ON HARMONIC ANALYSIS ASSOCIATED WITH THE HYPER-BESSEL OPERATOR ON THE COMPLEX PLANE

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DEDICATED TO PROFESSOR IVAN DIMOVSKI'S CONTRIBUTIONS

ABSTRACT. We investigate the harmonic analysis associated with the hyper-Bessel operator on $\mathbb{C}$, and we prove the chaotic character of the related convolution operators.

1. INTRODUCTION

In 1966, I. Dimovski [15] introduced the hyper-Bessel operators and started developing a Mikusinski-type approach to operational calculus related to them. Later on, with V. Kiryakova he continued a detailed study of these operators. Their collaboration lead to the theory of Generalized Fractional Calculus of Kiryakova (see [22, 23] and references therein).

In the present paper, we consider the hyper-Bessel operator

$$B_r := \frac{1}{z^{r-1}} \prod_{i=1}^{r-1} \left(z\frac{d}{dz} + (r\gamma_i + 1) \right) \frac{d}{dz}, \tag{1}$$

where r is an integer such that $r \geq 2$ and $\gamma = (\gamma_1, \ldots, \gamma_{r-1})$ is a vector index having $r-1$ real components $\gamma_k \geq -1 + \frac{k}{r}$, $k = 1, \ldots, r-1$. We establish some elements of harmonic analysis associated with the operator B_r in the complex plane. Namely, we introduce the generalized Fourier transform and prove a generalized Paley-Wiener theorem. As an application, we prove the chaotic character of the related convolution operators on the space of r-even and entire functions (see Definition 2.1 below).

It should be noted that Klyuchantsev [24] proved that Bessel functions of the vector index γ, called also Delerue functions, or hyper-Bessel functions, are eigenfunctions of the operator B_r. In [17] the authors studied the polynomial expansion for the solutions of the heat equation associated with the operator B_r, and recently, in [12] the operator B_r is connected with the theory of special functions associated with complex reflection groups. However, harmonic analysis associated with the operator B_r is not yet developed.

This paper is organized as follows: In Section 2 we briefly overview some notations and backgrounds including the Bessel functions of vector index and the space

$\mathcal{H}_r(\mathbb{C})$ of all r-even entire functions on $\mathbb{C}$. In Section 3 we introduce the generalized Fourier transform and then prove a generalized Paley-Wiener theorem. In Section 4 we study the generalized translation and generalized convolution on the space $\mathcal{H}_r(\mathbb{C})$ and its dual space, we also deals with the surjectivity of convolution operators.

We recall that a continuous linear operator T from a Fréchet space X into itself is said chaotic in the sense of Devaney, if

(a) T is hypercyclic. That is, there exists $x \in X$ (that is called hypercyclic vector of T) such that its orbit $\{x, Tx, T^2x, \dots\}$ is a dense subset of X,

(b) the set $Per(T) = \{x \in X; \ T^n x = x, \ \text{for some integer } n\}$ of periodic points of T is dense in X.

In 1991, G. Godefroy and J. Shapiro [18] generalized the classical works of Birkhoff [9] and MacLane [25] showing respectively that the operators of translation and differentiation on $\mathcal{H}(\mathbb{C})$, the space of all entire functions on $\mathbb{C}$, are hypercyclic, to show that every convolution operator on the space $\mathcal{H}(\mathbb{C}^N)$ that is not a scalar multiple of the identity is chaotic.

Since then, the chaotic character of convolution operators has been the object of extensive study. See for instance [2, 5, 7, 8, 11, 21] for some recent works and the monographs [4] and [19] for a general account of the theory.

In Section 5 we establish several characterizations of the continuous linear mappings from $\mathcal{H}_r(\mathbb{C})$ into itself that commute with the generalized translations, as a consequence, we prove the analogue of the results of Godefroy and Shapiro for the operator B_r.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the nonnegative integers $\{0, 1, \cdots\}$. We fix $r \in \mathbb{N}$ such that $r \geq 2$ and a vector index $\gamma = (\gamma_1, \dots, \gamma_{r-1})$ having $(r-1)$ real components satisfying

$$\gamma_k \geq -1 + \frac{k}{r}, \quad \text{for all} \quad k \in \{1, ..., r-1\}.$$

The Bessel operator of r-order (hyper-Bessel differential operator) given by (1) can also be written in the form

$$B_r = \frac{d^r}{dz^r} + \frac{a_1}{z}\frac{d^{r-1}}{dz^{r-1}} + \dots + \frac{a_{r-1}}{z^{r-1}}\frac{d}{dz}, \tag{2}$$

where

$$a_{r-k} = \frac{1}{(k-1)!} \sum_{j=1}^{k} (-1)^{k-j} \binom{j-1}{k-1} \prod_{i=1}^{r-1} (r\gamma_i + j), \tag{3}$$

for every $k \in \{1, \dots, r-1\}$. Simple examples of this operator are:

- The operator $\dfrac{d^r}{dz^r}$ when $\gamma_k = -1 + \dfrac{k}{r}, \ k \in \{1, \dots, r-1\}$;
- The classical Bessel operator of the second order

$$B_2 := \frac{d^2}{dz^2} + \frac{(2\gamma+1)}{z}\frac{d}{dz}, \quad \gamma \geq -\frac{1}{2}; \tag{4}$$

- The operator B_3 studied in [13] and [16]:

$$B_3 := \frac{d^3}{dz^3} + \frac{(3\,\nu)}{z}\frac{d^2}{dz^2} - \frac{(3\,\nu)}{z^2}\frac{d}{dz}, \quad \gamma_1 = -2/3, \ \gamma_2 = \nu - 1/3. \tag{5}$$

Definition 2.1. Let $w = e^{2i\pi/r}$ be the root of the unit. A function $u \colon \mathbb{C} \to \mathbb{C}$ is called r-even, if $u(w^k z) = u(z)$ for all $k \in \{1, \ldots, r-1\}$.

