

**INEQUALITIES OF SCHWARZ TYPE FOR  $n$ -TUPLES OF  
VECTORS IN INNER PRODUCT SPACES WITH  
APPLICATIONS**

SILVESTRU SEVER DRAGOMIR<sup>1,2</sup>

*DEDICATED TO PROFESSOR IVAN DIMOVSKI'S CONTRIBUTIONS*

**ABSTRACT.** In this paper some new inequalities of Schwarz and Buzano type for  $n$ -tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for  $n$ -tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

1. INTRODUCTION

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex numbers field  $\mathbb{K}$ . The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \quad (1.1)$$

The equality case holds in (1.1) if and only if there exists a constant  $\lambda \in \mathbb{K}$  such that  $x = \lambda y$ .

In 1985 the author [5] (see also [15]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \quad (1.2)$$

for any  $x, y, e \in H$  with  $\|e\| = 1$ .

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2|\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \quad (1.3)$$

that holds for any  $x, y, e \in H$  with  $\|e\| = 1$ .

---

2000 *Mathematics Subject Classification.* 47A05, 47A63; 47A99.

*Key words and phrases.* Schwarz inequality, Buzano inequality, Inner product spaces, Operator numerical radius, Operator norm inequality, Normal operators, Power series of normal operators.

©2017 Ilirias Research Institute, Prishtinë, Kosovë.

Submitted November 7, 2016. Published January 6, 2017.

For a probability distribution  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$ , i.e. we recall that  $p_i > 0$  for any  $i \in \{1, \dots, n\}$  and  $\sum_{i=1}^n p_i = 1$  we define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_p := \sum_{i=1}^n p_i \langle x_i, y_i \rangle$$

for  $n$ -tuples  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ . The attached norm is given by

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2}$$

for  $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ .

Let  $\mathbf{e} = (e_1, \dots, e_n) \in H^n$  with  $\sum_{i=1}^n p_i \|e_i\|^2 = 1$ . Making use of (1.2) and (1.3) for the inner product  $\langle \cdot, \cdot \rangle_p$  we have the inequalities

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right| + \left| \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right| \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, e_i \rangle \sum_{i=1}^n p_i \langle e_i, y_i \rangle \right|, \end{aligned} \quad (1.5)$$

for any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ .

If we take  $\mathbf{e} = (e, \dots, e) \in H^n$  with  $\|e\| = 1$ , then we get from (1.4) and (1.5) the inequalities

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right| + \left| \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right| \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \\ & \geq \left| \left\langle \sum_{i=1}^n p_i x_i, e \right\rangle \left\langle e, \sum_{i=1}^n p_i y_i \right\rangle \right|, \end{aligned} \quad (1.7)$$

for any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ .

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [13], [14], [18], [19], [20], [21], [22], [23], [24], [26] and the monographs [11] and [12].

Motivated by the above results, we establish in this paper some new inequalities of Schwarz and Buzano type for  $n$ -tuples of vectors in inner product spaces. Applications for norm and numerical radius inequalities for  $n$ -tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

## 2. MAIN RESULTS

Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex numbers field  $\mathbb{K}$ . For an  $n$ -tuple  $\mathbf{x} = (x_1, \dots, x_n) \in H^n$  and a probability distribution  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$ , we define the *average vector*

$$\bar{x}_p := \sum_{j=1}^n p_j x_j \in H.$$

In particular, for the *uniform probability*  $\mathbf{p} = (\frac{1}{n}, \dots, \frac{1}{n})$  we define

$$\bar{x} := \frac{1}{n} \sum_{j=1}^n x_j \in H.$$

We have the following result:

**Theorem 2.1.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex numbers field  $\mathbb{K}$ ,  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$  and  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$ . Then we have the inequalities*

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \max \left\{ \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle|, \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle x_i, \bar{y}_p \rangle| \right\} \\ & \geq \sum_{i=1}^n p_i \left[ \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle \bar{x}_p, y_i \rangle \right| + \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle x_i, \bar{y}_p \rangle \right| \right] \\ & \geq \begin{cases} \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - \langle \bar{x}_p, y_i \rangle - \langle x_i, \bar{y}_p \rangle|, \\ \sum_{i=1}^n p_i |\langle x_i, \bar{y}_p \rangle - \langle \bar{x}_p, y_i \rangle|. \end{cases} \end{aligned} \tag{2.1}$$

In particular, we have

$$\begin{aligned}
 & \left( \sum_{i=1}^n \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n \|y_i\|^2 \right)^{1/2} \\
 & \geq \max \left\{ \sum_{i=1}^n |\langle x_i, y_i \rangle - 2 \langle \bar{x}, y_i \rangle|, \sum_{i=1}^n |\langle x_i, y_i \rangle - 2 \langle x_i, \bar{y} \rangle| \right\} \\
 & \geq \sum_{i=1}^n \left[ \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle \bar{x}, y_i \rangle \right| + \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle x_i, \bar{y} \rangle \right| \right] \\
 & \geq \begin{cases} \sum_{i=1}^n |\langle x_i, y_i \rangle - \langle \bar{x}, y_i \rangle - \langle x_i, \bar{y} \rangle|, \\ \sum_{i=1}^n |\langle x_i, \bar{y} \rangle - \langle \bar{x}, y_i \rangle|. \end{cases}
 \end{aligned} \tag{2.2}$$

