Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 8 Issue 1(2017), Pages 170-185.

SOME INTEGRAL INEQUALITIES FOR m-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

GHULAM FARID, BUSHRA TARIQ

DEDICATED TO PROFESSOR IVAN DIMOVSKI'S CONTRIBUTIONS

Abstract. In this paper we have found some integral inequalities by applying concept of m-convexity introduced by Toader via Riemann-Liouville fractional integrals. Also we have derived several Hadamard-type inequalities as special cases.

1. Introduction

Convex functions play an important role in different fields of Mathematics, Science and Engineering. A class of functions related to convex functions is m-convex functions introduced by Toader in [22].

Definition 1.1. *A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If we take $m = 1$, then we recapture the concept of convex functions defined on $[0, b]$ and if we take $m = 0$, then we get the concept of starshaped functions on $[0, b]$. We recall that a function $f : [0, b] \to \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, 1] \text{ and } x \in [0, b].$$

If the set of m-convex functions on $[0, b]$ for which $f(0) < 0$ is denoted by $K_m(b)$, then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$.
Note that in the class $K_1(b)$ there are convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [6]).
Example: [16] The function $f : [0, \infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\left(4x^3 - 15x^2 + 18x - 5\right)$$

2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.
Key words and phrases. m-Convex functions; Hadamard inequality; Fejér-Hadamard inequality; fractional integrals; Hölder inequality.

Submitted November 7, 2016. Published January 10, 2017.

is $\frac{16}{17}-$convex function but it is not convex function.
For more results and inequalities related to m-convex functions one can consult for example [6, 7, 10, 14, 17] along with references.

The Hadamard and the Fejér-Hadamard inequalities are of great interest for researchers working in the field of integral inequalities, and their various extensions and generalizations have been found (see, [1–5, 9, 12, 13, 15, 20, 23] and references therein).
Let $f : I \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known in literature as the Hadamard inequality.

In [11] Fejér gave the following generalization of the Hadamrd inequality.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $g : [a, b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.2}$$

In literature above inequality is known as the Fejér-Hadamard inequality.

Now a days the Hadamard and the Fejér-Hadamard inequalities via fractional calculus are in focus. Recently a lot of papers by many researchers have been dedicated in this direction for example see [13, 19, 20] and references therein.

Two sided Riemann-Liouville fractional integral operator is defined as follows:
Let $f \in L_1[a, b]$. Then the Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

where $\Gamma(\alpha)$ is the Gamma function defined as:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt,$$

also

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In [21] authors have proved the following Fejér-Hadamard type inequalities for convex functions.

Theorem 1.3. *Let $f : I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then* *the following inequality holds*

$$\begin{aligned}\Bigg|\left(\int_a^b g(s)ds\right)^{\alpha}[f(a)+f(b)] &- \alpha\int_a^b\left(\int_a^t g(s)ds\right)^{\alpha-1}g(t)f(t)dt\\ &-\alpha\int_a^b\left(\int_t^b g(s)ds\right)^{\alpha-1}g(t)f(t)dt\Bigg| \\ &\le \frac{(b-a)^{\alpha+1}\|g\|_\infty^{\alpha}}{\alpha+1}[|f'(a)|+|f'(b)|].\end{aligned} \tag{1.3}$$

with $\alpha > 0$.

Theorem 1.4. *Let* $f : I^o \to \mathbb{R}$ *be a differentiable mapping on* I^o, $a, b \in I^o$ *with* $a < b$ *and let* $g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$. *If* $|f'|^q$, $q > 1$ *is convex on* $[a,b]$, *then the following inequality holds with* $\alpha > 0, \frac{1}{p}+\frac{1}{q} = 1$ *and* $\|g\|_\infty = sup|g(t)|$

$$\begin{aligned}\Bigg|\left(\int_a^b g(s)ds\right)^{\alpha}[f(a)+f(b)] &- \alpha\int_a^b\left(\int_a^t g(s)ds\right)^{\alpha-1}g(t)f(t)dt\\ &-\alpha\int_a^b\left(\int_t^b g(s)ds\right)^{\alpha-1}g(t)f(t)dt\Bigg| \\ &\le \frac{2(b-a)^{\alpha+1}\|g\|_\infty^{\alpha}}{(\alpha p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.\end{aligned} \tag{1.4}$$

In [19], authors have proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Theorem 1.5. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 \le a < b$. *If* $\left|f'\right|^q$, $q \ge 1$ *is convex on* $[a,b]$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}\Bigg|\frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)} &- \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\\ &\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\Bigg|\\ &\le \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{\left|f'(\lambda a+(1-\lambda)b)\right|^q+\left|f'(\lambda b+(1-\lambda)a)\right|^q}{2}\right]^{\frac{1}{q}},\end{aligned} \tag{1.5}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$ *with* $\alpha > 0$.