We denote by $\mathcal{H}_r(\mathbb{C})$ the space of all r-even entire functions on $\mathbb{C}$ endowed with the topology of the uniform convergence on compact subsets of $\mathbb{C}$. Recall that this topology is generated by the semi-norms

$$\|u\|_R := \sup_{|z| \le R} |u(z)|, \quad u \in \mathcal{H}_r(\mathbb{C}), \ R > 0. \tag{6}$$

Thus, $\mathcal{H}_r(\mathbb{C})$ is a Fréchet space.

If $u \in \mathcal{H}_r(\mathbb{C})$, then using integration by parts and taking into account that $u'(0) = \cdots = u^{(r-1)}(0) = 0$, we conclude from (2) that

$$B_r u(z) = u^{(r)}(z) + \sum_{k=1}^{r-1} \frac{a_k}{(k-1)!} \int_0^1 (1-t)^{k-1} u^{(r)}(tz)dt, \tag{7}$$

for all $z \ne 0$. Hence, by the analyticity theorem, we can extend the function $B_r u$ to the whole complex plane, so that $B_r u \in \mathcal{H}_r(\mathbb{C})$.

The normalized Bessel function of the vector index γ is the function $j_\gamma \colon \mathbb{C} \to \mathbb{C}$, defined by

$$j_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^{rn}}{\alpha_{rn}(\gamma)}, \quad z \in \mathbb{C}, \tag{8}$$

where

$$\alpha_{rn}(\gamma) = (r)^{rn} \, n! \prod_{i=1}^{r-1} \frac{\Gamma(\gamma_i + n + 1)}{\Gamma(\gamma_i + 1)}, \quad n \in \mathbb{N}. \tag{9}$$

See [17, 24] and [22, Chaps. 3-4] for more details.

For $\lambda \in \mathbb{C}$, define $j_\gamma(\lambda\cdot) \colon \mathbb{C} \to \mathbb{C}$ to be the function given by

$$[j_\gamma(\lambda\cdot)](z) = j_\gamma(\lambda z), \quad \text{for all } z \in \mathbb{C}. \tag{10}$$

Proposition 2.2. (see [17]) *For every $\lambda \in \mathbb{C}$, the function $j_\gamma(\lambda\cdot)$ is the unique solution in $\mathcal{H}_r(\mathbb{C})$ of the system*

$$\begin{cases} B_r u(z) & = -\lambda^r u(z) \\ u(0) & = 1 \\ D_z^k u(0) & = 0, \ \text{for all} \ k \in \{1, \ldots, r-1\} \end{cases}. \tag{11}$$

By induction on the integer n, we can see that $\alpha_{rn}(\gamma) \ge (rn)!$ for all $n \in \mathbb{N}$. Hence, by (8), the function j_γ satisfies the inequality

$$\left| j_\gamma(z) \right| \le G_\gamma(|z|) \le e^{|z|}, \quad \text{for all } z \in \mathbb{C}, \tag{12}$$

where

$$G_\gamma(z) = j_\gamma(e^{i\pi/r} z) = \sum_{n=0}^{+\infty} \frac{z^{rn}}{\alpha_{rn}(\gamma)}. \tag{13}$$

3. The generalized Fourier transform

Let $\mathcal{H}'_r(\mathbb{C})$ denote the strong dual space of the space $\mathcal{H}_r(\mathbb{C})$ and let $\mathrm{Exp}_r(\mathbb{C})$ denote the space of all r-even entire functions of exponential type. That is the space of all $u \in \mathcal{H}_r(\mathbb{C})$ for which there exists $a > 0$ such that

$$P_a(u) = \sup_{z \in \mathbb{C}} |u(z)| \, e^{-a|z|} < \infty. \tag{14}$$

For $a > 0$, let $\mathrm{Exp}_{r,a}(\mathbb{C})$ denote the Banach space of all $u \in \mathrm{Exp}_r(\mathbb{C})$ satisfying (14) provided with the norm P_a. Thus,

$$\mathrm{Exp}_r(\mathbb{C}) = \bigcup_{a>0} \mathrm{Exp}_{r,a}(\mathbb{C}).$$

We provide $\mathrm{Exp}_r(\mathbb{C})$ with the natural locally convex inductive limit topology.

Lemma 3.1. *Let* $u(z) = \sum_{n=0}^{+\infty} \dfrac{b_n}{\alpha_{rn}(\gamma)} z^{rn} \in \mathcal{H}_r(\mathbb{C})$. *Then* $u \in \mathrm{Exp}_r(\mathbb{C})$ *if and only if, there exist* $a, C > 0$ *such that* $|b_n| \leq Ca^{rn}$, *for all* $n \in \mathbb{N}$.

Proof. From [27, Page 43] we know that $u \in \mathrm{Exp}_r(\mathbb{C})$ if and only if

$$\limsup_{n \to \infty} \left| \frac{(rn)! b_n}{\alpha_{rn}(\gamma)} \right|^{\frac{1}{rn}} < \infty.$$

Using the Stirling formula, we can check that

$$\limsup_{n \to \infty} \left| \frac{(nr)!}{\alpha_{rn}(\gamma)} \right|^{\frac{1}{n}} = 1. \tag{15}$$

This implies that $u \in \mathrm{Exp}_r(\mathbb{C})$ if and only if $\limsup_{n \to \infty} |b_n|^{\frac{1}{rn}} < \infty$, and the conclusion of the lemma follows. $\qquad\square$

Lemma 3.2. *Let* $v(z) = \sum_{n=0}^{+\infty} \dfrac{b_n}{\alpha_{rn}(\gamma)} z^{rn} \in \mathrm{Exp}_r(\mathbb{C})$. *For* $u(z) = \sum_{n=0}^{+\infty} a_n z^{rn} \in \mathcal{H}_r(\mathbb{C})$, *let*

$$T_v(u) = \sum_{n=0}^{+\infty} a_n b_n. \tag{16}$$

Then, the series in (16) *converges absolutely and defines* $T_v \colon u \to T_v(u)$ *as an element of* $\mathcal{H}'_r(\mathbb{C})$.