*Proof.* Observe that, by the properties of inner product, we have

$$\begin{aligned}
 \sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\|^2 &= \sum_{i=1}^n p_i [ \|x_i\|^2 - 4\operatorname{Re} \langle x_i, \bar{x}_p \rangle + 4\|\bar{x}_p\|^2 ] \\
 &= \sum_{i=1}^n p_i \|x_i\|^2 - 4\operatorname{Re} \left\langle \sum_{i=1}^n p_i x_i, \bar{x}_p \right\rangle + 4\|\bar{x}_p\|^2 \\
 &= \sum_{i=1}^n p_i \|x_i\|^2 - 4\|\bar{x}_p\|^2 + 4\|\bar{x}_p\|^2 = \sum_{i=1}^n p_i \|x_i\|^2. \tag{2.3}
 \end{aligned}$$

By the Cauchy-Buniyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned}
 \left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} &= \left( \sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\
 &\geq \sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\| \|y_i\|.
 \end{aligned}$$

By the Schwarz inequality in  $(H, \langle \cdot, \cdot \rangle)$  we have for each  $i \in \{1, \dots, n\}$

$$\|x_i - 2\bar{x}_p\| \|y_i\| \geq |\langle x_i - 2\bar{x}_p, y_i \rangle| = |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle|$$

and then

$$\sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\| \|y_i\| \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle|.$$

Therefore

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle| \tag{2.4}$$

and, similarly

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle x_i, \bar{y}_p \rangle|. \tag{2.5}$$

Making use of (2.4) and (2.5) we get the first inequality in (2.1).

Since  $\max\{a, b\} \geq \frac{1}{2}(a + b)$  for positive numbers  $a$  and  $b$ , we get the second inequality in (2.1). The last part of (2.1) follows by the triangle inequality.  $\square$

**Remark.** Using the generalized triangle inequality we have

$$\sum_{i=1}^n p_i |\langle x_i, y_i \rangle - \langle \bar{x}_p, y_i \rangle - \langle x_i, \bar{y}_p \rangle| \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - 2 \langle \bar{x}_p, \bar{y}_p \rangle \right|$$

and by (2.1) we get the following string of inequalities

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \max \left\{ \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle|, \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle x_i, \bar{y}_p \rangle| \right\} \\ & \geq \sum_{i=1}^n p_i \left[ \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle \bar{x}_p, y_i \rangle \right| + \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle x_i, \bar{y}_p \rangle \right| \right] \\ & \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - \langle \bar{x}_p, y_i \rangle - \langle x_i, \bar{y}_p \rangle| \\ & \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - 2 \langle \bar{x}_p, \bar{y}_p \rangle \right|. \end{aligned} \quad (2.6)$$

The unweighted case is as follows

$$\begin{aligned} & \left( \sum_{i=1}^n \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n \|y_i\|^2 \right)^{1/2} \\ & \geq \max \left\{ \sum_{i=1}^n |\langle x_i, y_i \rangle - 2 \langle \bar{x}, y_i \rangle|, \sum_{i=1}^n |\langle x_i, y_i \rangle - 2 \langle x_i, \bar{y} \rangle| \right\} \\ & \geq \sum_{i=1}^n \left[ \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle \bar{x}, y_i \rangle \right| + \left| \frac{1}{2} \langle x_i, y_i \rangle - \langle x_i, \bar{y} \rangle \right| \right] \\ & \geq \sum_{i=1}^n |\langle x_i, y_i \rangle - \langle \bar{x}, y_i \rangle - \langle x_i, \bar{y} \rangle| \geq \left| \sum_{i=1}^n \langle x_i, y_i \rangle - 2 \langle \bar{x}, \bar{y} \rangle \right|. \end{aligned} \quad (2.7)$$

The inequality between the first and 4th and 5th terms in (2.7) was obtained in [25].

**Corollary 2.2.** With the assumptions of Theorem 2.1 we have

$$\frac{1}{2} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \right] \geq |\langle \bar{x}_p, \bar{y}_p \rangle| \quad (2.8)$$

and, in particular

$$\frac{1}{2n} \left[ \left( \sum_{i=1}^n \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n \langle x_i, y_i \rangle \right| \right] \geq |\langle \bar{x}, \bar{y} \rangle|. \quad (2.9)$$

*Proof.* By the triangle inequality we have

$$\left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - 2 \langle \bar{x}_p, \bar{y}_p \rangle \right| \geq 2 |\langle \bar{x}_p, \bar{y}_p \rangle| - \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right|$$

and from (2.6) we get

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \geq 2 |\langle \bar{x}_p, \bar{y}_p \rangle| - \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right|,$$

i.e.

$$\left( \sum_{i=1}^n p_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \geq 2 |\langle \bar{x}_p, \bar{y}_p \rangle|,$$

and the inequality (2.8) is proved.  $\square$

The following result also holds:

**Theorem 2.3.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex numbers field  $\mathbb{K}$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n) \in H^n$  and  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$ . Then we have the inequalities*

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2 \langle \bar{x}_p, y_i \rangle - 2 \langle x_i, \bar{y}_p \rangle + 4 \langle \bar{x}_p, \bar{y}_p \rangle| \\ & \geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle| \geq \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle \right| \end{aligned} \quad (2.10)$$

and, in particular

$$\begin{aligned} & \left( \sum_{i=1}^n \|x_i\|^2 \sum_{i=1}^n \|y_i\|^2 \right)^{1/2} \\ & \geq \sum_{i=1}^n |\langle x_i, y_i \rangle - 2 \langle \bar{x}, y_i \rangle - 2 \langle x_i, \bar{y} \rangle + 4 \langle \bar{x}, \bar{y} \rangle| \\ & \geq \sum_{i=1}^n |\langle x_i, y_i \rangle| \geq \left| \sum_{i=1}^n \langle x_i, y_i \rangle \right|. \end{aligned} \quad (2.11)$$

