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ON EXISTENCE AND ASSOCIATIVITY OF MULTI-CONVOLUTIONS OF BEURLING ULTRADISTRIBUTIONS

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DEDICATED TO PROFESSOR IVAN DIMOVSKI'S CONTRIBUTIONS

Abstract. We present certain results on existence of multi-convolutions of (tempered) distributions and Beurling (tempered) ultradistributions as well as on various forms of associativity of multi-convolutions of Beurling (tempered) ultradistributions. The results are formulated in terms of supports of distributions and ultradistributions, respectively.

1. Introduction

In this short note we are going to present without proofs some results on the existence of multi-convolutions of (ultra)distributions, being an extension of the classical convolution of two (ultra)distributons to the case of k (ultra)distributions, where $k \in \mathbb{N} \setminus \{1\}$. In case of distributions (see [22] and [6, 1, 26] such an extension is considered e.g. in [6], pp. 389–391, and plays an important role in formulating conditions for associativity of the convolution of distributions. We consider various forms of associativity of the (multi-)convolution of ultradistributions. The existence and associativity results are expressed in terms of supports of (ultra)distributions and the existence results in these terms have form of characterizations. For simplicity we restrict our discussions concerning the (multi-)convolution of ultradistributions to the Beurling case (see [2] and e.g. [16, 17, 3]) though analogous results can be formulated also for Roumieu ultradistributions (see [21] and e.g. [16, 17, 3]). The proofs of the results will be published separately.

2. Multi-convolutions of functions and distributions

We start with the definition of an extension of the convolution of two locally integrable functions to the case of k functions in L^1_{loc} (for $k = 3$ see [1], pp. 112–114). This extension will be called simply the *convolution* of k functions. Here and further on k denotes a fixed positive integer greater than 1. In case $k = 2$ the extension coincides with the convolution of two locally integrable functions.

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Definition 2.1. Let $F_1, F_2 \dots, F_k \in L^1_{\rm loc}(\mathbb{R}^d)$. By the (*functional*) *convolution* of $F_1, F_2 \dots, F_k \in L^1_{\rm loc}(\mathbb{R}^d)$ we mean the function $F_1 \star F_2 \star \dots \star F_k$ on $\mathbb{R}^d$ defined by

$$(F_1 \star F_2 \star \dots \star F_k)(x) := \int_{\mathbb{R}^{(k-1)d}} F(x,t)\, dt, \tag{2.1}$$

where

$$F(x,t) := F_1(x-t_1) \cdot F_2(t_1-t_2) \cdot \ \dots \ \cdot F_{k-1}(t_{k-2}-t_{k-1}) F_k(t_{k-1})$$

for $x \in \mathbb{R}^d$ and $t = (t_1, \dots, t_{k-1}) \in \mathbb{R}^{(k-1)d}$, whenever the function F under the integral signs in (2.1) as a function of the variable $t = (t_1, \dots, t_{k-1}) \in \mathbb{R}^{(k-1)d}$ is integrable on $\mathbb{R}^{(k-1)d}$ for almost all $x \in \mathbb{R}^d$. We say then that the *convolution* $F_1 \star F_2 \star \dots \star F_k$ *exists a.e. in* $\mathbb{R}^d$.

Since $L^1_{\rm loc} \subset \mathcal{D}' \subset \mathcal{D}'^{(M_p)}$, it is important to know whether the functional convolution defined in (2.1) is consistent with the convolution in the space $\mathcal{D}'$ of Schwartz distributions and in the space $\mathcal{D}'^{(M_p)}$ of Beurling ultradistributions.

There are various general definitions of the convolution in $\mathcal{D}'$, given (usually for $k = 2$) by C. Chevalley [4], L. Schwartz [23], R. Shiraishi [24], V. S. Vladimirov [26], J. Horvath [6], P. Dierolf-J. Voigt [5], A. Kamiński [8], R. Wawak [27], S. Mincheva-Kaminska [18, 19] and others. Most of the mentioned definitions are equivalent. They have counterparts in the space $\mathcal{S}'$ of tempered distributions (see e.g. [24], [5], [8], [27]) and in other subspaces of $\mathcal{D}'$, e.g. in the Gelfand-Shilov spaces $K'\{M_p\}$ (see e.g. [25], [15]). In the theory of ultradistributions and tempered ultradistributions of Beurling type (see [2], [16], [17], [3]) general notions of the convolution analogous to the mentioned definitions of the distributional convolution were studied in several publications (see e.g. [20], [10], [12], [13], [14], [3], [11]).