Theorem 1.6. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0 \le a < b$. *If* $\left|f'\right|^q$ *is convex on* $[a,b]$ *for some fixed* $q \ge 1$, *then the following inequality* *for fractional integrals holds:*

$$\begin{aligned}&\Bigg|\frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\\&\qquad\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\Bigg|\\&\leq\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[\frac{\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+\left|f'(\lambda b+(1-\lambda)a)\right|^{q}}{2}\right]^{\frac{1}{q}},\end{aligned}\tag{1.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ *and* $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Theorem 1.7. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $0\leq a<b$. *If* $\left|f'\right|^{q}$ *is convex on* $[a,b]$ *for some fixed* $q\geq1$, *then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Bigg|\frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}}\\&\qquad\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^{+}}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(\lambda b+(1-\lambda)a)\right]\Bigg|\\&\leq\left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\frac{1}{q}}\left[\frac{\left|f'(\lambda a+(1-\lambda)b)\right|^{q}+\left|f'(\lambda b+(1-\lambda)a)\right|^{q}}{2}\right]^{\frac{1}{q}},\end{aligned}\tag{1.7}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ *and* $\alpha>0$.

We organize the paper as follows:
In Section 2 we give the Fejér-Hadamard type inequalities for m-convex functions via Riemann-Liouville fractional integrals. Also we deduce results of [8, 21]. In Section 3 we give generalizations of the Hadamard-type inequalities for m-convex functions via Riemann-Liouville fractional integrals given in [18–20]. Moreover we deduce results of [8, 18–21] as special cases of our results.

2. Main results

We need the following lemma for our results.

Lemma 2.1. *Let* $f:I^{o}\to\mathbb{R}$ *be a differentiable mapping on* I^{o}, $a,b\in I^{o}$ *with* $a<mb$ *and let* $g:[a,mb]\to\mathbb{R}$ *be continuous on* $[a,mb]$. *If* $f',g\in L[a,b]$, *then for all* $t\in[a,mb]$ *the following equality holds*

$$\begin{aligned}&\left(\int_{a}^{mb}g(s)ds\right)^{\alpha}[f(a)+f(mb)]\\&-\alpha\int_{a}^{mb}\left(\int_{a}^{t}g(s)ds\right)^{\alpha-1}g(t)f(t)dt-\alpha\int_{a}^{mb}\left(\int_{t}^{mb}g(s)ds\right)^{\alpha-1}g(t)f(t)dt\\&\qquad=\int_{a}^{mb}\left(\int_{a}^{t}g(s)ds\right)^{\alpha}f'(t)dt-\int_{a}^{mb}\left(\int_{t}^{mb}g(s)ds\right)^{\alpha}f'(t)dt\end{aligned}\tag{2.1}$$

where $\alpha>0$

Proof. One can have

$$\begin{aligned}&\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha} f'(t)dt\\&=\left(\int_a^{mb} g(s)ds\right)^{\alpha} f(mb)-\alpha\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha-1} g(t)f(t)dt \qquad (2.2)\end{aligned}$$

and

$$\begin{aligned}&\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha} f'(t)dt\\&=-\left(\int_a^{mb} g(s)ds\right)^{\alpha} f(a)+\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt. \qquad (2.3)\end{aligned}$$

Subtracting (2.3) from (2.2) we get (2.1). □

Using above lemma we give the following generalization of Theorem 1.3.