*Proof.* By the Cauchy-Bunyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned} & \left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} \\ & = \left( \sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\|^2 \sum_{i=1}^n p_i \|y_i - 2\bar{y}_p\|^2 \right)^{1/2} \\ & \geq \sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\| \|y_i - 2\bar{y}_p\|. \end{aligned} \quad (2.12)$$

By the Schwarz inequality in  $(H, \langle \cdot, \cdot \rangle)$  we have

$$\begin{aligned} \|x_i - 2\bar{x}_p\| \|y_i - 2\bar{y}_p\| &\geq |\langle x_i - 2\bar{x}_p, y_i - 2\bar{y}_p \rangle| \\ &= |\langle x_i, y_i \rangle - 2\langle \bar{x}_p, y_i \rangle - 2\langle x_i, \bar{y}_p \rangle + 4\langle \bar{x}_p, \bar{y}_p \rangle| \end{aligned}$$

for any  $i \in \{1, \dots, n\}$ .

Therefore

$$\begin{aligned} &\sum_{i=1}^n p_i \|x_i - 2\bar{x}_p\| \|y_i - 2\bar{y}_p\| \tag{2.13} \\ &\geq \sum_{i=1}^n p_i |\langle x_i, y_i \rangle - 2\langle \bar{x}_p, y_i \rangle - 2\langle x_i, \bar{y}_p \rangle + 4\langle \bar{x}_p, \bar{y}_p \rangle| \\ &\geq \left| \sum_{i=1}^n p_i [\langle x_i, y_i \rangle - 2\langle \bar{x}_p, y_i \rangle - 2\langle x_i, \bar{y}_p \rangle + 4\langle \bar{x}_p, \bar{y}_p \rangle] \right| \\ &= \left| \sum_{i=1}^n p_i \langle x_i, y_i \rangle - 2 \left\langle \bar{x}_p, \sum_{i=1}^n p_i y_i \right\rangle - 2 \left\langle \sum_{i=1}^n p_i x_i, \bar{y}_p \right\rangle + 4 \langle \bar{x}_p, \bar{y}_p \rangle \right| \\ &= \sum_{i=1}^n p_i |\langle x_i, y_i \rangle|. \end{aligned}$$

Making use of (2.12) and (2.13) we get the desired result (2.10).  $\square$

The following corollary holds:

**Corollary 2.4.** *With the assumptions of Theorem 2.3 we have*

$$\begin{aligned} &\frac{1}{4} \left[ \left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \sum_{i=1}^n p_i |\langle x_i, y_i \rangle| \right] \tag{2.14} \\ &\geq \sum_{i=1}^n p_i \left| \langle \bar{x}_p, \bar{y}_p \rangle - \frac{1}{2} [\langle \bar{x}_p, y_i \rangle + \langle x_i, \bar{y}_p \rangle] \right|. \end{aligned}$$

*Proof.* If we add to the first inequality in (2.10) the quantity  $\sum_{i=1}^n p_i |\langle x_i, y_i \rangle|$  we have

$$\begin{aligned} &\left( \sum_{i=1}^n p_i \|x_i\|^2 \sum_{i=1}^n p_i \|y_i\|^2 \right)^{1/2} + \sum_{i=1}^n p_i |\langle x_i, y_i \rangle| \tag{2.15} \\ &\geq \sum_{i=1}^n p_i [|\langle x_i, y_i \rangle - 2\langle \bar{x}_p, y_i \rangle - 2\langle x_i, \bar{y}_p \rangle + 4\langle \bar{x}_p, \bar{y}_p \rangle| + |\langle x_i, y_i \rangle|] \\ &\geq \sum_{i=1}^n p_i |4\langle \bar{x}_p, \bar{y}_p \rangle - 2\langle \bar{x}_p, y_i \rangle - 2\langle x_i, \bar{y}_p \rangle| \\ &= 4 \sum_{i=1}^n p_i \left| \langle \bar{x}_p, \bar{y}_p \rangle - \frac{1}{2} [\langle \bar{x}_p, y_i \rangle + \langle x_i, \bar{y}_p \rangle] \right|. \end{aligned}$$

Dividing by 4 we get the desired result (2.14).  $\square$

We observe that, if we take  $H = \mathbb{C}$  with the inner product  $\langle z, w \rangle = z\bar{w}$  then by taking above  $x_i = a_i \in \mathbb{C}$  and  $y_i = \bar{b}_i, i \in \{1, \dots, n\}$ , then from (2.1) we get the inequality

$$\begin{aligned} & \left( \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} \\ & \geq \max \left\{ \sum_{i=1}^n p_i |a_i b_i - 2b_i \bar{a}_p|, \sum_{i=1}^n p_i |a_i b_i - 2a_i \bar{b}_p| \right\} \\ & \geq \sum_{i=1}^n p_i \left[ \left| \frac{1}{2} a_i b_i - b_i \bar{a}_p \right| + \left| \frac{1}{2} a_i b_i - a_i \bar{b}_p \right| \right] \\ & \geq \begin{cases} \sum_{i=1}^n p_i |a_i b_i - b_i \bar{a}_p - a_i \bar{b}_p|, \\ \sum_{i=1}^n p_i |a_i \bar{b}_p - b_i \bar{a}_p|, \end{cases} \end{aligned} \tag{2.16}$$

where  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  and

$$\bar{a}_p = \sum_{i=1}^n p_i a_i.$$

Utilising the inequality (2.8) we also have

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n p_i |b_i|^2 \right)^{1/2} + \left| \sum_{i=1}^n p_i a_i b_i \right| \right] \\ & \geq \left| \sum_{i=1}^n p_i a_i \right| \left| \sum_{i=1}^n p_i b_i \right| \end{aligned} \tag{2.17}$$

for any  $\mathbf{p} = (p_1, \dots, p_n) \in \mathcal{P}_n$  and  $a_i, b_i \in \mathbb{C}, i \in \{1, \dots, n\}$ .

