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ASYMPTOTIC LIMITS OF INFINITE INTEGRALS FROM AN EXPRESSION OF DIRAC FUNCTION

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Abstract. We use a limit relation between the Dirac delta function $\delta(x)$ and the Bessel function $J_\nu(x)$, to obtain limiting relations for some special functions.

1. Introduction

The Dirac delta function $\delta(x)$ is a powerful tool in several fields of physics, engineering and applied mathematics. Some years ago Lamborn [4, 5] proved that $\delta(x-1)$ may be expressed in terms of the asymptotic limit of the Bessel function $J_\nu(\nu x)$ of the first kind. Precisely he proved the relation

$$\lim_{\nu\to\infty}[\nu J_\nu(\nu x)] = \delta(x-1). \tag{1.1}$$

The limit relation is useful in treating relativistic dispersion relations in plasma physics in the limit of zero magnetic field [4].

From relation (1.1) it is possible to obtain new and useful results. Apelblat [2] obtained limiting relations for some special functions. In this paper we continue the investigation in this direction and obtain new limiting relations for many functions, but chiefly for Bessel functions and modified Bessel functions.

2. The results

Let us consider the relation (1.1) and multiply both the sides by a suitable function $f(tx)$. Then integrate from zero to infinity, with respect to x. We obtain

$$\int_0^\infty \delta(x-1)f(tx)dx = \lim_{\nu\to\infty}\left[\nu\int_0^\infty J_\nu(\nu x)f(tx)dx\right], \tag{2.1}$$

where we supposed that the integral on the right-hand side exists and that the order of integration and the limit can be reversed. Now, for the *sifting property* of the Delta function, the first integral in (2.1) is $f(t)$, thus we get

$$f(t) = \lim_{\nu\to\infty}\left[\nu\int_0^\infty J_\nu(\nu x)f(tx)dx\right]. \tag{2.2}$$

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This limit relation can be consider the starting point for several results. For example, if we replace $f(tx)$ by $K_\mu(tx)$, where K is the modified Bessel function of third kind, we obtain

$$K_\mu(t) = \lim_{\nu\to\infty}\left[\nu\int_0^\infty J_\nu(\nu x)K_\mu(tx)dx\right].$$

The main interest of (2.2) is when the integral on the right-hand side can be explicitely evaluated. Indeed, if we know that

$$h(t,\nu) = \int_0^\infty J_\nu(\nu x)f(tx)dx, \tag{2.3}$$

then we also have

$$f(t) = \lim_{\nu\to\infty}\left[\nu h(t,\nu)\right]. \tag{2.4}$$

For $y > 0$, let us consider the integral representation [3, p. 681; 6.552]

$$\int_0^\infty J_\nu(xy)\,\frac{dx}{(x^2+a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right)K_{\nu/2}\left(\frac{1}{2}ay\right), \qquad \Re a > 0,\ \Re\nu > -1,$$

where I_μ denotes the modified Bessel function of first kind. With $y=\nu$, we multiply both the sides by ν:

$$\nu\int_0^\infty J_\nu(x\nu)\,\frac{dx}{(x^2+a^2)^{1/2}} = \nu I_{\nu/2}\left(\frac{1}{2}a\nu\right)K_{\nu/2}\left(\frac{1}{2}a\nu\right).$$

Performing the limit for $\nu\to\infty$, we obtain

$$\lim_{\nu\to\infty}\left[\nu\int_0^\infty J_\nu(x\nu)\,\frac{dx}{(x^2+a^2)^{1/2}}\right] = \lim_{\nu\to\infty}\nu I_{\nu/2}\left(\frac{1}{2}a\nu\right)K_{\nu/2}\left(\frac{1}{2}a\nu\right)$$

and, using (2.2),

$$\frac{1}{(1+a^2)^{1/2}} = \lim_{\nu\to\infty}\nu I_{\nu/2}\left(\frac{1}{2}a\nu\right)K_{\nu/2}\left(\frac{1}{2}a\nu\right)$$

which, replacing $\nu/2$ by ν, can be written as

$$\lim_{\nu\to\infty}\nu I_\nu(a\nu)\,K_\nu(a\nu) = \frac{1}{2(1+a^2)^{1/2}}. \tag{2.5}$$