Proof. By Lemma 3.1 there exist $C, a > 0$ such that $|b_n| \leq Ca^{rn}$ for all $n \in \mathbb{N}$. On the other hand, by the Cauchy estimate, for $u(z) = \sum_{n=0}^{+\infty} a_n z^{rn} \in \mathcal{H}_r(\mathbb{C})$, we have $|a_n| \leq (2a)^{-rn} \|u\|_{2a}$, for all $n \in \mathbb{N}$. This implies that the series in (16) converges absolutely and $|T_v(u)| \leq 2C\|u\|_{2a}$ and hence, $T_v \in \mathcal{H}'_r(\mathbb{C})$. $\qquad\square$

Definition 3.3. For $T \in \mathcal{H}'_r(\mathbb{C})$, we define the generalized Fourier transform of T to be the function $\mathcal{F}_\gamma(T) \colon \mathbb{C} \to \mathbb{C}$ given by

$$\mathcal{F}_\gamma(T)(z) = \langle T(w), j_\gamma(wz) \rangle, \qquad z \in \mathbb{C}, \tag{17}$$

where j_γ is given by (8).

Theorem 3.4 (Palely-Wiener Theorem). *The transform* $\mathcal{F}_\gamma$ *is a topological isomorphism from* $\mathcal{H}'_r(\mathbb{C})$ *onto* $\mathrm{Exp}_r(\mathbb{C})$.

Proof. First, we prove that if $T \in \mathcal{H}'_r(\mathbb{C})$, then $\mathcal{F}_\gamma(T) \in \mathrm{Exp}_r(\mathbb{C})$. Let $T \in \mathcal{H}'_r(\mathbb{C})$. Since the series in (8) converges in $\mathcal{H}_r(\mathbb{C})$, it follows that

$$\mathcal{F}_\gamma(T)(z) = \langle T(w), j_\gamma(wz) \rangle = \sum_{n=0}^{+\infty} (-1)^n \frac{\langle T(w), w^{nr} \rangle}{\alpha_{rn}(\gamma)} z^{nr}, \tag{18}$$

for all $z \in \mathbb{C}$. Hence, $\mathcal{F}_\gamma(T) \in \mathcal{H}_r(\mathbb{C})$. On the other hand, the continuity of T infers that there exist $a, C > 0$ such that

$$|\langle T, u \rangle| \leq C\|u\|_a, \qquad \text{for all } u \in \mathcal{H}_r(\mathbb{C}),$$

which implies that $|\langle T(w), w^{nr} \rangle| \le Ca^{nr}$ for all $n \in \mathbb{N}$. It follows from (18) and Proposition 3.1 that $\mathcal{F}_\gamma(T) \in \mathrm{Exp}_r(\mathbb{C})$.

Next, we prove that $\mathcal{F}_\gamma$ is a continuous mapping from $\mathcal{H}'_r(\mathbb{C})$ into $\mathrm{Exp}_r(\mathbb{C})$. Since $\mathcal{H}'_r(\mathbb{C})$ is a bornological space, it is sufficient to show that $\mathcal{F}_\gamma(B)$ is a bounded set in $\mathrm{Exp}_r(\mathbb{C})$ whenever B is a bounded set of $\mathcal{H}'_r(\mathbb{C})$. Assume that B is a bounded subset of $\mathcal{H}'_r(\mathbb{C})$. Since $\mathcal{H}'_r(\mathbb{C})$ is barreled, it follows from the uniform boundedness principle that there exist $C, a > 0$ such that $|\langle T, u \rangle| \le C\|u\|_a$ for all $T \in B$ and all $u \in \mathcal{H}_r(\mathbb{C})$. Using (12), we conclude that if $T \in B$, then $|\mathcal{F}_\gamma(T)(z)| \le Ce^{a|z|}$ for all $z \in \mathbb{C}$, and hence $P_a(\mathcal{F}_\gamma(T)) \le C$. This implies that $\mathcal{F}_\gamma(B)$ is bounded in $\mathrm{Exp}_{r,a}(\mathbb{C})$ and hence bounded in $\mathrm{Exp}_r(\mathbb{C})$.

To see that the transform $\mathcal{F}_\gamma$ is one to one, assume that $T \in \mathcal{H}_r(\mathbb{C})$ is such that $\mathcal{F}_\gamma(T)(z) = 0$ for all $z \in \mathbb{C}$. Then, from (18) we conclude that $\langle T(w), w^{rn} \rangle = 0$ for all $n \in \mathbb{N}$. So, if $u(z) = \sum_{n=0}^{+\infty} b_n z^{rn} \in \mathcal{H}_r(\mathbb{C})$, then $\langle T, u \rangle = \sum_{n=0}^{+\infty} b_n \langle T, w^{rn} \rangle = 0$. Thus, $\mathcal{F}_\gamma$ is one to one.

For the surjectivity of $\mathcal{F}_\gamma$, let $v(z) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}(\gamma)} z^{rn} \in \mathrm{Exp}_r(\mathbb{C})$. Define $\tilde{v}(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n b_n}{\alpha_{rn}(\gamma)} z^{rn}$. Then $\mathcal{F}_\gamma(T_{\tilde{v}}) = v$, where $T_{\tilde{v}}$ is given by (16).

