = (\mathcal{F}_\alpha g(x))(w),$$

$$c_{\eta,a,b}(w) e^{-isign(w)|w|^{\frac{1}{\alpha}} y} |w|^{\frac{\eta}{\alpha}} (\mathcal{F}_\alpha z)(w)$$
$$+\ c_{\gamma,a,b}(w) e^{-isign(w)|w|^{\frac{1}{\alpha}} y} |w|^{\frac{\gamma}{\alpha}} (\mathcal{F}_\alpha z)(w) = (\mathcal{F}_\alpha g(x))(w),$$

$$(\mathcal{F}_\alpha z)(w) = \frac{e^{isign(w)|w|^{\frac{1}{\alpha}} y}}{c_{\eta,a,b}(w)\ |w|^{\frac{\eta}{\alpha}} + c_{\gamma,a,b}(w)\ |w|^{\frac{\gamma}{\alpha}}}\ (\mathcal{F}_\alpha g(x))(w),$$

where

$$c_{\eta,a,b}(w) = \left[(a+b)\cos\left(\frac{\eta\pi}{2}\right) + i\ sign(w)(a-b)\sin\left(\frac{\eta\pi}{2}\right) \right].$$

Using the inverse fractional Fourier transform, convolution properties and transforming the integral for the inverse fractional Fourier transform, we can represent the solution in the following form:

$$z(x) = \frac{\alpha}{\sqrt{2\pi}} \left(\int_0^{+\infty} e^{-i\tau(x+y)} \frac{1}{c^2_{\eta,a,b}\tau^\eta + c^2_{\gamma,a,b}\tau^\gamma}\, d\tau \right.$$
$$\left. + \int_0^{+\infty} e^{i\tau(x+y)} \frac{1}{c^1_{\eta,a,b}\tau^\eta + c^1_{\gamma,a,b}\tau^\gamma}\, d\tau \right) * g(x),$$

where

$$c^1_{\eta,a,b} = \left[(a+b)\cos\left(\frac{\eta\pi}{2}\right) + i\ (a-b)\sin\left(\frac{\eta\pi}{2}\right) \right]$$

and

$$c^2_{\eta,a,b} = \left[(a+b)\cos\left(\frac{\eta\pi}{2}\right) - i\ (a-b)\sin\left(\frac{\eta\pi}{2}\right) \right].$$

For some special values of the parameters a and b, the integrals at the right-hand side of the last formula can be essentially simplified or even written down in the closed form.

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