Remark. *The limit relation* (2.5) *has been already established by Apelblat* [2, p. 23; (28)], *using a result on the Whittaker functions. The same author proved this result* [2, p. 27; (58)] *using the Laplace transform. Actually our method seems to be the simplest.*

Consider now the integral representation [3, p. 712; 6.623]

$$\int_0^\infty e^{-ax}J_\nu(\nu x)\,\frac{dx}{x} = \frac{\left(\sqrt{a^2+\nu^2}-a\right)^\nu}{\nu^{\nu+1}}, \qquad \Re\nu > 0,\ \Re a > |\Im\nu|.$$

From this it follows

$$\lim_{\nu\to\infty}\left[\nu\int_0^\infty e^{-ax}J_\nu(\nu x)\,\frac{dx}{x}\right] = \lim_{\nu\to\infty}\left(\frac{\sqrt{a^2+\nu^2}-a}{\nu}\right)^\nu$$

and by (2.2)

$$\lim_{\nu\to\infty}\left(\frac{\sqrt{a^2+\nu^2}-a}{\nu}\right)^\nu = e^{-a}.$$

The next example, dealing with relation (2.2), is concerned with the modified Bessel functions I_μ.

From [3, p. 717; 6.631,7]

$$\int_0^\infty xe^{-ax^2}J_\nu(\beta x)dx = \frac{\sqrt{\pi}\beta}{8a^{3/2}}e^{-\frac{\beta^2}{8a}} \cdot \left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\beta^2}{8a}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\beta^2}{8a}\right)\right], \qquad \Re a > 0,\ \Re\nu > 0,$$

letting $\beta = \nu$ and with the same arguments used before, we get

$$\lim_{\nu\to\infty}\left[\nu\int_0^\infty xe^{-ax^2}J_\nu(\nu x)dx\right] = e^{-a} = \lim_{\nu\to\infty}\frac{\sqrt{\pi}\nu^2}{8a\sqrt{a}}e^{-\frac{\nu^2}{8a}}\left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\nu^2}{8a}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\nu^2}{8a}\right)\right],$$

and finally

$$\lim_{\nu\to\infty}\left\{\nu^2 e^{-\frac{\nu^2}{8a}}\left[I_{\frac{1}{2}(\nu-1)}\left(\frac{\nu^2}{8a}\right) - I_{\frac{1}{2}(\nu+1)}\left(\frac{\nu^2}{8a}\right)\right]\right\} = \frac{8a\sqrt{a}}{\sqrt{\pi}e^a}.$$

The next example provides an elementary, but interesting, limit.

From [3, p. 730; 6.671,1]

$$\int_0^\infty \sin(\beta x)J_\nu(\nu x)dx = \frac{\sin\left(\nu \arcsin\frac{\beta}{\nu}\right)}{\sqrt{\nu^2-\beta^2}}, \qquad \beta < \nu,$$

we get

$$\lim_{\nu\to\infty}\left[\nu\int_0^\infty \sin(\beta x)J_\nu(\nu x)dx\right] = \sin\beta = \lim_{\nu\to\infty}\frac{\nu\sin\left(\nu\arcsin\frac{\beta}{\nu}\right)}{\sqrt{\nu^2-\beta^2}}.$$

Finally, in this example, we provide a limit relation for the ratio of two gamma functions.

From [3, p. 684; 6.561, 14]

$$\int_0^\infty x^\mu J_\nu(\nu x)dx = 2^\mu\nu^{-\mu-1}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}, \qquad -\Re\nu - 1 < \Re\mu < \frac{1}{2},\ \nu > 0,$$

we find

$$1 = \lim_{\nu\to\infty}2^\mu\nu^{-\mu}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)},$$

or

$$\lim_{\nu\to\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}\frac{1}{\nu^\mu} = 2^{-\mu},$$

in accordance with the Tricomi-Erdelyi asymptotic result [1, p. 257; 6.1.47]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} + \cdots, \qquad x\to\infty.$$

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