We recall here, for simplicity, only one of the equivalent versions of definitions of convolution in case of distributions and ultradistributions of Beurling type, corresponding to the Vladimirov sequential definition of the convolution in $\mathcal{D}'$ based on the notion defined below that some authors call "special approximate unit", but we use for it the shorter name "approximate unit" since no other types of these sequences will be discussed in this note (see [26], p. 63, and [5]).

Definition 2.2. We call a sequence (η_n) of elements of $\mathcal{D}(\mathbb{R}^d)$ an *approximate unit* on $\mathbb{R}^d$, if for every compact set in $\mathbb{R}^d$ (in symbols $K \subset\subset \mathbb{R}^d$) there exists an $n_0 \in \mathbb{N}$ such that $\eta_n(x) = 1$ for $x \in K$, $n \geq n_0$ (hence $\eta_n \to 1$ in $\mathcal{E}(\mathbb{R}^d)$) and, in addition,

$$\sup_{n \in \mathbb{N}} \| \eta_n^{(i)} \|_\infty =: \gamma_i < \infty \quad \text{for every} \quad i \in \mathbb{N}_0^d,$$

where $\| \cdot \|_\infty$ means the L^∞-norm. We denote the set of all approximate units on $\mathbb{R}^d$ by $\mathbb{U}(\mathbb{R}^d)$.

The Vladimirov type definitions of multi-convolutions, i.e. the convolution of k distributions in $\mathcal{D}'$ and of the convolution of k tempered distributions in $\mathcal{S}'$ for any $k \in \mathbb{N} \setminus \{1\}$, can be formulated as follows (cf. [26], p. 61-64, and [5]):

Definition 2.3. For given distributions $S_1, \dots, S_k \in \mathcal{D}'(\mathbb{R}^d)$ we define the *convolution* $S_1 * \dots * S_k$ *in* $\mathcal{D}'$ by

$$\langle S_1 * \dots * S_k,\ \varphi \rangle := \lim_{n\to\infty} \langle S_1 \otimes \dots \otimes S_k,\ \eta_n\, \varphi^{\triangle} \rangle, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{2.2}$$

where

$$\varphi^{\triangle}(x_1, \dots, x_k) := \varphi(x_1 + \dots + x_k), \qquad x_1, \dots, x_k \in \mathbb{R}^d, \ \ \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{2.3}$$

whenever the limit exists for all $(\eta_n) \in \mathbb{U}(\mathbb{R}^{kd})$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$; we say then that the *convolution* $S_1 * \ldots * S_k$ *exists in* $\mathcal{D}'$.

Definition 2.4. For given tempered distributions $S_1, \ldots, S_k \in \mathcal{S}'(\mathbb{R}^d)$ we define the *convolution* $S_1 * \ldots * S_k$ *in* $\mathcal{S}'$ by

$$\langle S_1 * \ldots * S_k,\ \psi\rangle := \lim_{n\to\infty} \langle S_1 \otimes \ldots \otimes S_k,\ \eta_n\, \psi^{\triangle}\rangle, \qquad \psi \in \mathcal{S}(\mathbb{R}^d) \tag{2.4}$$

with $\psi^{\triangle}$ defined as in (2.3), whenever the limit exists for all $(\eta_n) \in \mathbb{U}(\mathbb{R}^{kd})$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$; we say then that the *convolution* $S_1 * \ldots * S_k$ *exists in* $\mathcal{S}'$.

Since $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ can be considered as a subspace of $\mathcal{D}'(\mathbb{R}^d)$ by means of the imbedding:

$$L^1_{\mathrm{loc}}(\mathbb{R}^d) \ni F \mapsto T_F \in \mathcal{D}'(\mathbb{R}^d),$$

where

$$(T_F, \varphi) := \int_{\mathbb{R}^d} F(x)\varphi(x)\, dx, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

we may treat given functions $F_1, \ldots, F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and their convolution $F_1 \star \ldots \star F_k$ defined in (2.1) if only the following assumption is satisfied:

(a) $\quad F_1 \star \ldots \star F_k$ exists a.e. in $\mathbb{R}^d$ $\quad$ & $\quad F_1 \star \ldots \star F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

However condition (a) does not imply, in general, that the Vladimirov type convolution of the distributions represented by the functions $F_1, \ldots, F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ exists in $\mathcal{D}'$. This is true under the following stronger condition:

(b) $\quad F_1 \star \ldots \star F_k$ exists a.e. in $\mathbb{R}^d$ $\quad$ & $\quad |F_1| \star \ldots \star |F_k| \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Definition 2.5. Let $F_1, \ldots, F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We say that the *convolution* $F_1 \star \ldots \star F_k$ *exists in* L^1_{loc} if condition (b) holds. Moreover, we say then that the *convolutions* $F_1 \star \ldots \star F_k$ and $F_1 * \ldots * F_k$ *are consistent in* $\mathcal{D}'$ and write

$$F_1 \star \ldots \star F_k \sim F_1 * \ldots * F_k,$$

whenever

$$\langle F_1 * \ldots * F_k, \varphi\rangle = \langle F_1 \star \ldots \star F_k, \varphi\rangle \quad (= \langle F_1 \otimes \ldots \otimes F_k, \varphi^{\triangle}\rangle)$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Theorem 2.6. *Let $F_1, \ldots, F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If the convolution $F_1 \star \ldots \star F_k$ exists in L^1_{loc}, then $F_1 * \ldots * F_k$ exists in $\mathcal{D}'$ and both the convolutions are consistent in $\mathcal{D}'$, i.e.*

$$F_1 \star \ldots \star F_k \sim F_1 * \ldots * F_k. \tag{2.5}$$

It should be noted that the assumptions of Theorem 2.6, i.e. the conditions in (b), are essential while the conditions in (a) are not sufficient to get (2.5). As a matter of fact one can prove that there exist (smooth) functions $F_1, \ldots, F_k \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying (a) such that $|F_1| \star \ldots \star |F_k| \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and the functional convolution $F_1 \star \ldots \star F_k$ is a function with arbitrarily large singularity at a given point in the following sense:

Theorem 2.7. *For an arbitrary non-negative continuous function Φ on the ball $B(x_0, 1) \subset \mathbb{R}^d$ there exist a C^∞ function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that the kth convolution power $\varphi^{\star k}$ of φ exists a.e. on $\mathbb{R}^d$, but*

$$|\varphi^{\star k}(x)| > \Phi(x) \qquad \textit{for all} \quad x \in B(x_0, 1). \tag{2.6}$$

Remark 2.8. *We may take in Theorem 2.7 such a* Φ *that* $|\Phi(x)| \to \infty$ *as* $x \to x_0$ *as quickly as we wish. In particular, it follows from Theorem 2.7 that there is a smooth function* φ *on* $\mathbb{R}^d$ *such that the* k*th convolution power* $\varphi^{\star k}$ *of* φ *exists a.e. on* $\mathbb{R}^d$ *but* $\varphi^{\star k}$ *is not a locally integrable function and moreover, it cannot be identified with any distribution or ultradistribution.*

There are various sufficient conditions for existence of convolutions of functions and distributions; we discuss here only these expressed in terms of their supports (cf. [6], p. 389, and [1], p. 126).

Theorem 2.9. *Let* $A_1, \dots A_k \subseteq \mathbb{R}^d$ *be arbitrary sets for* $k \geq 2$. *The following conditions are equivalent:*

(C) *the set* $(A_1 \times \ldots \times A_k) \cap K^{\Delta}$ *is bounded in* $\mathbb{R}^{kd}$ *for every* K *bounded in* $\mathbb{R}^d$, *where*

$$K^{\Delta} := \{(x_1, \ldots, x_k) \in \mathbb{R}^{kd} :\ x_1 + \ldots + x_k \in K\};$$

(c) *the following implication holds:*

$$\lim_{n\to\infty} \sum_{i=1}^{k} |x_{i,n}| = \infty \quad \Rightarrow \quad \lim_{n\to\infty} |\sum_{i=1}^{k} x_{i,n}| = \infty,$$

whenever $x_{i,n} \in A_i$ *for* $i \in \{1, \ldots, k\}$ *and* $n \in \mathbb{N}$.

Definition 2.10. Sets $A_1, \dots A_k \subseteq \mathbb{R}^d$ are called *compatible*, if any of the two equivalent conditions (C), (c) in Theorem 2.9 is satisfied.

The following two theorems show that compatibility of supports of given functions (distributions) guarantees the existence of their convolution in L^1_{loc} (in $\mathcal{D}'$) (cf. [6], pp. 390–391, and [1], pp. 124–128):

Theorem 2.11. *If functions* $F_1, \ldots, F_k \in L^1_{\text{loc}}(\mathbb{R}^d)$ *have supports contained in compatible sets* $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, *respectively, then the convolution* $F_1 \star \ldots \star F_k$ *exists in* L^1_{loc} *and* $\operatorname{supp}(F_1 \star \ldots \star F_k) \subseteq A_1 + \ldots + A_k$.