Finally, the continuity of $(\mathcal{F}_\gamma)^{-1}$ follows from the open mapping theorem, [26, Th. 24.30]. $\qquad \square$

Corollary 3.5. *Suppose Λ is any subset of $\mathbb{C}$ with a limit point in $\mathbb{C}$, and let $J(\Lambda)$ be the linear span of the functions $j_\gamma(\lambda \cdot)$ with $\lambda \in \Lambda$. Then $J(\Lambda)$ is dense in $\mathcal{H}_r(\mathbb{C})$.*

Proof. Suppose that $T \in \mathcal{H}'_r(\mathbb{C})$ is such that $\langle T(w), j_\gamma(\lambda w) \rangle = 0$, for all $\lambda \in \Lambda$. Then $\mathcal{F}_\gamma(T)(\lambda) = 0$ for all $\lambda \in \Lambda$. Since, by Theorem 3.4, $\mathcal{F}_\gamma(T)$ is entire on $\mathbb{C}$ and Λ has a limit point in $\mathbb{C}$, it follows that $\mathcal{F}_\gamma(T)$ vanishes on $\mathbb{C}$. Hence, Theorem 3.4 also implies that T vanishes on $\mathcal{H}_r(\mathbb{C})$, so by the Hahn-Banach theorem, $J(\Lambda)$ is dense in $\mathcal{H}_r(\mathbb{C})$. $\qquad \square$

4. The generalized translation and generalized convolution

In this section we study the generalized translation and generalized convolution associated with the operator B_r.

4.1. The generalized translation. We begin with the following useful lemma.

Lemma 4.1. *If $u \in \mathcal{H}_r(\mathbb{C})$ and $R > 0$, then*

$$\|B_r^n u\|_R \le \frac{M^n (nr)!}{R^{nr}} \|u\|_{2R}, \tag{19}$$

for all $n \in \mathbb{N}$, where

$$M = 1 + \sum_{k=1}^{r-1} \frac{|a_k|}{k!}. \tag{20}$$

Proof. First, we prove by induction on n that

$$\|B_r^n u\|_R \le M^n \|u^{(nr)}\|_R, \quad \text{for all} \ \ n \in \mathbb{N}. \tag{21}$$

The result is trivially true for $n = 0$. Assume that (21) is satisfied for the integer n. Then,

$$\|(B_r)^{n+1} u\|_R = \|B_r^n (B_r u)\|_R \le M^n \|(B_r u)^{(nr)}\|_R. \tag{22}$$

Using (7) we conclude that

$$(B_r u)^{(nr)}(z) = u^{((n+1)r)}(z) + \sum_{k=1}^{r-1} \frac{a_k}{(k-1)!} \int_0^1 (1-t)^{k-1} t^{nr} u^{((n+1)r)}(tz) dt,$$

for all $z \in \mathbb{C}$. This implies that $\|(B_r u)^{(nr)}\|_R \leq M \|u^{(n+1)r}\|_R$. Hence,

$$\|(B_r)^{n+1} u\|_R \leq M^{n+1} \|u^{(n+1)r}\|_R,$$

and the proof of (21) is complete.

Now, according to the Cauchy integral formula, we have

$$\left\| u^{(k)} \right\|_R \leq \frac{k!}{R^k} \|u\|_{2R}, \qquad \text{for all } k \in \mathbb{N}. \tag{23}$$

This together with (21) yield (19). $\qquad\square$

Corollary 4.2. *The operator B_r is continuous from $\mathcal{H}_r(\mathbb{C})$ into itself.*

Proof. The result follows immediately from (19) by taking $n = 1$. $\qquad\square$

Following J. Delsarte [14], we define the generalized translation operator as follows

$$(T_z^\gamma u)(w) = \sum_{n=0}^{+\infty} \frac{w^{rn}}{\alpha_{rn}(\gamma)} B_r^n u(z), \quad w, z \in \mathbb{C}, \ u \in \mathcal{H}_r(\mathbb{C}). \tag{24}$$

Note that by Proposition 2.2, we have the product formula

$$j_\gamma(\lambda z) j_\gamma(\lambda w) = T_z^\gamma (j_\gamma(\lambda.))(w), \quad w, z \in \mathbb{C}. \tag{25}$$

Proposition 4.3. *The series in (24) converges on compact subsets of $\mathbb{C}$ and defines T_z^γ as a continuous linear operator from $\mathcal{H}_r(\mathbb{C})$ into itself, for evry $z \in \mathbb{C}$.*

Proof. Let $R, R' > 0$ and let $u \in \mathcal{H}_r(\mathbb{C})$. It follows from Lemma 4.1 that for all w, $z \in \mathbb{C}$ such that $|w| \leq R$ and $|z| \leq R'$ we have

$$|\frac{w^{rn}}{\alpha_{rn}(\gamma)} B_r^n u(z)| \leq \frac{R^{nr} M^n (nr)!}{(R')^{nr} \alpha_{rn}(\gamma)} \|u\|_{2R'}, \quad n \in \mathbb{N}. \tag{26}$$

Since

$$\limsup_{n \to \infty} \left| \frac{(nr)!}{\alpha_{rn}(\gamma)} \right|^{\frac{1}{n}} = 1, \tag{27}$$

for sufficiently large R', we have $(R/R')^r M < 1$, and then, the series in (24) converges uniformly on the closed polydisk

$$\{(w, z) \in \mathbb{C} \times \mathbb{C} \mid |w| \leq R, |z| \leq R'\}. \tag{28}$$

Thus, the series converges uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$. Now, if we fix $z \in \mathbb{C}$ and we chose R' sufficiently large so that $|z| \leq R'$ and $(R/R')^r M < 1$, from (26) we conclude that there exists $C > 0$ such that

$$\|T_z^\gamma u\|_R \leq C \|u\|_{2R'}, \qquad \text{for all } u \in \mathcal{H}_r(\mathbb{C}). \tag{29}$$

