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FRACTIONAL TYPE POISSON EQUATIONS BY RADIAL BASIS FUNCTIONS KANSA APPROACH

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Abstract. In this paper we propose a numerical scheme for the solution of fractional order of Poisson equation in $\mathbb{R}^2$. The new scheme uses the Radial basis functions (RBFs) method to benefit the desired properties of mesh free techniques such as no need to generate any mesh and easily applied to high dimensions. In the numerical solution approach the Kansa's collocation method is used to discrete fractional derivative terms with the multiquadric basis function. The numerical experiments two dimensional cases are presented and discussed, which conform well with the corresponding exact solutions.

1. Introduction

Poisson equation is one of the most popular elliptic differential equations with broad utility in theoretical physics, mechanical engineering and electrostatics. However a number of physical systems could only be modelled by using the non-integer order of derivatives and integrals. A lot of analytical and numerical methods of such systems have been proposed in academia such as variation iteration method [9], fractional finite difference method [2],[12], homotopy perturbation method [11],[16] and Adomian decomposition method [18],[4].

In conjunction with these methods radial basis functions method is one of the more practical ways of solving fractional order of models. The most significant property of an RBF technique is that there is no need to generate any mesh so it called meshfree method. One only requires the pairwise distance between points for an RBF approximation. Therefore it can be easily applied to high dimensional problems since the computation of distance in any dimensions is straightforward. On the other hand in order to solve partial differential equations (PDEs) in [6], [7] Kansa proposed RBF collocation method which is mesh-free and easy-to-handle in comparison with the other methods. Not only integer order PDEs [19] but also Kansa's approach has been used fractional order of PDEs [3].

This prospective study was designed to investigate the use of radial basis functions methods to solve the fractional Poisson equations via Kansa's collocation method. The remaining part of the paper proceeds as follows: the second section of this paper will review the basic tools of fractional calculus, Poisson equation

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and RBFs. The third section begins by laying out the numerical discretization formulation and looks at how can be computed the fractional order of RBFs. The fourth section presents the findings of the research by numerically. The fifth section concludes this study with some remarks.

2. Preliminaries

2.1. **Fractional calculus.** Here we review the Riemann-Liouville [15], [14], Caputo [10] and conformable fractional derivatives [8], [1].

Definition 2.1. *The Riemann-Liouville fractional derivative of order α of function $u(t)$ is described as*

$$ {}_a^{\alpha}\mathcal{D}^t u(t) = \frac{1}{\Gamma(\tau-\alpha)}\frac{d^\tau}{dt^\tau}\int_a^t (t-\xi)^{\tau-\alpha-1}u(\xi)d\xi, \qquad t > a \tag{2.1} $$

where $\tau = \lceil \alpha \rceil$.

Definition 2.2. *The Caputo fractional derivative of order α of function $u(t)$ is described as*

$$ {}_a^{\alpha}\mathcal{D}_C^t u(t) = \frac{1}{\Gamma(\tau-\alpha)}\int_a^t (t-\xi)^{\tau-\alpha-1}u^{(\tau)}(\xi)d\xi, \qquad t > a \tag{2.2} $$

where $\tau = \lceil \alpha \rceil$.

Definition 2.3. *The conformable fractional derivative of order α of function $u(t)$ is described as*

$$ {}^{\alpha}D^t u(t) = \lim_{\xi\to 0}\frac{u\left(t+\xi t^{1-\alpha}\right)-u\left(t\right)}{\xi} $$

provided the limits exits. Note that if u is fully differentiable at t, then the derivative is ${}^{\alpha}D^t u(t) = t^{1-\alpha}u'(t)$.

2.2. **Poisson Equation.** The general form of Poisson equation on a finite domain $\Omega = \{(x,y)|(x,y)\in[0,1]\times[0,1]\}$ is

$$ \nabla^2 u(x,y) = f(x,y), \tag{2.3} $$

where ∇^2 is the Laplace operator. In two dimensional Cartesian coordinates the Poisson equation takes the form

$$ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y) = f(x,y). \tag{2.4} $$

In the case of $f(x,y)=0$, Poisson equation convert to Laplace's equation. Here we begin by briefly reviewing the fractional Poisson equation. The fractional order of Poisson equation can be given as follows:

$$ \nabla^{\alpha,\beta}u(x,y) = \left(\frac{\partial^\alpha}{\partial x^\alpha}+\frac{\partial^\beta}{\partial y^\beta}\right)u(x,y) = f(x,y), \qquad 1<\alpha, \quad \beta\leqslant 2, \tag{2.5} $$

with Dirichlet boundary conditions. In order to provide mesh-free numerical solution of equation (4.3), we will use the radial basis function method which will be summarized below.