Theorem 2.2. *Let $f : I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ and let $g : [a, mb] \to \mathbb{R}$ be continuous on $[a, mb]$. If $|f'|$ is m-convex on $[a, mb]$ with $a < mb$, then for $\alpha > 0$ and $\|g\|_\infty = sup|g(t)|$ the following inequality holds*

$$\begin{aligned}&\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.\\&\left.\qquad-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right| \qquad (2.4)\\&\qquad\le\frac{(mb-a)^{\alpha+1}\|g\|_\infty^{\alpha}}{\alpha+1}[|f'(a)|+m|f'(b)|].\end{aligned}$$

Proof. Using (2.1) of Lemma 2.1 we get

$$\begin{aligned}&\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.\\&\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|\\&\le\int_a^{mb}\left|\int_a^t g(s)ds\right|^{\alpha}|f'(t)|dt+\int_a^{mb}\left|\int_t^{mb} g(s)ds\right|^{\alpha}|f'(t)|dt.\end{aligned}$$

As $g(t) \leq \|g\|_\infty$, so we have

$$\left|\left(\int_a^{mb} g(s)ds\right)^\alpha [f(a)+f(mb)] - \alpha\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|$$
$$\leq \|g\|_\infty^\alpha\left[\int_a^{mb}(t-a)^\alpha|f'(t)|dt + \int_a^{mb}(mb-t)^\alpha|f'(t)|dt\right]$$
$$= \|g\|_\infty^\alpha\left[\int_a^{mb}(t-a)^\alpha\left|f'\left(\frac{mb-t}{mb-a}a + m\frac{t-a}{mb-a}b\right)\right|dt\right.$$
$$\left.+\int_a^{mb}(mb-t)^\alpha\left|f'\left(\frac{mb-t}{mb-a}a + m\frac{t-a}{mb-a}b\right)\right|dt\right].$$

Using m-convexity of $|f'|$ we have

$$\left|\left(\int_a^{mb} g(s)ds\right)^\alpha [f(a)+f(mb)] - \alpha\int_a^{mb}\left(\int_a^t g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|$$
$$\leq \|g\|_\infty^\alpha\left[\int_a^{mb}(t-a)^\alpha\left(\frac{mb-t}{mb-a}|f'(a)| + m\frac{t-a}{mb-a}|f'(b)|\right)dt\right.$$
$$\left.+\int_a^{mb}(mb-t)^\alpha\left(\frac{mb-t}{mb-a}|f'(a)| + m\frac{t-a}{mb-a}|f'(b)|\right)dt\right].$$

From which after some calculations we get the required result. □

Corollary 2.3. *For $g(s) \equiv 1$ we have the following Hadamard-type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[J_{a+}^\alpha f(mb) + J_{mb-}^\alpha f(a)\right]\right|$$
$$\leq \frac{mb-a}{2(\alpha+1)}\left[|f'(a)| + m|f'(b)|\right], \quad (2.5)$$

with $\alpha > 0$.

Remark. *In Theorem 2.2*
(i) if we take $m = 1$, then we get Theorem 1.3.
(ii) if we take$g(s) \equiv 1$ along with $m = 1$, then we get [21, Corollary 2].
(iii) if we take $\alpha = 1$ along with $m = 1$, then we get [21, Corollary 3].

Theorem 2.4. *Let $f : I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ and let $g : [a, mb] \to \mathbb{R}$ be continuous on $[a, mb]$. If $|f'|^q$ is m-convex on $[a, mb]$, $q > 1$, with $a < mb$, then the following inequality with $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|g\|_\infty = sup|g(t)|$*

holds

$$\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^{t} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right| \tag{2.6}$$
$$\leq \frac{2(mb-a)^{\alpha+1}\|g\|_{\infty}^{\alpha}}{(\alpha p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q+m|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

Proof. Using (2.1) of Lemma 2.1 we get

$$\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^{t} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|$$
$$\leq \int_a^{mb}\left|\int_a^{t} g(s)ds\right|^{\alpha}|f'(t)|dt+\int_a^{mb}\left|\int_t^{mb} g(s)ds\right|^{\alpha}|f'(t)|dt.$$

Using Hölder's inequality on the right hand side of the above inequality we have

$$\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^{t} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|$$
$$\leq \left(\int_a^{mb}\left|\int_a^{t} g(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^{mb}\left|\int_t^{mb} g(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}.$$

As $g(t)\leq \|g\|_{\infty}$, so we get

$$\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^{t} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1} g(t)f(t)dt\right|$$
$$\leq \|g\|_{\infty}^{\alpha}\left[\left(\int_a^{mb}|t-a|^{\alpha p}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\alpha p}dt\right)^{\frac{1}{p}}\right]\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}.$$