Theorem 2.12. *If distributions* $S_1, \ldots, S_k \in \mathcal{D}'(\mathbb{R}^d)$ *have supports contained in compatible sets* $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, *respectively, then the convolution* $S_1 * \ldots * S_k$ *exists in* $\mathcal{D}'$ *and* $\operatorname{supp}(S_1 * \ldots * S_k) \subseteq A_1 + \ldots + A_k$.

On the other hand, it follows from Theorem 2.13 below that the counterpart of Theorem 2.12 for the convolution in $\mathcal{S}'$ is not true under the condition of compatibility of supports of the considered tempered distributions (cf. [7]):

Theorem 2.13. *For each nonnegative continuous function* Φ *on* $\mathbb{R}^d$ (*of an arbitrary increase*) *there is a* C^∞ *function* φ *on* $\mathbb{R}^d$, *bounded together with all derivatives* (*moreover* $\lim_{|x|\to\infty} \varphi(x) = 0$) *such that the* k*th convolution power* φ^{*k} *of* φ *exists everywhere in* $\mathbb{R}^d$ *and*

$$|(\varphi^{*k})(x)| > \Phi(x) \qquad \text{for all } \ x \in \mathbb{R}^d. \tag{2.7}$$

Moreover, the sets $A_1, \ldots, A_k$, *where* $A_i := \operatorname{supp} \varphi$ *for* $i = 1, \ldots, k$, *are compatible.*

Theorem 2.7 claims that the convolution may arbitrarily increase near points in $\mathbb{R}^d$ while Theorem 2.13 says about such an increase at infinity: the existence of the convolution in L^1_{loc} does not imply any restriction of its growth at infinity.

Remark 2.14. *Theorem 2.13 implies, in particular, that there exist tempered distributions $S_1, \ldots, S_k \in \mathcal{S}'(\mathbb{R}^d)$, represented by smooth functions $\varphi_1, \ldots, \varphi_k$, bounded together with all its derivatives, such that the convolution $S_1 * \ldots * S_k$ exists in $\mathcal{D}'$ but not in $\mathcal{S}'$:*

$$\varphi_1 \star \ldots \star \varphi_k \sim S_1 * \ldots * S_k \in \mathcal{D}'(\mathbb{R}^d) \setminus \mathcal{S}'(\mathbb{R}^d).$$

But Theorem 2.13 has a more general value: remarks analogous to the above concern other spaces of generalized functions, e.g. the subspace $\mathcal{P}$ of slowly increasing functions of the space L^1_{loc}, the Gelfand-Shilov subspaces $\mathcal{K}'(M_p)$ of $\mathcal{D}'$, the space $\mathcal{S}'^* \subset \mathcal{D}'^*$ of tempered ultradistributions of Beurling and Roumieu type etc.

Theorem 2.13 and Remark 2.14 indicate that the condition of compatibility requires modifications in other spaces of generalized functions than $\mathcal{D}'$ (cf. [7, 9]).

Definition 2.15. We call sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$ *polynomially compatible*, if there exists a positive polynomial P on $[0, \infty)$ such that the following implication holds:

$$(P) \qquad x_1 \in A_1, \ldots, x_k \in A_k \Rightarrow |x_1| + \ldots + |x_k| \le P\,(|x_1 + \ldots + x_k|).$$

Theorem 2.16. *Let $F_1, \ldots, F_k \in \mathcal{P}(\mathbb{R}^d)$. If the supports of $F_1, \ldots, F_k$ are contained in polynomially compatible sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, respectively, then the convolution $F_1 \star \ldots \star F_k$ exists in L^1_{loc} and $F_1 \star \ldots \star F_k \in \mathcal{P}(\mathbb{R}^d)$. Moreover,* $\operatorname{supp}(F_1 \star \ldots \star F_k) \subseteq A_1 + \ldots + A_k$.

Theorem 2.17. *Let $S_1, \ldots, S_k \in \mathcal{S}'(\mathbb{R}^d)$. If the supports of $S_1, \ldots, S_k$ are contained in polynomially compatible sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, respectively, then the convolution $S_1 * \ldots * S_k$ exists in $\mathcal{S}'$ and* $\operatorname{supp}(S_1 * \ldots * S_k) \subseteq A_1 + \ldots + A_k$.