If in (2.17) we take  $p_i = \frac{q_i}{\sum_{k=1}^n q_k}$  with  $q_i \geq 0, i \in \{1, \dots, n\}$  and  $\sum_{k=1}^n q_k > 0$ , then we get

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n q_i |a_i|^2 \right)^{1/2} \left( \sum_{i=1}^n q_i |b_i|^2 \right)^{1/2} + \left| \sum_{i=1}^n q_i a_i b_i \right| \right] \sum_{k=1}^n q_k \\ & \geq \left| \sum_{i=1}^n q_i a_i \right| \left| \sum_{i=1}^n q_i b_i \right|. \end{aligned} \tag{2.18}$$

Moreover, by taking  $b_i = a_i, i \in \{1, \dots, n\}$  in (2.18) we get

$$\frac{1}{2} \left[ \sum_{i=1}^n q_i |a_i|^2 + \sum_{i=1}^n q_i a_i^2 \right] \sum_{k=1}^n q_k \geq \left| \sum_{i=1}^n q_i a_i \right|^2. \tag{2.19}$$

If  $r_i \in \mathbb{R} \setminus \{0\}$  and  $z_i \in \mathbb{C}, i \in \{1, \dots, n\}$ , then by taking  $q_i = r_i^2$  and  $a_i = \frac{z_i}{r_i}, i \in \{1, \dots, n\}$  we get the well known *de Bruijn inequality* [2]

$$\frac{1}{2} \left[ \sum_{i=1}^n |z_i|^2 + \sum_{i=1}^n z_i^2 \right] \sum_{i=1}^n r_i^2 \geq \left| \sum_{i=1}^n r_i a_i \right|^2. \tag{2.20}$$

The other vector inequalities from above have similar versions for complex numbers. However the details are not presented here.



3. APPLICATIONS FOR  $n$ -TUPLES OF OPERATORS

If in (2.8) we take  $p_i = \frac{q_i}{\sum_{k=1}^n q_k}$  with  $q_i \geq 0, i \in \{1, \dots, n\}$  and  $\sum_{k=1}^n q_k > 0$ , then we get

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n q_i \|x_i\|^2 \right)^{1/2} \left( \sum_{i=1}^n q_i \|y_i\|^2 \right)^{1/2} + \left| \sum_{i=1}^n q_i \langle x_i, y_i \rangle \right| \right] \sum_{k=1}^n q_k \\ & \geq \left| \left\langle \sum_{i=1}^n q_i x_i, \sum_{i=1}^n q_i y_i \right\rangle \right| \end{aligned} \quad (3.1)$$

for any  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in H^n$ .

Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space. The *numerical range* of an operator  $T$  is the subset of the complex numbers  $\mathbb{C}$  given by [17, p. 1]:

$$W(T) = \{ \langle Tx, x \rangle, x \in H, \|x\| = 1 \}.$$

The *numerical radius*  $w(T)$  of an operator  $T$  on  $H$  is given by [17, p. 8]:

$$w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}. \quad (3.2)$$

It is well known that  $w(\cdot)$  is a norm on the Banach algebra  $B(H)$  of all bounded linear operators on the Hilbert space  $H$ . This norm is equivalent with the operator norm. In fact, the following more precise result holds [17, p. 9]:

**Theorem 3.1** (Equivalent norm). *For any  $T \in B(H)$  one has*

$$w(T) \leq \|T\| \leq 2w(T). \quad (3.3)$$

The following result holds:

**Theorem 3.2.** *Let  $(A_1, \dots, A_n), (B_1, \dots, B_n)$  be two  $n$ -tuple of bounded linear operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$  and  $q_i \geq 0, i \in \{1, \dots, n\}$  with  $\sum_{k=1}^n q_k > 0$ . Then*

$$\begin{aligned} & \left\| \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i \right\| \\ & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |B_i|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n q_i B_i^* A_i \right\| \right] \sum_{k=1}^n q_k \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} & w \left( \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i \right) \\ & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |B_i|^2 \right\|^{1/2} + w \left( \sum_{i=1}^n q_i B_i^* A_i \right) \right] \sum_{k=1}^n q_k. \end{aligned} \quad (3.5)$$

*Proof.* We take in inequality (3.1)  $x_i = A_i x$ ,  $y_i = B_i y$ ,  $i \in \{1, \dots, n\}$  to get

$$\begin{aligned} & \frac{1}{2} \left[ \left( \sum_{i=1}^n q_i \|A_i x\|^2 \right)^{1/2} \left( \sum_{i=1}^n q_i \|B_i y\|^2 \right)^{1/2} + \left| \sum_{i=1}^n q_i \langle A_i x, B_i y \rangle \right| \right] \sum_{k=1}^n q_k \\ & \geq \left| \left\langle \sum_{i=1}^n q_i A_i x, \sum_{i=1}^n q_i B_i y \right\rangle \right|, \end{aligned}$$

for any  $x, y \in H$ .