2.3. **Radial basis function method.** One of the properly approach to solving PDE is radial basis functions (RBFs). The main idea of the RBFs is to calculate distance to any fixed center points x_i with the form $\varphi(\|x - x_i\|_2)$. Additionally RBF may also have scaling parameter called shape parameter ε. This can be done in the manner that $\varphi(r)$ is replaced by $\varphi(\varepsilon r)$. Generally shape parameter have been chosen arbitrarily because there are no exact consequence about how to choose best shape parameter. Some of the RBFs are listed in Table 1.

RBFs	$\varphi(r)$
Multiquadric (MQ)	$\sqrt{1+r^2}$
Inverse Multiquadric (IMQ)	$\frac{1}{\sqrt{1+r^2}}$
Inverse Quadratic (IQ)	$\frac{1}{1+r^2}$
Gaussian (GA)	e^{-r^2}

TABLE 1. Radial basis functions

The main advantageous of RBF technique is that it does not require any mesh hence it called mesh-free method. Therefore the RBF interpolation can be represent as a linear combination of RBFs as follows:

$$s = \sum_{i=1}^{N} a_i \varphi(\|x - x_i\|_2) \tag{2.6}$$

where the a_i's the coefficients which are usually calculated by collocation technique. Some of the greatest advantages of RBF interpolation method lies in its practicality in almost any dimension and their fast convergence to the approximated target function.

3. Meshfree numerical approximation method

In this section we present a numerical scheme to solve fractional elliptic partial differential equation via non-symmetric method with radial basis functions. Let take the Poisson equation of the form

$$\nabla^{\alpha,\beta} u(x,y) = f(x,y), \qquad (x,y) \quad \text{in} \quad \Omega \tag{3.1}$$

$$u(x,y) = g(x,y), \qquad (x,y) \quad \text{on} \quad \partial\Omega \tag{3.2}$$

with Dirichlet boundary conditions where $\Omega \in \mathbb{R}^2$. Thus we are trying to compute u while f and g are fixed. We can now use Kansa's RBF collocation method [6], [7]. We build in a simple one-dimensional model. Let propose an approximation solution u of the form

$$\mathrm{u} = \sum_{i=1}^{N} a_i \varphi(\|x - x_i\|_2) \tag{3.3}$$

where $\mathfrak{X} = x_1, x_2, \ldots, x_N$ are the set of nodes in Ω. Then the collocation matrix which constructed by using Poisson equation (3.1) and boundary condition (3.2) to the collocation points $\mathfrak{X}$ will be of the form

$$[A] = \begin{pmatrix} \nabla^{\alpha,\beta}[\varphi] \\ \varphi \end{pmatrix}, \tag{3.4}$$

where the two blocks are constituted of entries:

$$\nabla^{\alpha,\beta}[\varphi]_{i,j} = \nabla^{\alpha,\beta}\varphi(\|x_i - x_j\|_2), \qquad x_i \in \mathfrak{I}, \qquad x_j \in \mathfrak{X} \tag{3.5}$$

$$\varphi_{i,j} = \varphi(\|x_i - x_j\|_2), \qquad x_i \in \mathfrak{B}, \qquad x_j \in \mathfrak{X} \tag{3.6}$$

where $\mathfrak{I}$ and $\mathfrak{B}$ represent a set of interior and a set of boundary points of the set of $\mathfrak{X}$ collocation points respectively (i.e., $\mathfrak{X} = \mathfrak{I} \cup \mathfrak{B}$). The problem described above is called well-posed (or correctly-set) if the linear matrix system $A\mathbf{a} = \mathbf{F}$, where $\mathbf{F}$ is composed of $\mathbf{f} = [f(x_i)]$, $x_i \in \mathfrak{I}$, and $\mathbf{g} = [g(x_i)]$, $x_i \in \mathfrak{B}$, has a unique solution. Since the outstanding properties of multiquadrics in terms of certainty and complexity Kansa particularly suggest to use it in (3.3).

The main difference between numerical solution of integer and non-integer order of elliptic PDE's is calculation of RBF derivatives. In other words one need to compute the Riemann-Liouville, Caputo or conformable fractional derivatives of any radial basis functions, say multiquadric. In [13], Riemann-Liouville and Caputo fractional derivative versions of some common radial basis functions have been computed explicitly. In addition to this, the conformable fractional versions of radial basis function have been presented in [17]. For instance Riemann-Liouville and Caputo fractional derivative of multiquadric basis function can be described as follows for $a \neq 0$ and $t > a$:

$${}_a^{\alpha}\mathcal{D}^t\sqrt{1+t^2} = \sum_{n=0}^{\infty}\frac{(-1)^n C_n^{2n}}{(1-2n)4^n}(2n)!(x-a)^{-\alpha}\sum_{k=0}^{2n}\frac{a^{2n-k}(x-a)^k}{(2n-k)!\Gamma(k-\alpha+1)}$$

and

$${}_a^{\alpha}\mathcal{D}_C^t\sqrt{1+t^2} = \sum_{n=0}^{\infty}\frac{(-1)^n C_n^{2n}}{(1-2n)4^n}(2n)!a^{\tau}(x-a)^{\tau-\alpha}\sum_{k=0}^{2n-\tau}\frac{a^{2n-k}(x-a)^k}{(2n-\tau-k)!\Gamma(\tau+k-\alpha+1)}$$

respectively. Another example is conformable fractional derivative of multiquadric which given below

$${}^{\alpha}\mathfrak{D}\sqrt{1+t^2} = t^{1-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n C_n^{2n}2n}{(1-2n)4^n}t^{2n-1}. \tag{3.7}$$

Then these results are used straight-forwardly in the collocation radial basis functions for solving fractional PDEs. Although there appears infinite sums in the previous formulas, one can truncate the terms once they are smaller than the machine precision.

4. Numerical experiments

Now in order to verify proposed method in the previous sections we will give some numerical experiments results of some fractional Poisson equations. In these experiments we use the multiquadric basis function and take the $\varepsilon = 4$.

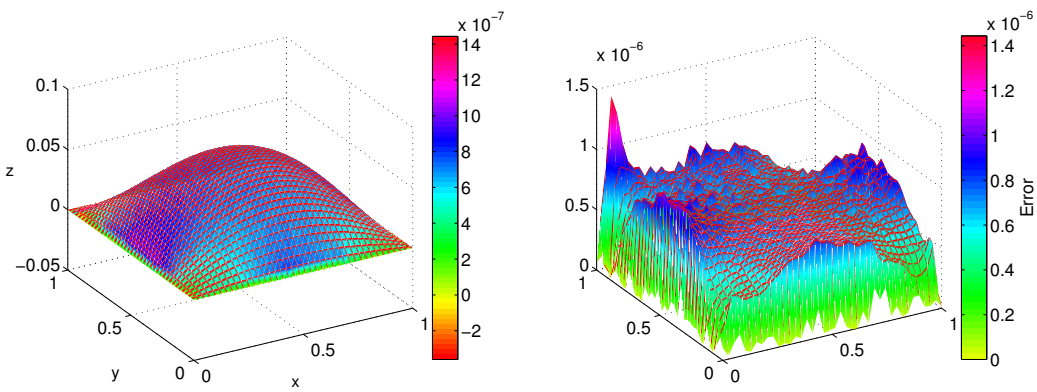
4.1. **Experiment 1.** Let consider the Riemann-Liouville fractional Poisson equation

$$\left(\frac{\partial^{4/3}}{\partial x^{4/3}} + \frac{\partial^{3/2}}{\partial y^{3/2}}\right)u(x,y) = \frac{(1080x^{1/3} - 1800x^{-2/3})y(y-1)}{360\Gamma(2/3)} + \frac{(4\sqrt{y} - y^{-1/2})x(x-1)}{\sqrt{\pi}}$$

on a finite domain $(x, y) \in \Omega = [0, 1]^2$ with the boundary condition

$$u(x, y) = 0 \qquad (x, y) \in \partial\Omega. \tag{4.1}$$

The exact solution is given by $u(x, y) = x(1 - x)y(1 - y)$.



(A) Numerical solution (B) Maximum error for RBF solution

FIGURE 1. Approximate solution of target function $u(x, y)$ (left) and maximum error for RBF solution (right).

4.2. **Experiment 2.** Let consider the Caputo fractional Poisson equation [5]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^{3/2}}{\partial y^{3/2}}\right) u(x, y) = -4\sin(2\pi x)\pi^2 y(y-1/2)(y-1) + \frac{4\sqrt{y}\sin(\pi x)\cos(\pi x)(4y-3)}{\sqrt{\pi}}$$

on a finite domain $(x, y) \in \Omega = [0, 1]^2$ with the boundary condition

$$u(x, y) = 0 \qquad (x, y) \in \partial\Omega. \tag{4.2}$$

The exact solution is given by $u(x, y) = \sin(2\pi x)y(y - 1/2)(y - 1)$.