The conditions of compatibility and polynomial compatibility for existence of the convolutions in $\mathcal{D}'$ and in $\mathcal{S}'$, respectively, are sharp in terms of supports, since the following inverse results to Theorems 2.12 and 2.17, respectively, hold (cf. [9]):

Theorem 2.18. *Let $A_1, \ldots, A_k \subseteq \mathbb{R}^d$. Assume that for arbitrary distributions $S_1, \ldots, S_k \in \mathcal{D}'(\mathbb{R}^d)$ such that* $\operatorname{supp} S_1 \subseteq A_1, \ldots, \operatorname{supp} S_k \subseteq A_k$, *respectively, the convolution $S_1 * \ldots * S_k$ exists in $\mathcal{D}'$. Then the sets $A_1, \ldots, A_k$ are compatible.*

Theorem 2.19. *Let $A_1, \ldots, A_k \subseteq \mathbb{R}^d$. Assume that for arbitrary tempered distributions $S_1, \ldots, S_k \in \mathcal{S}'(\mathbb{R}^d)$ such that* $\operatorname{supp} S_1 \subseteq A_1, \ldots, \operatorname{supp} S_k \subseteq A_k$, *respectively, the convolution $S_1 * \ldots * S_k$ exists in $\mathcal{S}'$. Then the sets $A_1, \ldots, A_k$ are polynomially compatible.*

3. Beurling ultradistributions

Let us briefly recall definitions and present some theorems concerning ultradistributions of Beurling type.

Fix a sequence $(M_p) = (M_p)_{p \in \mathbb{N}_0}$ of positive numbers such that

$(M.1) \quad M_p^2 \le M_{p-1} M_{p+1}, \quad p \in \mathbb{N};$

$(M.2) \quad M_p \le a\, b^p\, M_q M_{p-q}, \quad p,\, q \in \mathbb{N},\ 0 \le q \le p;$

$(M.3) \quad \sum_{p=q+1}^{\infty} M_{p-1} M_p^{-1} \le a\, q\, M_q M_p^{-1}, \quad q \in \mathbb{N},$

where $a, b > 0$ are constants.

For example, the Gevrey sequences (M_p) of the form: 1° $M_p = (p!)^s$; 2° $M_p = p^{ps}$; 3° $M_p = \Gamma(1 + ps)$ for $s > 1$ and $p \in \mathbb{N}$, satisfy conditions $(M.1)$-$(M.3)$.

It is convenient to extend the sequence $(M_p)_{p\in\mathbb{N}_0}$ to its multidimensional counterpart $(M_k)_{k\in\mathbb{N}_0^d}$ as follows:

$$M_k := M_{\kappa_1+\ldots+\kappa_d}, \qquad k=(\kappa_1,\ldots,\kappa_d)\in\mathbb{N}_0^d.$$

By the *associated function* corresponding to the sequence (M_p) we mean the function $M\colon [0,\infty)\to[0,\infty)$ such that $M(0)=0$ and

$$M(t) := \sup_{p\in\mathbb{N}_0} \log_+(t^p/M_p), \quad \log_+ := \max\{\log t, 0\}, \qquad t>0.$$

Let the symbol $K\subset\subset V$ for an open set $V\subseteq\mathbb{R}^d$ mean that the set K is compact and $K\subset V$. Denote first by $\mathcal{D}_{(M_p),K,h}(\mathbb{R}^d)$, for given $h>0$ and $K\subset\subset\mathbb{R}^d$, the space of all C^∞ functions φ on $\mathbb{R}^d$ with supports contained in K such that

$$\|\varphi\|_{K,h} := \sup_{k\in\mathbb{N}_0^d}\sup_{x\in K} \mid \varphi^{(k)}(x)\mid (h^k M_k)^{-1}<\infty, \tag{3.1}$$

endowed with the topology of the norm $\|\cdot\|_{K,h}$ defined in (3.1).

Now we may define the space $\mathcal{D}^{(M_p)}(\mathbb{R}^d)$ of test functions for Beurling ultradistributions as follows:

$$\mathcal{D}^{(M_p)}(\mathbb{R}^d) := \underset{K\subset\subset\mathbb{R}^d}{\text{ind lim}}\ \underset{h\to 0}{\text{proj lim}}\ \mathcal{D}_{(M_p),K,h}(\mathbb{R}^d).$$

The known theorem of Denjoy-Carleman-Mandelbrojt guarantees that the space $\mathcal{D}^{(M_p)}$ contains sufficiently many functions.

The space $\mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ of Beurling ultradistributions is defined as the strong dual of $\mathcal{D}^{(M_p)}(\mathbb{R}^d)$.