Since

$$\begin{aligned} \sum_{i=1}^n q_i \|A_i x\|^2 &= \sum_{i=1}^n q_i \langle A_i x, A_i x \rangle = \sum_{i=1}^n q_i \langle A_i^* A_i x, x \rangle \\ &= \sum_{i=1}^n q_i \langle |A_i|^2 x, x \rangle = \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle, \end{aligned}$$

$$\sum_{i=1}^n q_i \|B_i y\|^2 = \left\langle \sum_{i=1}^n q_i |B_i|^2 y, y \right\rangle,$$

$$\sum_{i=1}^n q_i \langle A_i x, B_i y \rangle = \left\langle \sum_{i=1}^n q_i B_i^* A_i x, y \right\rangle$$

and

$$\left\langle \sum_{i=1}^n q_i A_i x, \sum_{i=1}^n q_i B_i y \right\rangle = \left\langle \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i x, y \right\rangle,$$

then we have

$$\begin{aligned} & \left| \left\langle \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i x, y \right\rangle \right| \tag{3.6} \\ & \leq \frac{1}{2} \left[ \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^n q_i |B_i|^2 y, y \right\rangle^{1/2} \right. \\ & \quad \left. + \left| \left\langle \sum_{i=1}^n q_i B_i^* A_i x, y \right\rangle \right| \right] \sum_{k=1}^n q_k \end{aligned}$$

for any  $x, y \in H$ .

This inequality is of interest in itself.

We know that for any bounded operator  $T$  we have

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y \rangle|.$$

Taking the supremum in (3.6) over  $\|x\| = \|y\| = 1$  we have

$$\begin{aligned}
& \left\| \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i \right\| & (3.7) \\
&= \sup_{\|x\|=\|y\|=1} \left| \left\langle \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i x, y \right\rangle \right| \sum_{k=1}^n q_k \\
&\leq \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[ \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^n q_i |B_i|^2 y, y \right\rangle^{1/2} \right. \\
&\quad \left. + \left| \left\langle \sum_{i=1}^n q_i B_i^* A_i x, y \right\rangle \right| \right] \sum_{k=1}^n q_k \\
&\leq \frac{1}{2} \left[ \sup_{\|x\|=1} \left[ \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle^{1/2} \sup_{\|y\|=1} \left\langle \sum_{i=1}^n q_i |B_i|^2 x, x \right\rangle^{1/2} \right] \right. \\
&\quad \left. + \sup_{\|x\|=\|y\|=1} \left| \left\langle \sum_{i=1}^n q_i B_i^* A_i x, y \right\rangle \right| \right] \sum_{k=1}^n q_k \\
&= \frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |B_i|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n q_i B_i^* A_i \right\| \right] \sum_{k=1}^n q_k,
\end{aligned}$$

because  $\sum_{i=1}^n q_i |A_i|^2$  and  $\sum_{i=1}^n q_i |B_i|^2$  are positive selfadjoint operators.

This proves (3.4).

If we put in (3.6)  $y = x$  and then take the supremum over  $\|x\| = 1$ , we get

$$\begin{aligned}
& w \left( \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i \right) & (3.8) \\
&= \sup_{\|x\|=1} \left| \left\langle \sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i x, x \right\rangle \right| \\
&\leq \frac{1}{2} \left[ \sup_{\|x\|=1} \left[ \left( \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle \right)^{1/2} \left( \left\langle \sum_{i=1}^n q_i |B_i|^2 x, x \right\rangle \right)^{1/2} \right] \right. \\
&\quad \left. + \sup_{\|x\|=1} \left| \left\langle \sum_{i=1}^n q_i B_i^* A_i x, x \right\rangle \right| \right] \sum_{k=1}^n q_k.
\end{aligned}$$

Since

$$\begin{aligned}
 & \sup_{\|x\|=1} \left[ \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^n q_i |B_i|^2 x, x \right\rangle^{1/2} \right] \\
 & \leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^n q_i |A_i|^2 x, x \right\rangle^{1/2} \sup_{\|x\|=1} \left\langle \sum_{i=1}^n q_i |B_i|^2 x, x \right\rangle^{1/2} \\
 & = \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |B_i|^2 \right\|^{1/2},
 \end{aligned} \tag{3.9}$$

we get from (3.8) and (3.9) the desired result (3.5).  $\square$

**Corollary 3.3.** *Let  $(A_1, \dots, A_n)$  be an  $n$ -tuple of bounded linear operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$  and  $q_i \geq 0, i \in \{1, \dots, n\}$  with  $\sum_{k=1}^n q_k > 0$ . Then*

$$\begin{aligned}
 & \left\| \left( \sum_{i=1}^n q_i A_i \right)^2 \right\| \\
 & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |A_i^*|^2 \right\|^{1/2} + \left\| \sum_{i=1}^n q_i A_i^2 \right\| \right] \sum_{k=1}^n q_k
 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
 & w \left( \left( \sum_{i=1}^n q_i A_i \right)^2 \right) \\
 & \leq \frac{1}{2} \left[ \left\| \sum_{i=1}^n q_i |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^n q_i |A_i^*|^2 \right\|^{1/2} + w \left( \sum_{i=1}^n q_i A_i^2 \right) \right] \sum_{k=1}^n q_k.
 \end{aligned} \tag{3.11}$$

**Remark.** *If we take  $n = 2$  and  $q_1 = q_2 = 1$  in (3.4), then we get*

$$\begin{aligned}
 & \|(B_1^* + B_2^*)(A_1 + A_2)\| \\
 & \leq \left\| |A_1|^2 + |A_2|^2 \right\|^{1/2} \left\| |B_1|^2 + |B_2|^2 \right\|^{1/2} + \|B_1^* A_1 + B_2^* A_2\|
 \end{aligned} \tag{3.12}$$

for any operators  $A_1, A_2, B_1, B_2$ .