From m-convexity of $|f'|^q$ we have

$$\left|\left(\int_a^{mb} g(s)ds\right)^{\alpha}[f(a)+f(mb)]-\alpha\int_a^{mb}\left(\int_a^{t} g(s)ds\right)^{\alpha-1}g(t)f(t)dt\right.$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)ds\right)^{\alpha-1}g(t)f(t)dt\right|$$
$$\leq \|g\|_\infty^{\alpha}\left[\left(\int_a^{mb}|t-a|^{\alpha p}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\alpha p}dt\right)^{\frac{1}{p}}\right]$$
$$\left(\int_a^{b}\left(\frac{mb-t}{mb-a}|f'(a)|^q+m\frac{t-a}{mb-a}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}.$$

From which one can have inequality (2.6). □

Corollary 2.5. *If we take $g(s)\equiv 1$ we have the following Hadamard-type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[J_{a+}^{\alpha}f(mb)+J_{mb-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{(mb-a)}{(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+m|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Remark. *In Theorem 2.4*
(*i*) *if we take $m=1$ we get Theorem 1.4.*
(*ii*) *if we take $\alpha=1$ along with $m=1$ we get [21, Corollary 5].*
(*iii*) *if we take $\alpha=1$, $g(s)\equiv 1$ along with $m=1$ we get [8, Theorem 2.3].*

3. Generalized Hadamard-type integral inequalities for m-convex functions via Riemann-Liouville fractional integrals

In this section we are interested to give some Hadamard-type inequalities for differentiable m-convex functions via fractional integrals. Also we deduce some results of [18–20].

We need the following lemma for our results.

Lemma 3.1. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on (a,mb) with $0\leq a<mb$. If $f'\in L[a,mb]$, then the following equality holds*

$$-\frac{f(m\lambda b+(1-\lambda)a)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}+\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}$$
$$\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^+}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^-}f(m\lambda b+(1-\lambda)a)\right]$$
$$=\int_0^1[(1-t)^{\alpha}-t^{\alpha}]f'\left(t(\lambda a+m(1-\lambda)b)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)dt, \tag{3.1}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ and $\alpha>0$.

Proof. Since

$$\begin{aligned}
&\int_0^1 [(1-t)^\alpha - t^\alpha] f'\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt \\
&= \int_0^1 (1-t)^\alpha f'\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt \\
&- \int_0^1 t^\alpha f'\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt. \qquad (3.2)
\end{aligned}$$

First term of right hand side is calculated as

$$\begin{aligned}
&\int_0^1 (1-t)^\alpha f'\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt \\
&= -\frac{f\left(m\lambda b + (1-\lambda)a\right)}{(1-2\lambda)(mb-a)} \\
&+ \frac{\alpha}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}} \int_{m\lambda b+(1-\lambda)a}^{\lambda a+m(1-\lambda)b} \left(\lambda a + m(1-\lambda)b - x\right)^{\alpha-1} f(x)dx \\
&= -\frac{f(m\lambda b + (1-\lambda)a)}{(1-2\lambda)(mb-a)} + \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}} J^\alpha_{(m\lambda b+(1-\lambda)a)^+} f(\lambda a + m(1-\lambda)b),
\end{aligned}$$

while second term of the right side is calculated as

$$\begin{aligned}
&\int_0^1 t^\alpha f'\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt \\
&= \frac{f\left(\lambda a + m(1-\lambda)b\right)}{(1-2\lambda)(mb-a)} \\
&- \frac{\alpha}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}} \int_{m\lambda b+(1-\lambda)a}^{\lambda a+m(1-\lambda)b} \left(x - m\lambda b + (1-\lambda)a\right)^{\alpha-1} f(x)dx \\
&= \frac{f(\lambda a + m(1-\lambda)b)}{(1-2\lambda)(mb-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}} J^\alpha_{(\lambda a+m(1-\lambda)b)^-} f(m\lambda b + (1-\lambda)a).
\end{aligned}$$

Now using these values in (3.2), we get required result. □

Remark. *In Lemma 3.1*
(i) if we take $m = 1$, then we get [19, Lemma 2.1].
(ii) if we take $\lambda = 0$ or $\lambda = 1$ along with $m = 1$, then we get [20, Lemma 2].
(iii) if we take $\alpha = m = 1$ along with $\lambda = 0$ or $\lambda = 1$, then we get [8, Lemma 2.1].