Beurling ultradistributions can be characterized as follows (see [16]):

Theorem 3.1. *$S\in\mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ if and only if for each open relatively compact set V in $\mathbb{R}^d$ there exist measures $\mu_\alpha\in\mathcal{C}'(\overline{V})$ for $\alpha\in\mathbb{N}_0^d$ such that for every $K\subset\subset V$ there are constants L_K and $c_K>0$ satisfying the inequalities:*

$$S|_V=\sum_{\alpha\in\mathbb{N}_0^d}\mu_\alpha^{(\alpha)} \quad \textit{and} \quad \|\mu_\alpha\|_{\mathcal{C}'(\overline{V})}\le c_K L_K^\alpha/M_\alpha \quad \textit{for} \quad \alpha\in\mathbb{N}_0^d. \tag{3.2}$$

For a fixed $m>0$, let us consider the space $\mathcal{S}_2^{(M_p),m}(\mathbb{R}^d)$ of all C^∞ functions ψ on $\mathbb{R}^d$ such that

$$\sigma_{m,2}(\psi) := \left(\sum_{\alpha,\beta\in\mathbb{N}_0^d}\left(\frac{m^{\alpha+\beta}}{M_\alpha M_\beta}\,\|\langle\cdot\rangle^\beta\psi^{(\alpha)}\|_2\right)^2\right)^{1/2}<\infty, \tag{3.3}$$

where $\langle\cdot\rangle$ is the function defined by $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x\in\mathbb{R}^d$ and $\|\cdot\|_2$ is the L^2-norm, endowed with the topology of the norm $\sigma_{m,2}$ defined in (3.3).

Now we define the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ of test functions for Beurling tempered ultradistributions as follows:

$$\mathcal{S}^{(M_p)}(\mathbb{R}^d) := \underset{m\to\infty}{\text{proj lim}}\ \mathcal{S}_2^{(M_p),m}(\mathbb{R}^d).$$

Beurling tempered ultradistributions are defined as elements of the strong dual $\mathcal{S}'^{(M_p)}(\mathbb{R}^d)$ of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and can be characterized as follows (see [14]):

Theorem 3.2. *$S \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d)$ if and only if there are functions F_α in $L^2(\mathbb{R}^d)$ for $\alpha \in \mathbb{N}_0^d$ and positive constants λ, L and c such that*

$$S|_U = \sum_{\alpha \in \mathbb{N}_0^d} (E_\lambda F_\alpha)^{(\alpha)} \quad \text{and} \quad \|F_\alpha\|_2 \leq cL^\alpha / M_\alpha \quad \text{for} \quad \alpha \in \mathbb{N}_0^d,$$

where $E_\lambda(x) = e^{M(\lambda|x|)}$ for $x \in \mathbb{R}^d$.

4. Multi-convolutions of Beurling ultradistributions

There are several equivalent general definitions of the convolution of two Beurling ultradistributions in $\mathcal{D}'^{(M_p)}$ and of the convolution of two Beurling tempered ultradistributions in $\mathcal{S}'^{(M_p)}$, corresponding to the mentioned general definitions of the convolution in $\mathcal{D}'$ and in $\mathcal{S}'$, respectively.

We consider here only the counterparts of the sequential definitions of Vladimirov type based on the notion of (M_p)-approximate unit.

Definition 4.1. A sequence (η_n) of elements of $\mathcal{D}^{(M_p)}(\mathbb{R}^d)$ is called (M_p)-*approximate unit* on $\mathbb{R}^d$, if for every $K \subset\subset \mathbb{R}^d$ there is an $n_0 \in \mathbb{N}$ such that $\eta_n(x) = 1$ for $x \in K$, $n \geq n_0$ and, in addition, if there is a constant $h > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{k \in \mathbb{N}_0^d} \left(\frac{h^k}{M_k} \parallel \eta_n^{(k)} \parallel_\infty \right) < \infty.$$

Let $\mathbb{U}^{(M_p)}(\mathbb{R}^d)$ denote the set of all (M_p)-approximate units on $\mathbb{R}^d$.

We now define multi-convolutions of Beurling ultradistributions in $\mathcal{D}'^{(M_p)}$ and Beurling tempered ultradistributions in $\mathcal{S}'^{(M_p)}$ of Vladimirov type, i.e. the convolution of Vladimirov type of k Beurling ultradistributions in $\mathcal{D}'^{(M_p)}$ as well as the convolution of Vladimirov type of k Beurling tempered ultradistributions in $\mathcal{S}'^{(M_p)}$.