Since T_z^γ is linear, it is continuous. $\qquad\square$

Remark 4.4. We define a generalized addition formula associated with the operator B_r, for $z, w \in \mathbb{C}$ and $n \in \mathbb{C}$, as follows

$$(z \oplus_\gamma w)^{rn} := \sum_{k=0}^n \binom{\alpha_{rn}}{\alpha_{rk}} w^{rk} z^{r(n-k)}, \qquad (30)$$

where

$$\binom{\alpha_{rn}}{\alpha_{rk}} := \frac{\alpha_{rn}}{\alpha_{rk}\, \alpha_{r(n-k)}}$$

is the generalized binomial. It is easy to check that for all $n \in \mathbb{N}$ and all $z, w \in \mathbb{C}$ we have

$$(z \oplus_\gamma w)^{rn} = z^{rn}\, {}_rF_{r-1}\left[\begin{array}{c} -n, -(n+\gamma_i) \\ \gamma_i+1 \end{array} \bigg| \left(-\frac{w}{z}\right)^r\right], \qquad (31)$$

where ${}_rF_{r-1}$ is the hypergeometric function (see [1]). This addition in terms of hypergeometric functions is an analogue of the addition for Bessel functions presented by Bochner [10] and F. M. Cholewinski and J. A. Reneke [13] (see also [16]). For $u(t) = \sum_{n=0}^{+\infty} b_n t^{nr} \in \mathcal{H}_r(\mathbb{C})$, define

$$u(z \oplus_\gamma w) = \sum_{n=0}^{+\infty} b_n \,(z \oplus_\gamma w)^{rn}, \qquad z, w \in \mathbb{C}. \qquad (32)$$

Then, it is easy to check that $T_z^\gamma u(w) = u(z \oplus_\gamma w)$ for all $z, w \in \mathbb{C}$.

In the following proposition, we give some properties of the generalized translation operators T_z^γ for $z \in \mathbb{C}$.

Proposition 4.5. *For $u \in \mathcal{H}_r(\mathbb{C})$ and $z, w \in \mathbb{C}$ we have*

   (i) $T_0^\gamma u(z) = u(z)$.
   (ii) $T_z^\gamma u(w) = T_w^\gamma u(z)$.
   (iii) $B_r T_z^\gamma u = T_z^\gamma B_r u$.
   (iv) $T_z^\gamma \circ T_w^\gamma u = T_w^\gamma \circ T_z^\gamma u$.

Proof. These properties are satisfied when $u = j_\gamma(\lambda \cdot)$ where $\lambda \in \mathbb{C}$ and hence, by Corollary 3.5, are satisfied by all $u \in \mathcal{H}_r(\mathbb{C})$. $\qquad \square$

Proposition 4.6. *Let $u \in \mathcal{H}_r(\mathbb{C})$. Then, the mapping $F_u \colon z \mapsto T_z^\gamma u$ is continuous from $\mathbb{C}$ into $\mathcal{H}_r(\mathbb{C})$.*

Proof. By Proposition 4.5(ii), we can write

$$T_z^\gamma u(w) = T_w^\gamma u(z) = \sum_{n=0}^{+\infty} \frac{z^{rn}}{\alpha_{rn}(\gamma)} B_r^n u(w), \quad \text{for all } z, w \in \mathbb{C}, \qquad (33)$$

or

$$T_z^\gamma u = \sum_{n=0}^{+\infty} \frac{z^{rn}}{\alpha_{rn}(\gamma)} B_r^n u, \quad \text{for all } z \in \mathbb{C}. \qquad (34)$$

Thus, the mapping F_u possesses a power series expansion with coefficients in the Fréchet space $\mathcal{H}_r(\mathbb{C})$ and radius of convergence $R = +\infty$ (see [3, Apend. A]. So it is continuous on $\mathbb{C}$. $\qquad \square$

4.2. **The generalized convolution.**

Definition 4.7. For $T \in \mathcal{H}'_r(\mathbb{C})$ and $u \in \mathcal{H}_r(\mathbb{C})$ we define the generalized convolution of T and u to be the function $T \star_\gamma u \colon \mathbb{C} \to \mathbb{C}$ given by

$$T \star_\gamma u(z) = \langle T, T_z^\gamma u \rangle, \qquad z \in \mathbb{C}. \tag{35}$$

Proposition 4.8. *For every* $T \in \mathcal{H}'_r(\mathbb{C})$, *the mapping* $u \to T \star_\gamma u$ *is continuous from* $\mathcal{H}_r(\mathbb{C})$ *into itself.*

Proof. Let $u \in \mathcal{H}_r(\mathbb{C})$. Since, by Proposition 4.3, the series in (34) converges in $\mathcal{H}_r(\mathbb{C})$ it follows that

$$T \star_\gamma u(z) = \sum_{n=0}^{+\infty} \frac{\langle T, B_r^n u \rangle}{\alpha_{rn}(\gamma)} z^{rn}, \quad \text{for all} \ \ z \in \mathbb{C}.$$

Hence, $T \star_\gamma u \in \mathcal{H}_r(\mathbb{C})$. On the other hand, by the continuity of T, there exist C, $R' > 0$ such that

$$|\langle T, \varphi \rangle| \leq C \|\varphi\|_{R'}, \qquad \text{for all } \varphi \in \mathcal{H}_r(\mathbb{C}). \tag{36}$$

Now, let $R > 0$. Using Lemma 4.1 we conclude that

$$\left| \frac{\langle T, B_r^n u \rangle}{\alpha_{rn}(\gamma)} z^{rn} \right| \leq C \frac{R^{nr} M^n (nr)!}{(R')^{nr} \alpha_{rn}(\gamma)} \|u\|_{2R'}, \tag{37}$$

for all $z \in \mathbb{C}$ such that $|z| \leq R$ and all $n \in \mathbb{N}$. The rest of the proof runs in a similar way as the proof of Proposition 4.3. $\qquad \square$

Definition 4.9. If $T, S \in \mathcal{H}_r(\mathbb{C})$, the convolution $T \star_\gamma S$ is the element of $\mathcal{H}'_r(\mathbb{C})$ defined by

$$\langle T \star_\gamma S, u \rangle = \langle T, S \star_\gamma u \rangle, \qquad u \in \mathcal{H}_r(\mathbb{C}).$$

In the following proposition, we give some algebraic properties of the generalized convolution. The proof follows from Theorem 3.4.