Using the above lemma we give following Hadamard-type inequality for m-convex functions.

Theorem 3.2. *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable mapping on (a, mb) with $0 \leq a < mb$. If $\left|f'\right|^q$, $q \geq 1$ is m-convex on $[a, mb]$, then the following inequality* *holds*

$$\begin{aligned}
&\Bigg|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\\
&\quad\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\Bigg|\\
&\le\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{\left|f^{'}(\lambda a+m(1-\lambda)b)\right|^{q}+m\left|f^{'}(\lambda b+(1-\lambda)\frac{a}{m})\right|^{q}}{2}\right]^{\frac{1}{q}} \qquad (3.3)
\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ *with* $\alpha>0$.

Proof. Firstly, we suppose that $q=1$. Using Lemma 3.1 and m-convexity of $\left|f^{'}\right|$, we find

$$\begin{aligned}
&\Bigg|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\\
&\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\Bigg|\\
&\le\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f^{'}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt\\
&\le\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t\left|f^{'}\left(\lambda a+m(1-\lambda)b\right)\right|+m(1-t)\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\right]dt\\
&=\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[t\left|f^{'}\left(\lambda a+m(1-\lambda)b\right)\right|+m(1-t)\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\right]dt\\
&+\int_{\frac{1}{2}}^1\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[t\left|f^{'}\left(\lambda a+m(1-\lambda)b\right)\right|+m(1-t)\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\right]dt. \qquad (3.4)
\end{aligned}$$

By calculation we have

$$\begin{aligned}
&\left|f^{'}(\lambda a+m(1-\lambda)b)\right|\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}t-t^{\alpha+1}\right]dt\\
&\qquad+m\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha+1}-(1-t)t^{\alpha}\right]dt\\
&\qquad=\left|f^{'}(\lambda a+m(1-\lambda)b)\right|\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]\\
&\qquad\qquad+m\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\left[\frac{1}{\alpha+2}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]
\end{aligned}$$

and

$$\left|f^{'}(\lambda a+m(1-\lambda)b)\right|\left[\int_{\frac{1}{2}}^{1}t^{\alpha+1}dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha}tdt\right]$$
$$+m\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\left[\int_{\frac{1}{2}}^{1}t^{\alpha}(1-t)dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+1}dt\right]$$
$$=\left|f^{'}(\lambda a+m(1-\lambda)b)\right|\left[\frac{1}{(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]$$
$$+m\left|f^{'}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right].$$

Therefore from (3.4) we have

$$\Bigg|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}$$
$$\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\Bigg|$$
$$\leq\frac{2}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\left[\frac{\left|f^{'}(\lambda a+m(1-\lambda)b)\right|+m\left|f^{'}(\lambda b+(1-\lambda)\frac{a}{m})\right|}{2}\right].$$

Secondly, we suppose that $q>1$. Using Lemma 3.1 and power mean inequality we have

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f^{'}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f^{'}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Using m-convexity of $\left|f^{'}\right|^{q}$ we get

$$\left|\frac{f(m\lambda b+(1-\lambda)a)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left(t(\lambda a+m(1-\lambda)b)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}|(1-t)^{\alpha}-t^{\alpha}|\,dt+\int_{\frac{1}{2}}^1|t^{\alpha}-(1-t)^{\alpha}|\,dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left(t(\lambda a+m(1-\lambda)b)\right|^q+m(1-t)\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{1-\frac{1}{q}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{\frac{1}{q}}$$
$$\left[\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}$$
$$\leq 2^{\frac{q-1}{q}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

From which one can have (3.3). □

Remark. *In Theorem 3.2*

(i) if we take $m=1$, then the inequality (3.3) reduces to the inequality (1.5) of Theorem 1.5.

(ii) if we take $m=1$ along with $\lambda=0$ or $\lambda=1$, then we get [19, Corollary 2.4].

(iv) if we take $\alpha=m=1$ along with $\lambda=0$ or $\lambda=1$, then we get [18, Theorem 1].