Definition 4.2. For given Beurling ultradistributions $S_1, \ldots, S_k$ on $\mathbb{R}^d$ we define the *convolution* $S_1 * \ldots * S_k$ *in* $\mathcal{D}'^{(M_p)}$ by

$$\langle S_1 * \ldots * S_k,\ \varphi \rangle := \lim_{n \to \infty} \langle S_1 \otimes \ldots \otimes S_k,\ \eta_n \varphi^\triangle \rangle,$$

whenever the above limit exists for all $(\eta_n) \in \mathbb{U}^*(\mathbb{R}^{kd})$ and for all $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^d)$. We say then that the *convolution* $S_1 * \ldots * S_k$ *exists in* $\mathcal{D}'^{(M_p)}$ (cf. [10]).

Definition 4.3. For given Beurling tempered ultradistributions $S_1, \ldots, S_k$ on $\mathbb{R}^d$ we define the *convolution* $S_1 * \ldots * S_k$ *in* $\mathcal{S}'^{(M_p)}$ by

$$\langle S_1 * \ldots * S_k,\ \psi \rangle := \lim_{n \to \infty} \langle S_1 \otimes \ldots \otimes S_k,\ \eta_n \psi^\triangle \rangle,$$

whenever the above limit exists for all $(\eta_n) \in \mathbb{U}^{(M_p)}(\mathbb{R}^{kd})$ and for all $\psi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$. We say then that the *convolution* $S_1 * \ldots * S_k$ *exists in* $\mathcal{S}'^{(M_p)}$ (cf. [14]).

Similarly as in case of the convolution in L^1_{loc} and in $\mathcal{D}'$, for the existence of the convolution of k ultradistributions in $\mathcal{D}'^{(M_p)}$ compatibility of their supports is a sufficient condition (cf. [12]):

Theorem 4.4. *If $S_1, \ldots, S_k \subseteq \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ and the supports of the ultradistributions $S_1, \ldots, S_k$ are contained in compatible sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, respectively, then the convolution $S_1 * \ldots * S_k$ exists in $\mathcal{D}'^{(M_p)}$ and* $\operatorname{supp}(S_1 * \ldots * S_k) \subseteq A_1 + \ldots + A_k$.

Also the following inverse result analogous to Theorem 2.18 is true (cf. [11]):

Theorem 4.5. *Let $A_1, \dots, A_k \subseteq \mathbb{R}^d$. Suppose that the convolution $S_1 * \dots * S_k$ exists in $\mathcal{D}'^{(M_p)}$ for arbitrary ultradistributions $S_1, \dots, S_k \in \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ such that* $\operatorname{supp} S_1 \subseteq A_1, \dots, \operatorname{supp} S_k \subseteq A_k$*, respectively. Then $A_1, \dots, A_k$ are compatible.*

In case of the convolution in $\mathcal{S}'^{(M_p)}$ the condition of compatibility requires a modification (cf. [11]).

Definition 4.6. Let $M: [0,\infty) \to [0,\infty)$ be a nondecreasing function. Sets $A_1, \dots A_k \subseteq \mathbb{R}^d$ are called *M-compatible*, if there is a constant $\theta > 0$ such that

$$(M) \qquad x_i \in A_i \;\; (i = 1, \dots, k) \;\Rightarrow\; \sum_{i=1}^{k} M(|x_i|) \;\le\; M\big(\theta \,|\sum_{i=1}^{k} x_i|\big) \,+\, \theta\,.$$

Let (M_p) be a sequence of positive numbers satisfying conditions $(M.1)$-$(M.3)$ and let M be the associated function for the sequence (M_p).

Theorem 4.7. *Let $S_1, \dots, S_k \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d)$. If the supports of $S_1, \dots, S_k$ are contained in M-compatible sets $A_1, \dots, A_k \subseteq \mathbb{R}^d$, then the convolution $S_1 * \dots * S_k$ exists in $\mathcal{S}'^{(M_p)}$. Moreover,* $\operatorname{supp}(F_1 \star \dots \star F_k) \subseteq A_1 + \dots + A_k$.