Assume that  $T, V$  are bounded linear operators and consider the Cartesian decomposition

$$T = A + iB, \quad V = C + iD$$

with the selfadjoint operators  $A, B, C, D$  given by

$$A = \frac{1}{2}(T^* + T), \quad B = \frac{1}{2i}(T - T^*)$$

and

$$C = \frac{1}{2}(V^* + V), \quad D = \frac{1}{2i}(V - V^*).$$

Take  $A_1 = A, A_2 = iB, B_1 = C$  and  $B_2 = -iD$ . Then

$$\begin{aligned}
 & (B_1^* + B_2^*)(A_1 + A_2) = VT, \\
 & |A_1|^2 + |A_2|^2 = A^2 + B^2 = \frac{1}{2}(|T|^2 + |T^*|^2),
 \end{aligned}$$

$$|B_1|^2 + |B_2|^2 = C^2 + D^2 = \frac{1}{2} (|V|^2 + |V^*|^2)$$

and

$$\begin{aligned} B_1^* A_1 + B_2^* A_2 &= CA + (-iD)^* (iB) = CA - DB \\ &= \frac{1}{2} (V^* + V) \frac{1}{2} (T^* + T) - \frac{1}{2i} (V - V^*) \frac{1}{2i} (T - T^*) \\ &= \frac{1}{4} [(V^* + V)(T^* + T) + (V - V^*)(T - T^*)] \\ &= \frac{1}{2} (VT + V^*T^*). \end{aligned}$$

Then by (3.12) we get

$$\|VT\| \leq \frac{1}{2} \left[ \left\| |V|^2 + |V^*|^2 \right\|^{1/2} \left\| |T|^2 + |T^*|^2 \right\|^{1/2} + \|VT + V^*T^*\| \right]. \quad (3.13)$$

If we replace in this inequality  $V$  with  $V^*$  and  $T$  with  $T^*$ , then we get the dual inequality

$$\|V^*T^*\| \leq \frac{1}{2} \left[ \left\| |V|^2 + |V^*|^2 \right\|^{1/2} \left\| |T|^2 + |T^*|^2 \right\|^{1/2} + \|VT + V^*T^*\| \right]. \quad (3.14)$$

Adding these inequalities give us

$$\|VT\| + \|V^*T^*\| \leq \left\| |V|^2 + |V^*|^2 \right\|^{1/2} \left\| |T|^2 + |T^*|^2 \right\|^{1/2} + \|VT + V^*T^*\|, \quad (3.15)$$

which implies that

$$\begin{aligned} 0 &\leq \|VT\| + \|V^*T^*\| - \|VT + V^*T^*\| \\ &\leq \left\| |V|^2 + |V^*|^2 \right\|^{1/2} \left\| |T|^2 + |T^*|^2 \right\|^{1/2}, \end{aligned} \quad (3.16)$$

for any bounded linear operators  $V, T$ .

Using the inequality (3.5) for  $n = 2$  we can derive in a similar way the numerical radius inequality

$$\begin{aligned} 0 &\leq w(VT) + w(V^*T^*) - w(VT + V^*T^*) \\ &\leq \left\| |V|^2 + |V^*|^2 \right\|^{1/2} \left\| |T|^2 + |T^*|^2 \right\|^{1/2}, \end{aligned} \quad (3.17)$$

for any bounded linear operators  $V, T$ .

If we take in (3.16) and (3.17)  $V = T$ , then we get

$$0 \leq \|V^2\| + \|(V^*)^2\| - \|V^2 + (V^*)^2\| \leq \left\| |V|^2 + |V^*|^2 \right\|, \quad (3.18)$$

and

$$0 \leq w(V^2) + w((V^*)^2) - w(V^2 + (V^*)^2) \leq \left\| |V|^2 + |V^*|^2 \right\| \quad (3.19)$$

for any bounded linear operators  $V$ .

4. APPLICATIONS FOR FUNCTIONS OF NORMAL OPERATORS

Recall some examples of power series with nonnegative coefficients

$$\begin{aligned}
 \frac{1}{1-\lambda} &= \sum_{n=0}^{\infty} \lambda^n, \quad \lambda \in D(0, 1); \\
 \ln \frac{1}{1-\lambda} &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n, \quad \lambda \in D(0, 1); \\
 \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \quad \lambda \in \mathbb{C}; \\
 \sinh \lambda &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}, \quad \lambda \in \mathbb{C}; \\
 \cosh \lambda &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}, \quad \lambda \in \mathbb{C}.
 \end{aligned}
 \tag{4.1}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\begin{aligned}
 \frac{1}{2} \ln \left( \frac{1+\lambda}{1-\lambda} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\
 \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \quad \lambda \in D(0, 1); \\
 \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \\
 {}_2F_1(\alpha, \beta, \gamma, \lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \quad \alpha, \beta, \gamma > 0, \\
 &\lambda \in D(0, 1)
 \end{aligned}
 \tag{4.2}$$

where  $\Gamma$  is *Gamma function*.

The following result for power series with nonnegative coefficients holds:

**Theorem 4.1.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with nonnegative coefficients  $a_n \geq 0$  for  $n \in \mathbb{N}$  and having the radius of convergence  $R > 0$  or  $R = \infty$ . If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\alpha > 0$  such that  $\alpha < R$  and  $\|U\|, \|V\| \leq 1$ , then*

$$\begin{aligned}
 &|\langle f(\alpha U^*) f(\alpha V) x, y \rangle| \\
 &\leq \frac{1}{2} f(\alpha) \left[ \left\langle f(\alpha |V|^2) x, x \right\rangle^{1/2} \left\langle f(\alpha |U|^2) y, y \right\rangle^{1/2} + |\langle f(\alpha U^*V) x, y \rangle| \right]
 \end{aligned}
 \tag{4.3}$$

for any  $x, y \in H$ .