(iv) if we take $\alpha=m=q=1$ along with $\lambda=0$ or $\lambda=1$, then we get [8, Theorem 2.2].

Theorem 3.3. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on (a,mb) with $0\leq a<mb$. If $\left|f'\right|^q$ is m-convex on $[a,mb]$ for some fixed $q\geq 1$, then the following inequality holds*

$$\left|\frac{f(m\lambda b+(1-\lambda)a)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{q}}\left[\frac{\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}} \tag{3.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. By Using Lemma 3.1, Hölder inequality and m-convexity of $|f'|^q$ respectively we have

$$\begin{aligned}
&\Bigg|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\\
&\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\Bigg|\\
&\leq\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt\\
&\leq\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq\left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]^p\,dt+\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]^p\,dt\right)^{\frac{1}{p}}\\
&\left(\int_0^1\left(t\left|f'\left(\lambda a+m(1-\lambda)b\right)\right|^q+m(1-t)\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\leq\left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha p}-t^{\alpha p}]\,dt+\int_{\frac{1}{2}}^1[t^{\alpha p}-(1-t)^{\alpha p}]\,dt\right)^{\frac{1}{p}}\\
&\left(\frac{\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right)^{\frac{1}{q}}\\
&\leq\left(\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right)^{\frac{1}{p}}\left(\frac{\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Also we have used $(A-B)^q\leq A^q-B^q,\quad A\geq B\geq 0.$ □

Remark. *In Theorem 3.3*

(i) if we take $m=1$, then the inequality (3.5) reduces to the inequality (1.6) of Theorem 1.6.

(ii) if we take $m=1$ along with $\lambda=0$ or $\lambda=1$, then we get [19, Corollary 2.7].

(iii) if we take $\alpha=m=1$ along with $\lambda=0$ or $\lambda=1$, then we get [19, Corollary 2.8].

Theorem 3.4. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on (a,mb) with $0\leq a<mb$. If $\left|f'\right|^q$ is m-convex on $[a,mb]$ for some fixed $q\geq 1$, then the following*

inequality holds

$$\left|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\frac{1}{q}}\left[\frac{\left|f'(\lambda a+m(1-\lambda)b)\right|^{q}+m\left|f'(\lambda b+(1-\lambda)\frac{a}{m})\right|^{q}}{2}\right]^{\frac{1}{q}}\tag{3.6}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ *and* $\alpha>0$.

Proof. By Using Lemma 3.1, Hölder inequality and m-convexity of $|f'|^q$ respectively we have

$$\left|\frac{f\left(m\lambda b+(1-\lambda)a\right)+f(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}-\frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(mb-a)^{\alpha+1}}\right.$$
$$\left.\times\left[J^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f(\lambda a+m(1-\lambda)b)+J^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt$$
$$\leq\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1|(1-t)^{\alpha}-t^{\alpha}|^q\right.$$
$$\left.\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}|(1-t)^{\alpha}-t^{\alpha}|^q\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1|t^{\alpha}-(1-t)^{\alpha}|^q\left|f'\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\left|f'\left(\lambda a+m(1-\lambda)b\right)\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha}t-t^{q\alpha+1}\right]dt\right.$$
$$+m\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha+1}-t^{q\alpha}(1-t)\right]dt$$
$$+\left|f'\left(\lambda a+m(1-\lambda)b\right)\right|^q\int_{\frac{1}{2}}^1\left[t^{q\alpha+1}-(1-t)^{q\alpha}t\right]dt$$
$$\left.+m\left|f'\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\int_{\frac{1}{2}}^1\left[t^{q\alpha}(1-t)-(1-t)^{q\alpha+1}\right]dt\right)^{\frac{1}{q}}$$
$$\leq\left[\frac{1}{(q\alpha+1)}\left(1-\frac{1}{2^{q\alpha}}\right)\right]^{\frac{1}{q}}\left[\left|f'(\lambda a+m(1-\lambda)b)\right|^q+m\left|f'(\lambda b+(1-\lambda)\frac{a}{m})\right|^q\right]^{\frac{1}{q}}.$$

We have used $(A-B)^q\leq A^q-B^q$ for any $A\geq B\geq 0$ and $q\geq 1$.