Proposition 4.10. *Let* $T, S, R \in \mathcal{H}'_r(\mathbb{C})$. *Then*
   (i) $\mathcal{F}_\gamma(T \star_\gamma S) = \mathcal{F}_\gamma(T)\mathcal{F}_\gamma(S)$.
   (ii) $T \star_\gamma S = S \star_\gamma T$.
   (iii) $T \star_\gamma (S \star_\gamma R) = (T \star_\gamma S) \star_\gamma R$.
   (iv) $T \star_\gamma \delta = T$, *where* δ *denotes the Dirac functional.*
   (v) $B_r(T \star_\gamma S) = (B_r T) \star_\gamma S = T \star_\gamma (B_r S)$, *where* B_r *is defined on* $\mathcal{H}'_r$ *by transposition.*

Proposition 4.11. *Let* $T \in \mathcal{H}'_r(\mathbb{C})$. *If* T *is nonzero, then the map* $T\star_\gamma \colon u \mapsto T \star_\gamma u$ *from* $\mathcal{H}_r(\mathbb{C})$ *into itself is surjective.*

Proof. From Lemma 23.31 and Theorem 26.3 of [26], we know that the present statement is equivalent to the two properties that the dual map $(T\star_\gamma)' \colon \mathcal{H}'_r(\mathbb{C}) \to \mathcal{H}'_r(\mathbb{C})$ is injective and has closed image. The first condition follows from Proposition 4.10. Let us prove the second condition. Since $\mathcal{H}'_r(\mathbb{C})$ has the structure of a Montel DF-space, by [20, Th. 15.12], it is enough to prove that $(T\star_\gamma)'$ has a sequentially closed image. Suppose that we have a sequence $(S_n)_n$ in $\mathcal{H}'_r(\mathbb{C})$ and $S \in \mathcal{H}'_r(\mathbb{C})$ such that $(S_n \star_\gamma T) \to S$ in $\mathcal{H}'_r(\mathbb{C})$. Then by Theorem 3.4 $\mathcal{F}_\gamma(S_n)\mathcal{F}_\gamma(T) \to \mathcal{F}_\gamma(S)$ in $\mathrm{Exp}_r(\mathbb{C})$. Since $\mathrm{Exp}_r(\mathbb{C})$ is continuously embedded in $\mathcal{H}_r(\mathbb{C})$, the convergence holds also uniformly over compact sets. Therefore, the entire function $\mathcal{F}_\gamma(S)$ vanishes at all the zeros of $\mathcal{F}_\gamma(T)$, with at least the same multiplicity. Hence, $f = \mathcal{F}_\gamma(S)/\mathcal{F}_\gamma(T)$ is an entire function. So, Lindelöf's theorem

[6, §4.5.7] ensures that f is of exponential type. Using Theorem 3.4, we conclude that there is $R \in \mathcal{H}'_r(\mathbb{C})$ such that $\mathcal{F}_\gamma(R) = f$ and hence,

$$\mathcal{F}_\gamma(R)\mathcal{F}_\gamma(T) = \mathcal{F}_\gamma(S).$$

Thus, $S = T \star_\gamma R$. This completes the proof. $\square$

5. Chaotic character of the generalized convolution operators

We begin by characterizing the continuous linear mappings from $\mathcal{H}_r(\mathbb{C})$ into itself that commute with generalized translation operators.

Proposition 5.1. *If $\mathcal{L}$ is a continuous linear mapping from $\mathcal{H}_r(\mathbb{C})$ into itself, then following are equivalent:*

(i) *The operator $\mathcal{L}$ commutes with the operator B_r.*
(ii) *The operator $\mathcal{L}$ commutes with T_z^γ for all $z \in \mathbb{C}$.*
(iii) *There exists a unique $T \in \mathcal{H}'_r(\mathbb{C})$ such that*

$$\mathcal{L}u = T \star_\gamma u, \quad \text{for all } u \in \mathcal{H}_r(\mathbb{C}). \tag{38}$$

(iv) *There exists $\Phi(z) = \sum_{n=0}^{+\infty} \dfrac{b_n}{\alpha_{rn}} z^{rn} \in \mathrm{Exp}_r(\mathbb{C})$ such that $\mathcal{L} = \Phi(B_r)$. That is, for every $u \in \mathcal{H}_r(\mathbb{C})$,*

$$\mathcal{L}(u)(z) = [\Phi(B_r)u](z) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}} B_r^n u(z), \qquad z \in \mathbb{C}. \tag{39}$$

Moreover, the series in (39) converges in $\mathcal{H}_r(\mathbb{C})$.

Proof. (i)⇒(ii): Since, by Proposition 4.3, the series in (24) converges in $\mathcal{H}_r(\mathbb{C})$ and $\mathcal{L}$ is continuous on $\mathcal{H}_r(\mathbb{C})$ it follows that

$$\mathcal{L}(T_z^\gamma u) = \sum_{n=0}^{+\infty} \frac{z^{rn}}{\alpha_{rn}(\gamma)} \mathcal{L}(B_r^n u) = \sum_{n=0}^{+\infty} \frac{z^{rn}}{\alpha_{rn}(\gamma)} B_r^n \mathcal{L}(u) = T_z^\gamma \mathcal{L}(u).$$

(ii)⇒(iii): Suppose that $T \in \mathcal{H}'_r(\mathbb{C})$ is such that (38) holds. Then, obviously, $T(u) = (\mathcal{L}u)(0)$ for all $u \in \mathcal{H}_r(\mathbb{C})$. Hence T is unique. Conversely, the mapping $T\colon u \mapsto \mathcal{L}(u)(0)$ belong to $\mathcal{H}'_r(\mathbb{C})$ and, by Proposition 4.5 (i) and (ii), for all $u \in \mathcal{H}_r(\mathbb{C})$ we have

$$\mathcal{L}(u)(z) = [T_0^\gamma(\mathcal{L}(u)](z) = [T_z^\gamma(\mathcal{L}(u)](0) = \mathcal{L}(T_z^\gamma u)(0) = \langle T, T_z^\gamma u \rangle.$$

for all $z \in \mathbb{C}$. Thus, $\mathcal{L}(u) = T \star_\gamma u$.