*Proof.* Using the inequality (3.6) we can state that

$$\begin{aligned} & \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^*)^i \sum_{i=0}^n a_i \alpha^i V^i x, y \right\rangle \right| \\ & \leq \frac{1}{2} \left[ \left\langle \sum_{i=0}^n a_i \alpha^i |V^i|^2 x, x \right\rangle \right]^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i |(U^*)^i|^2 y, y \right\rangle^{1/2} \\ & + \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^*)^i V^i x, y \right\rangle \right| \sum_{i=0}^n a_i \alpha^i \end{aligned} \quad (4.4)$$

for any  $x, y \in H$  and  $n \geq 1$ .

Since  $U, V$  are normal operators, then for  $i \geq 1$

$$|V^i|^2 = (V^i)^* V^i = (V^*)^i V^i = (V^* V)^i = |V|^{2i}$$

and

$$|(U^*)^i|^2 = |U|^{2i}.$$

Also, since  $U^* V = V U^*$ , then

$$(U^*)^i V^i = (U^* V)^i$$

for any  $i \geq 1$ .

Therefore from (4.4) we get

$$\begin{aligned} & \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^*)^i \sum_{i=0}^n a_i \alpha^i V^i x, y \right\rangle \right| \\ & \leq \frac{1}{2} \left[ \left\langle \sum_{i=0}^n a_i \alpha^i |V|^{2i} x, x \right\rangle \right]^{1/2} \left\langle \sum_{i=0}^n a_i \alpha^i |U|^{2i} y, y \right\rangle^{1/2} \\ & + \left| \left\langle \sum_{i=0}^n a_i \alpha^i (U^* V)^i x, y \right\rangle \right| \sum_{i=0}^n a_i \alpha^i, \end{aligned} \quad (4.5)$$

for any  $x, y \in H$  and  $n \geq 1$ .

Since  $\|\alpha |V|^2\| = \alpha \|V\|^2 < R$ ,  $\|\alpha |U|^2\| = \alpha \|U\|^2 < R$ ,  $\|\alpha U^* V\| \leq \alpha \|U\| \|V\| < R$ ,  $\|\alpha U^*\| < R$  and  $\|\alpha V\| < R$ , then the series

$$\sum_{i=0}^{\infty} a_i \alpha^i (U^*)^i, \quad \sum_{i=0}^{\infty} a_i \alpha^i V^i, \quad \sum_{i=0}^{\infty} a_i \alpha^i |V|^{2i}, \quad \sum_{i=0}^{\infty} a_i \alpha^i |U|^{2i}, \quad \sum_{i=0}^{\infty} a_i \alpha^i (U^* V)^i$$

are convergent in  $B(H)$  and  $\sum_{i=0}^{\infty} a_i \alpha^i$  is convergent in  $\mathbb{R}$ .

Taking the limit over  $n \rightarrow \infty$  in (4.5) we get the desired result (4.3).  $\square$

**Corollary 4.2.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with nonnegative coefficients  $a_n \geq 0$  for  $n \in \mathbb{N}$  and having the radius of convergence  $R > 0$  or  $R = \infty$ . If  $V$  is normal operator on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$  and  $\alpha > 0$  such that  $\alpha < R$  and  $\|V\| \leq 1$ , then

$$\begin{aligned} & |\langle f^2(\alpha V) x, y \rangle| \\ & \leq \frac{1}{2} f(\alpha) \left[ \langle f(\alpha |V|^2) x, x \rangle \right]^{1/2} \left\langle f(\alpha |V|^2) y, y \right\rangle^{1/2} + |\langle f(\alpha V^2) x, y \rangle| \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} & \left| \left\langle |f(\alpha V)|^2 x, y \right\rangle \right| \tag{4.7} \\ & \leq \frac{1}{2} f(\alpha) \left[ \left\langle f(\alpha |V|^2) x, x \right\rangle^{1/2} \left\langle f(\alpha |V|^2) y, y \right\rangle^{1/2} + \left| \left\langle f(\alpha |V|^2) x, y \right\rangle \right| \right] \end{aligned}$$

for any  $x, y \in H$ .

*Proof.* It follows by Theorem 4.1 by choosing  $U = V^*$  and  $U = V$ . □

**Example:** a. If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\alpha > 0$  then

$$\begin{aligned} & \left| \left\langle \exp(\alpha(U^* + V)) x, y \right\rangle \right| \tag{4.8} \\ & \leq \frac{1}{2} \left[ \left\langle \exp(\alpha |V|^2) x, x \right\rangle^{1/2} \left\langle \exp(\alpha |U|^2) y, y \right\rangle^{1/2} \right. \\ & \quad \left. + \left| \left\langle \exp(\alpha U^*V) x, y \right\rangle \right| \exp(\alpha) \right] \end{aligned}$$

for any  $x, y \in H$ .

b. If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\|U\|, \|V\| < 1$ ,  $\alpha \in (0, 1)$  then

$$\begin{aligned} & \left| \left\langle (1_H - \alpha U^*)^{-1} (1_H - \alpha V)^{-1} x, y \right\rangle \right| \tag{4.9} \\ & \leq \frac{1}{2} \left[ \left\langle (1 - \alpha |V|^2)^{-1} x, x \right\rangle^{1/2} \left\langle (1 - \alpha |U|^2)^{-1} y, y \right\rangle^{1/2} \right. \\ & \quad \left. + \left| \left\langle (1 - \alpha U^*V)^{-1} x, y \right\rangle \right| \right] (1 - \alpha)^{-1} \end{aligned}$$

for any  $x, y \in H$ .