Also the following inverse result being a counterpart of Theorem 2.19 is true (cf. [11]):

Theorem 4.8. *Let $A_1, \dots, A_k \subseteq \mathbb{R}^d$. Suppose that the convolution $S_1 * \dots * S_k$ exists in $\mathcal{S}'^{(M_p)}$ for arbitrary tempered ultradistributions $S_1, \dots, S_k \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d)$ such that* $\operatorname{supp} S_1 \subseteq A_1, \dots, \operatorname{supp} S_k \subseteq A_k$*, respectively. Then the sets $A_1, \dots, A_k$ are M-compatible.*

Let us notice that compatibility (M-compatibility) of k sets can be reduced to compatibility (M-compatibility) of a smaller than k number of sets, due to the following characterization (cf. [15]):

Theorem 4.9. *Assume that sets $A_1, \dots A_k \subseteq \mathbb{R}^d$ are compatible (M-compatible). Then for every integer $m, 1 \le m \le k$, and for every partition $\{K_j\}_{1 \le j \le m}$ of the set $K := \{1, \dots, k\}$ the following two assertions hold:*

(a) for every $j = 1, \dots, m$, the sets A_i with indices $i \in K_j$ are compatible (M-compatible);

(b) the sets $B_1, \dots, B_m$, where

$$B_j := \sum_{i \in K_j} A_i := \{\sum_{i \in K_j} x_i : \; x_i \in A_i\} \qquad \text{for } \; j = 1, \dots, m,$$

are compatible (M-compatible).

Conversely, if for some $m, 1 \le m \le k$, there is a partition $\{K_j\}_{1 \le j \le m}$ of the set K such that conditions (a) and (b) are satisfied, then the sets are compatible (M-compatible).

Let us conclude now two particular assertions resulting from Theorem 4.9.

Corollary 4.10. *Let $A_1, \dots A_k \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

(a) the sets $A_1, \dots A_k$ are compatible (M-compatible);

(b) for every index $j = 1, \dots, k-1$ the two sets $A_1 + \dots + A_j$ and A_{j+1} are compatible (M-compatible).

Corollary 4.11. *Let $A, B, C \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

(a) *the sets A, B, C are compatible (M-compatible);*

(b) *the two sets $A + B$, C are compatible (M-compatible) and the two sets A, B are compatible (M-compatible).*

Let $K := \{1, \ldots, k\}$ be a finite set of indices and let $K_0 := \{i_1, \ldots, i_j\}$ its subset, where $1 \leq i_1 < \ldots < i_j \leq k$ $(1 \leq j \leq k)$. Let $S_i \in \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ $(S_i \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d))$ for $i \in K$. Then we use the following convenient notation for the convolution of $S_1, \ldots, S_k$ and the convolution of $S_{i_1} * \ldots * S_{i_j}$:

$$*_{i \in K} \; S_i := S_1 * \ldots * S_k; \qquad *_{i \in K_0} \; S_i := S_{i_1} * \ldots * S_{i_j}.$$

Theorem 4.12. *Let $S_1, \ldots S_k \in \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ $(S_1, \ldots S_k \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d))$. Assume that supports of $S_1, \ldots S_k$ are contained in compatible (M-compatible) sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, respectively. Then for every partition $\{K_j\}_{1 \leq j \leq m}$ $(1 \leq m \leq k)$ of the set $K := \{1, \ldots, k\}$ all the convolutions in the following equality exist in $\mathcal{D}'^{(M_p)}$ (exist in $\mathcal{S}'^{(M_p)}$):*

$$S_1 * \ldots * S_k = *_{1 \leq j \leq m} \; (*_{i \in K_j} \; S_i)$$

and the equality holds true.

Corollary 4.13. *Let $S_1, \ldots S_k \in \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ $(S_1, \ldots S_k \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d))$. Assume that supports of $S_1, \ldots S_k$ are contained in compatible (M-compatible) sets $A_1, \ldots, A_k \subseteq \mathbb{R}^d$, respectively. Then all the convolutions in the following equality exist in $\mathcal{D}'^{(M_p)}$ (exist in $\mathcal{S}'^{(M_p)}$):*

$$S_1 * \ldots * S_k = (\ldots((S_1 * S_2) * S_3) * \ldots * S_{k-1}) * S_k$$

and the equality holds true.

Corollary 4.14. *Let $R, S, T \in \mathcal{D}'^{(M_p)}(\mathbb{R}^d)$ $(R, S, T \in \mathcal{S}'^{(M_p)}(\mathbb{R}^d))$. Assume that supports of R, S, T are contained in compatible (M-compatible) sets $A, B, C \subseteq \mathbb{R}^d$, respectively. Then all the convolutions in the following equality exist in $\mathcal{D}'^{(M_p)}$ (exist in $\mathcal{S}'^{(M_p)}$):*

$$R * S * T = (R * S) * T = R * (S * T)$$

and the equality holds true.

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