(iii)⇒(iv): Suppose that $T \in \mathcal{H}'_r(\mathbb{C})$ is such that (38) holds. Since, by Proposition 4.3 the series in (24) converges in $\mathcal{H}_r(\mathbb{C})$ with respect to w, we have

$$(\mathcal{L}u)(z) = \sum_{n=0}^{+\infty} \frac{\langle T, w^{rn} \rangle}{\alpha_{rn(\gamma)}} B_r^n u(z), \qquad \text{for all } z \in \mathbb{C}. \tag{40}$$

For $n \in \mathbb{N}$, set $b_n = \langle T, w^{rn} \rangle$. Using the continuity of T, we can see that there exist $C, a > 0$ such that $|b_n| \leq Ca^{rn}$ for all $n \in \mathbb{N}$. Hence, by Lemma 3.1, $\Phi(z) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}} z^{rn} \in \mathrm{Exp}_r(\mathbb{C})$. The convergence in $\mathcal{H}_r(\mathbb{C})$ of the series in (40) can be established similarly to the proof of Proposition 4.3 using Lemma 4.1.

(iv)⇒(i): Since B_r is continuous on $\mathcal{H}_r(\mathbb{C})$ it follows that

$$B_r(\mathcal{L}u) = B_r \left(\sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}} B_r^n u \right) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}} B_r^{n+1} u = \mathcal{L}(B_r u),$$

for all $u \in \mathcal{H}_r(\mathbb{C})$. This completes the proof. $\square$

One of the most important result in theory of hypercyclic and chaotic operator is the Godefroy-Shapiro criterion given by the following theorem.

Theorem 5.2 (Godefroy-Shapiro)**.** (See [19, Th. 3.1, Page 69]) *Let X a be a Fréchet space and Let T be an operator on X. Suppose that the subspaces*

$$X_0 := \operatorname{span}\{x \in X \mid Tx = \lambda x, \quad \text{for some } \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda| < 1\}, \qquad (41)$$

$$Y_0 := \operatorname{span}\{x \in X \mid Tx = \lambda x, \quad \text{for some } \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda| > 1\} \qquad (42)$$

are dense in X. Then T is hypercyclic. If, moreover, the subspace

$$Z_0 := \operatorname{span}\{x \in X \mid Tx = e^{i\pi\alpha}x, \quad \text{for some } \alpha \in \mathbb{Q}\} \qquad (43)$$

is dense in X, then T is chaotic.

An operator $\mathcal{L}$ from $\mathcal{H}_r(\mathbb{C})$ into itself that satisfies the conditions in the statement of Proposition 5.1 is called convolution operator. In the following theorem we obtain the chaos of the convolution operators associated with the operator B_r on $\mathcal{H}_r(\mathbb{C})$.

Theorem 5.3. *Let $\mathcal{L}$ be a convolution operator associated with the operator B_r on $\mathcal{H}_r(\mathbb{C})$. If $\mathcal{L}$ is not a scalar multiple of the identity, then it is a chaotic operator.*

Proof. By virtue of Proposition 5.1, we can find an entire function $\Phi \in \operatorname{Exp}_r(\mathbb{C})$ such that $\mathcal{L} = \Phi(B_r)$. If we write $\Phi(z) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}(\gamma)} z^{rn}$, then by Proposition 2.2, for every $\lambda \in \mathbb{C}$ we have

$$\mathcal{L}j_\gamma(\lambda\cdot) = \Phi(B_r)j_\gamma(\lambda\cdot) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_{rn}(\gamma)}(-\lambda^r)^n j_\gamma(\lambda\cdot) = \Phi(e^{i\pi/r}\lambda)j_\gamma(\lambda\cdot),$$

where $j_\gamma(\lambda\cdot)$ is given by (10). To simplify, let $\Psi(\lambda) = \Phi(e^{i\pi/r}\lambda)$, for all $\lambda \in \mathbb{C}$. It follows that for every $\lambda \in \mathbb{C}$, the function $j_\gamma(\lambda\cdot)\colon z \mapsto j_\gamma(\lambda z)$ is an eigenfunction of $\mathcal{L}$ associated with the eigenvalue $\Psi(\lambda)$. Since $\mathcal{L}$ is not a scalar multiple of the identity, the function Ψ is not constant. So, the sets $A = \{z \in \mathbb{C} \mid |\Psi(z)| < 1\}$ and $B = \{z \in \mathbb{C} \mid |\Psi(z)| > 1\}$ are open and nonempty. Hence, according to Corollary 3.5,

$$\operatorname{span}\{f \in \mathcal{H}_r(\mathbb{C}) \mid \mathcal{L}f = \Psi(\lambda)f, \quad \text{for some } \lambda \in A\},$$

and

$$\operatorname{span}\{f \in \mathcal{H}_r(\mathbb{C}) \mid \mathcal{L}f = \Psi(\lambda)f, \quad \text{for some } \lambda \in B\}$$