By taking the supremum in (4.3) over  $\|x\| = \|y\| = 1$  we can state the following norm inequality:

**Corollary 4.3.** *Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with nonnegative coefficients  $a_n \geq 0$  for  $n \in \mathbb{N}$  and having the radius of convergence  $R > 0$  or  $R = \infty$ . If  $U, V$  are normal operators on the Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ ,  $U^*V = VU^*$  and  $\alpha > 0$  such that  $\alpha < R$  and  $\|U\|, \|V\| \leq 1$ , then*

$$\begin{aligned} & \|f(\alpha U^*) f(\alpha V)\| \tag{4.10} \\ & \leq \frac{1}{2} f(\alpha) \left[ \left\| f(\alpha |V|^2) \right\|^{1/2} \left\| f(\alpha |U|^2) \right\|^{1/2} + \|f(\alpha U^*V)\| \right]. \end{aligned}$$

The reader may state various particular inequalities of interest. However the details are omitted here.

#### REFERENCES

- [1] J. M. Aldaz, *Strengthened Cauchy-Schwarz and Hölder inequalities*. J. Inequal. Pure Appl. Math. **10** (2009), no. 4, Article 116, 6 pp.
- [2] N. G. de Bruijn, *Problem 12*, Wisk. Opgaven, **21** (1960), 12-14.
- [3] M. L. Buzano, *Generalizzazione della diseguaglianza di Cauchy-Schwarz*. (Italian), Rend. Sem. Mat. Univ. e Politech. Torino, **31** (1971/73), 405-409 (1974).
- [4] A. De Rossi, *A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets*. Math. Inequal. Appl. **2** (1999), no. 2, 263-282.
- [5] S. S. Dragomir, *Some refinements of Schwartz inequality*, Simpozionul de Matematici și Aplicații, Timișoara, Romania, 1-2 Noiembrie 1985, 13-16.



- [6] S. S. Dragomir, *Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality*. Integral Transforms Spec. Funct. **20** (2009), no. 9-10, 757–767.
- [7] S. S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces. I*. J. Inequal. Pure Appl. Math. **6** (2005), no. 3, Article 59, 15 pp.
- [8] S. S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces. II*. J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Article 14, 11 pp.
- [9] S. S. Dragomir, *Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces*. J. Inequal. Pure Appl. Math. **5** (2004), no. 3, Article 76, 18 pp.
- [10] S. S. Dragomir, *New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces*. Aust. J. Math. Anal. Appl. **1** (2004), no. 1, Art. 1, 18 pp.
- [11] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*. Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.
- [12] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*. Nova Science Publishers, Inc., New York, 2007. xii+243 pp. ISBN: 978-1-59454-903-8; 1-59454-903-6.
- [13] S. S. Dragomir and Anca C. Goşa, *Quasilinearity of some composite functionals associated to Schwarz's inequality for inner products*. Period. Math. Hungar. **64** (2012), no. 1, 11–24.
- [14] S. S. Dragomir and B. Mond, *Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces*. Soochow J. Math. **21** (1995), no. 4, 413–426.
- [15] S. S. Dragomir and I. Sándor, *Some inequalities in pre-Hilbertian spaces*. Studia Univ. Babeş-Bolyai Math. **32** (1987), no. 1, 71–78.
- [16] H. Gunawan, *On  $n$ -inner products,  $n$ -norms, and the Cauchy-Schwarz inequality*. Sci. Math. Jpn. **55** (2002), no. 1, 53–60.
- [17] K. E. Gustafson and D. K. M. Rao, *Numerical Range*, Springer-Verlag, New York, Inc., 1997.
- [18] E. R. Lorch, *The Cauchy-Schwarz inequality and self-adjoint spaces*. Ann. of Math. (2) **46**, (1945). 468–473.
- [19] C. Lupu and D. Schwarz, *Another look at some new Cauchy-Schwarz type inner product inequalities*. Appl. Math. Comput. **231** (2014), 463–477.
- [20] M. Marcus, *The Cauchy-Schwarz inequality in the exterior algebra*. Quart. J. Math. Oxford Ser. (2) **17** 1966 61–63.
- [21] P. R. Mercer, *A refined Cauchy-Schwarz inequality*. Internat. J. Math. Ed. Sci. Tech. **38** (2007), no. 6, 839–842.
- [22] F. T. Metcalf, *A Bessel-Schwarz inequality for Gramians and related bounds for determinants*. Ann. Mat. Pura Appl. (4) **68** 1965 201–232.
- [23] T. Precupanu, *On a generalization of Cauchy-Buniakowski-Schwarz inequality*. An. Şti. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. (N.S.) **22** (1976), no. 2, 173–175.
- [24] K. Trenčevski and R. Malčeski, *On a generalized  $n$ -inner product and the corresponding Cauchy-Schwarz inequality*. J. Inequal. Pure Appl. Math. **7** (2006), no. 2, Article 53, 10 pp.
- [25] L. Tuo, *Generalizations of Cauchy-Schwarz inequality in unitary spaces*, submitted.
- [26] G.-B. Wang and J.-P. Ma, *Some results on reverses of Cauchy-Schwarz inequality in inner product spaces*. Northeast. Math. J. **21** (2005), no. 2, 207–211.

S. S. DRAGOMIR

<sup>1</sup>MATHEMATICS, SCHOOL OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA., <sup>2</sup>SCHOOL OF COMPUTATIONAL & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA  
*E-mail address:* sever.dragomir@vu.edu.au