□

Remark. *In Theorem 3.4*

(*i*) *if we take* $m = 1$*, then the inequality (3.6) reduces to the inequality (1.7) of Theorem 1.7.*

(*ii*) *if we take* $m = 1$ *along with* $\lambda = 0$ *or* $\lambda = 1$*, then we get [19, Corollary 2.11].*

(*iii*) *if we take* $\alpha = m = 1$ *along with* $\lambda = 0$ *and* $\lambda = 1$*, then we get [19, Corollary 2.12].*

References

[1] A. Azócar, K. Nikodem and G. Roa, *Fejér-type inequalities for strongly convex functions*, Ann. Math. Sil. **26** (2012) 43–54.

[2] A.G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colomb. Mat. **28**(1) (1994) 7–12.

[3] M. Alomari and M. Darus, *On the Hadamard's inequality for log convex functions on coordinates*, J. Inequal. Appl., **2009** (2009) 13 pp. Article ID 283147.

[4] M.K. Bakula and J. Pečarić, *Note on some Hadamard-type inequalities*, J. Ineq. Pure Appl. Math. **5**(3) (2004).

[5] M.K. Bakula, M.E. Ozdemir and J. Pečarić, *Hadamard tpye inequalities for m-convex and* (α, m)*-convex functions*, J. Ineq. Pure Appl. Math. **9**(4) (2008) 12 pp.

[6] S.S. Dragomir, *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Turk. J. Math. **33**(1) (2002) 45–55.

[7] S.S. Dragomir and G. H. Toader, *Some inequalities for m-convex functions*, Stud. Univ. Babeş-Bolyai Math. **38**(1) (1993) 21–28.

[8] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11**(5) (1998) 91–95.

[9] S.S. Dragomir and C.E.M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.

[10] G. Farid, M. Marwan and A.U. Rehman, *New mean value theorems and generalization of Hadamard inequality via coordinated m-convex functions*, J. Inequal. Appl. **2015** (2015) 11pp. Article ID 283.

[11] L. Fejér, *Uber die fourierreihen, II*, Math. Naturwiss Anz. Ungar. Akad. Wiss. **24** (1906) 369–390. (In Hungarian)

[12] P.M. Gill, C.E.M. Pearce and J. Pečarić, *Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl. **215**(2) (1997) 461–470.

[13] İ. İşcan, *Hermite Hadamard Fejér type inequalities for convex functions via fractional integrals*, Stud. Univ. Babeş-Bolyai Math. **60**(3) (2015) 355–366.

[14] İ. İşcan, *New estimates on generalization of some integral inequalities for* (α, m)*-convex functions*, Contemp. Anal. Appl. Math. **1**(2) (2013) 253–264.

[15] U.S. Kirmaci, M.K. Bakula, M. E. Ozdemir and J. Pečarić, *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput., **193** (2007) 26–35.

[16] P.T. Mocanu, I. Serb and G. Toader, *Real star-convex functions*, Stud. Univ. Babes-Bolyai Math. **42**(3) (1997) 65–80.

[17] M.E. Ozdemir, M. Avci and E. Set, *On some inequalities of Hermite-Hadamard type via m-convexity*, Appl. Math. Lett. **23**(9) (2010) 1065–1070.

[18] C.E.M. Pearce and J. Pečarić, *Inequalities for diferentiable mappings with application to special means and quadrature formalae*, Appl. Math. Lett. **13**(2) (2000) 51-55.

[19] M.Z. Sarikaya and H. Budak, *Generalized Hermite–Hadamard type integral inequalities for fractional integrals*, Filomat. **30**(5) (2016) 1315–1326.

[20] M.Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model. **57** (2013) 2403–2407.

[21] M.Z. Sarikaya and S. Erden, *On the Hermite-Hadamard Féjer-type integral inequality for convex function*, TJANT. **2**(3) (2014) 85–89.

[22] G.H. Toader, *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim. Cluj-Naploca. (Romania) (1984), 329–338.

[23] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl. **2013** (2013) 15 pp. Article ID 325.

G. Farid
Department of Mathematics, COMSATS Institute of Information Technology, Attock Campus, Pakistan
E-mail address: faridphdsms@hotmail.com,ghlmfarid@ciit-attock.edu.pk

B. Tariq
Army public school and college, Attock, Pakistan
E-mail address: bushratariq38@yahoo.com