are dense in $\mathcal{H}_r(\mathbb{C})$. So, Godefroy-Shapiro subspaces X_0 and Y_0 given by (41) and (42) are dense in $\mathcal{H}_r(\mathbb{C})$. Hence, by Theorem 5.2 the operator $\mathcal{L}$ is hypercyclic. On the other hand, using the same method as in [19, Page 108], we see that the set

$$\{\lambda \in \mathbb{C} \mid \Psi(\lambda) = e^{\alpha i\pi}, \text{ for some } \alpha \in \mathbb{Q}\}$$

has an accumulation point. Hence, by Corollary 3.5, we conclude that

$$\operatorname{span}\{j_\gamma(\lambda\cdot) \mid \lambda \in \mathbb{C}, \quad \text{such that } \Psi(\lambda) = e^{\alpha i\pi} \text{ for some } \alpha \in \mathbb{Q}\}.$$

is dense in $\mathcal{H}_r(\mathbb{C})$. So the Godefroy-Shapiro subspace Z_0 given by (43) is dense in $\mathcal{H}_r(\mathbb{C})$. Hence, Theorem 5.2 implies that $\mathcal{L}$ is chaotic. $\qquad \square$

Remark 5.4. Under the hypotheses of Theorem 5.2, by Birkoff transitivity theorem [4]*Theorem 1.2, Page 2, the set of hypercyclic vector for $\mathcal{L}$ is a dense G_δ set of $\mathcal{H}_r(\mathbb{C})$ and by Herrero-Bourdon theorem [19, Th. 2.55], there is a linear space $\mathcal{M}$ of $\mathcal{H}_r(\mathbb{C})$ such that every element of $\mathcal{M} \setminus \{0\}$ is hypercyclic for $\mathcal{L}$.

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications Vol. 71, Cambridge Univ. Press (1999).

[2] R. Aron, D. Markose, *On universal functions*, J. Korean Math. Soc. **41**, 1 (2004) 65–76.

[3] E. Barletta, S. Dragomir, *Vector valued holomorphic functions*, Bull. Math. Soc. Sci. Math. Roumanie **52(100)**, 3 (2009) 211–226.

[4] F. Bayart, É. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Math. Vol. 179 (2009).

[5] M. Belhadj, J.J. Betancor, *Hankel convolution operators on entire functions and distributions*, J. Math. Anal. Appl. **276** (2002) 40–63.

[6] C. A. Berenstein, R. Gay, *Complex Variables: An Introduction*, 2nd ed., Graduate Texts in Math. Vol. 125, Springer-Verlag, New York (1997).

[7] J. J. Betancor, J. D. Betancor, J. M. Méndez, *Hypercyclic and chaotic convolution operators in Chébli-Triméche hypergroups*, Rocky Mountain J. Math. **34** (2004) 1207–1238.

[8] J. J. Betancor, M. Sifi, K. Triméche, *Hypercyclic and chaotic convolution operators associated with the Dunkl operators on* $\mathbb{C}$, Acta Mathematica Hungarica **106**, 1-2 (2005) 101–116.

[9] G. D. Birkhoff, *Demonstration d'un théorème élementaire sur les fonctions entières*, C. R. Acad. Sci. Paris **189** (1929) 473-–475.

[10] S. Bochner, *Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated Bessel functions*, Proc. of the Conf. on Diff. Equations, University of Maryland (1955) 23–48.

[11] J. Bonet, *Hypercyclic and chaotic convolution operators*, J. London Month. Soc. **62**, 2 (2000) 252–262.

[12] F. Bouzeffour, *Special functions associated with complex reflection groups*, The Ramanujan Journal (2013) 7–18.

[13] F. M. Cholewinski and J. A. Reneke, *The generalized Airy diffusion equation*, Electronic J. of Differential Equations **2003**, 87 (2003) 1–64.

[14] J. Delsarte, *Sur une extension de la formule de Taylor*, Journ. de Math. **XVII**, fasc. III (1938).

[15] I. Dimovski, *Operational calculus for a class of differential operators*, C. R. Acad. Bulg. Sci. **19**, 2 (1966) 1111–1114.

[16] A Fitouhi, M. S. Ben Hammouda, and W. Binous, *On a third singular differential operator and transmutation*, Far East J. Math. Sci. (2005).

[17] A. Fitouhi, N. H. Mahmoud, S. A. Ould Ahmed Mahmoud, *Polynomial expansions for solutions of higher-order Bessel heat equations*, J. Math. Anal. Appl. **206** (1997) 155–167.

[18] G. Godefroy, J. H. Shapiro, *Operators with dense, invariant cyclic vector manifolds*, J. Funct. Analysis **98**, 2 (1991) 229–269.

[19] K.-G. Grosse-Erdmann, A. Peris, *Linear Chaos*, Universitext, Springer-Verlag, London (2011).

[20] J. Kakol, W. Kubiś, L. Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Developments in Mathematics Vol. 24, Springer-Verlag, New York (2011).

[21] V. E. Kim, *Hypercyclicity and chaotic character of generalized convolution operators generated by Gelfond-Leontev operators*, Mat. Zametki **85**, 6 (2009) 849–856.

[22] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, Harlow; John Wiley, N. York (1994).

[23] V. Kiryakova, *From the hyper-Bessel operators of Dimovski to the generalized fractional calculus*, Fractional Calculus and Applied Analysis **17**, 4 (2014) 977—1000.

[24] M. I. Klyuchantsev, Singular differential operators with r-1 parameters and Bessel functions of vector index, Siberian Math. **24** (1983) 53–366.

[25] G. R. MacLane, *Sequences of derivatives and normal families*, J. Anal. Math. **2**, 1 (1952) 72–87.

[26] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Transl. by M. S. Ramanujan, Oxford Graduate Texts in Mathematics, Clarendon Press, Oxford (1997).

[27] L. A. Rubel, J. E. Colliander, *Entire and Mermorphic Functions*, Universitext, Springer-Verlag, New York (1996).

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