4.3. **Experiment 3.** Let consider the Conformable fractional Poisson equation

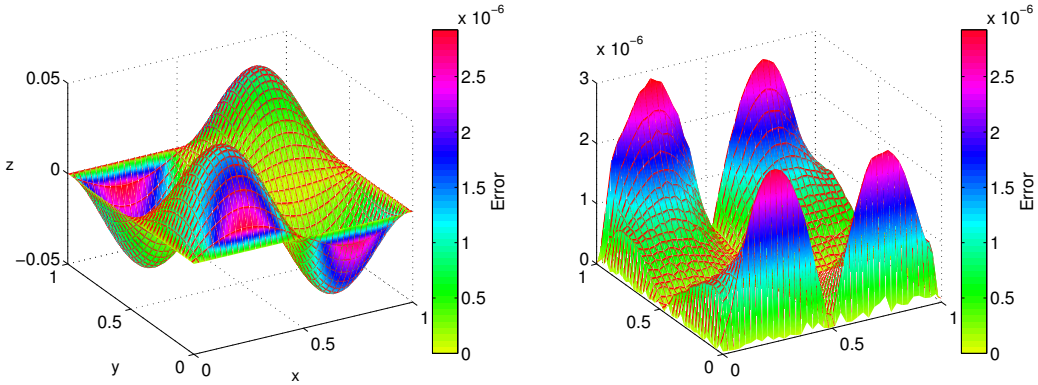
$$\begin{aligned}\left(\frac{\partial^{3/2}}{\partial x^{3/2}} + \frac{\partial^{4/3}}{\partial y^{4/3}}\right) u(x, y) &= \pi\cos(\pi y/2)\left(\frac{\cos(\pi x)}{4\sqrt{x}} + \pi\sqrt[4]{x}\sin(\pi x)\right) \\ &- \frac{\pi\sin(\pi x)}{2}\left(\frac{\sin(\pi y/2)}{3\sqrt[3]{y}} + \frac{\pi\sqrt[3]{y^2}\cos(\pi y/2)}{2}\right).\end{aligned}$$

on a finite domain $(x, y) \in \Omega = [0, 1]^2$ with the boundary condition

$$\begin{aligned} u(x, y) &= \sin(\pi x) \qquad (x, y) \in \Omega_1, \\ u(x, y) &= 0 \qquad (x, y) \in \Omega_2. \end{aligned}$$

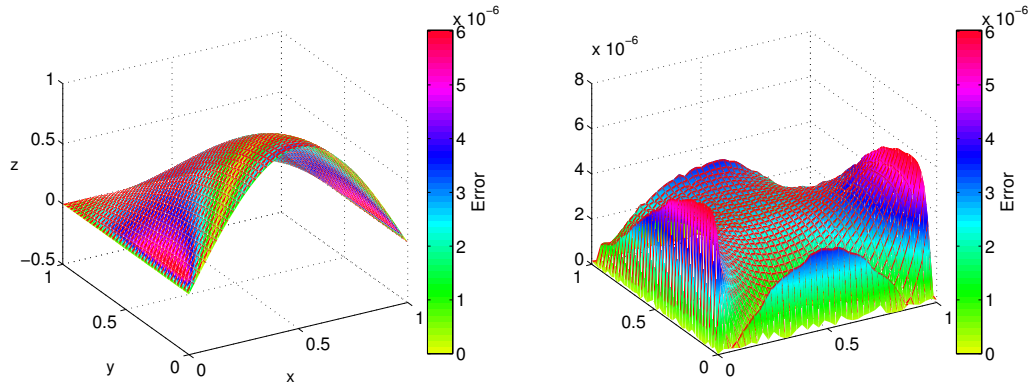
where $\Omega_1 = \{(x, y)|x \in [0, 1], y = 0\}$ and $\Omega_2 = \partial\Omega\backslash\Omega_1$. The exact solution is given by $u(x, y) = \sin(\pi x)\cos(\pi y/2)$.

In Figure 1, 2 and 3, we present the multiquadric solution of Riemann-Louville, Caputo and conformable Poisson equations along with its maximum error respectively. These figures show that the RBF method has been successfully applied to



(A) Numerical solution (B) Maximum error for RBF solution

FIGURE 2. Approximate solution of target function $u(x, y)$ (left) and maximum error for RBF solution (right).



(A) Numerical solution (B) Maximum error for RBF solution

FIGURE 3. Approximate solution of target function $u(x, y)$ (left) and maximum error for RBF solution (right).

the numerical solution problem of fractional order Poisson equation in $\mathbb{R}^2$ with encouraging performance. These results confirm the superior performance of RBF methods for numerical solution of fractional PDEs.

5. Concluding remark

In this investigation, the aim was to present a numerical scheme to solve fractional order of PDEs via collocation technique. The methods used for solution of Poisson equation may be applied to other PDEs elsewhere in the world. Numerical experiments confirm the efficiency and high accuracy of this technique. Because of RBFs multidimensional property future research should therefore concentrate on the investigation of numerical solution of high dimensional PDEs.

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