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A CLASS OF MIXED VARIATIONAL–LIKE INEQUALITIES AND EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. We use generalized weakly relaxed η–α monotonicity to analyze mixed variational-like inequalities, involving a nonlinear bifunction, in a Banach space. Existence of the solution to the problem is established using KKM theorem. An iterative algorithm is obtained using auxiliary principle technique. Also strong convergence of the iterates to the exact solution is established. We have applied our results to corresponding equilibrium problems.

1. Introduction

Let K be a nonempty compact convex subset of a real reflexive Banach space E with dual space E^*. If $N : E \times E \to E^*$, $b : E \times E \to \mathbb{R}$ and $\eta : K \times K \to E$, then the mixed variational-like inequality problem (in short MVLIP) is to find,

$$w \in K : \langle N(w, y), \eta(v, w)\rangle + b(w, v) - b(w, w) \geq 0, \forall y, v \in K. \tag{1.1}$$

This kind of problem was studied earlier under different kinds of generalized monotonicities in Banach or Hilbert space settings. The concept of monotonicity and its generalizations have been very useful in the study of the property of the solution set and convergence analysis of the approximate solutions to the exact solution [14, 19]. However, in order to analyze a wider range of real life problems, monotonicity sometimes becomes a stronger assumption to be satisfied. This fact motivated the study of different kinds of generalized monotonicities with the aim to weaken the monotonicity condition, involved in the variational or variational–like inequality problems. In this paper we study the problem (1.1) under generalized weakly relaxed η–α monotonicity, which is a weaker assumption than monotonicity, relaxed η–α monotonicity and generalized relaxed α-monotonicity, studied earlier.

MVLIP (1.1) was studied by Huang and Deng [9] for set-valued maps under strongly η–α monotonicity in Hilbert space and they provided an iterative algorithm using auxiliary principle technique. There are substantial number of results on existence and uniqueness of variational inequalities under Hilbert space setting. Generalizing the concepts to Banach space, Fang and Huang [7] studied variational-like inequalities introducing a new concept of relaxed η–α monotonicity. Later, this work was extended by Bai et al. [2], under relaxed η–α pseudomonotonicity.

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Kutbi and Sintunavarat [10] proved some existence results for variational-like inequalities involving a single operator using weakly relaxed η–α monotone mapping. Recently Mahato and Nahak [11] introduced the concepts of the generalized relaxed α-monotone and generalized relaxed α-pseudomonotone mappings for equilibrium problems, closely related to variational inequality. In all these studies [2, 7, 10] the underlying map is a single map from the associated Banach or Hilbert space E to its nonempty compact convex subset K. Very recently Alleche and Radulescu [1] investigated set-valued equilibrium problems while focusing on weakening the semi-continuity. All these approaches mainly emphasize on qualitative aspects, involving the study of the property of the solution set, while the thrust on the numerical approach is low.

Motivated by these works, we introduce the concept of generalized weakly relaxed η–α monotonicity, which can be regarded as a proper generalization of monotonicity, weakly relaxed η–α monotonicity and generalized relaxed α-monotonicity. We have extended the results of [2], [10] and [11] to nonlinear mixed variational-like inequality under generalized weakly relaxed η–α monotonicity. Along with the qualitative aspect, that is, the study of existence of solution, we study the numerical aspect and propose an iterative algorithm to approximate the exact solution using auxiliary principle technique which is due to Glowinski et al. [17]. Following the ideas of Ding [5], we analyze the convergence criteria for the proposed iterative algorithm. The results obtained are extended to corresponding equilibrium problem using the concept of trifunction variational inequality studied in [3].

2. Preliminaries

Throughout the paper we assume that K is a nonempty compact convex subset of a real reflexive Banach space E and E^* be its dual space.

Definition 1. *If $N : E\times E \to E^*$, $\eta : K\times K \to E$, $\alpha : E\times E \to \mathbb{R}$, $t > 0, z \in E$ and $p > 1$ is a constant, then N is said to be generalized weakly relaxed η–α monotone if $\langle N(v,y)-N(w,y), \eta(v,w)\rangle \geq \alpha(v,w)$, $\forall v, w \in K$, where $\alpha(w,w) = 0$, $\lim_{t\to 0} \frac{d}{dt}\alpha(tv+(1-t)w, w) = 0$.*

Remark 1. *It follows from the above definitions that,*

(i) If $\alpha(v,w) = \beta(v-w)$, with $\lim_{t\to 0}\beta(tz) = 0$ and $\lim_{t\to 0}\frac{d}{dt}\beta(tz) = 0$, generalized weakly relaxed η–α monotonicity reduces to weakly relaxed η–α monotonicity. This was studied in the context of mixed variational-like inequality in [15] *and for variational-like inequality involving a single operator in* [10].

(ii) As $\alpha(w,w) = 0$, $\lim_{t\to 0}\frac{d}{dt}\alpha(tv+(1-t)w, w) = 0$, then using L'Hospital's rule, we get $\lim_{t\to 0}\frac{\alpha(tv+(1-t)w,w)}{t} = 0$ and hence it reduces to generalized relaxed η–α monotonicity ([11]*).*

(iii) It reduces to relaxed η–α monotonicity if $\alpha(v,w) = \beta(v-w)$, with $\beta(tz) = t^p\beta(z)$, for $t > 0, z \in E$.

(iv) If $\alpha \equiv 0$, then N becomes η-monotone, i.e.,

$$\langle N(v,y) - N(w,y), \eta(v,w)\rangle \geq 0$$

and if $\eta \equiv v - w$, then N is said to be simply monotone, i.e.,

$$\langle N(v,y) - N(w,y), v - w\rangle \geq 0.$$

So it follows that monotonicity $\Longrightarrow$ relaxed η–α monotonicity $\Longrightarrow$ weakly relaxed η–α monotonicity $\Longrightarrow$ generalized weakly relaxed η–α monotonicity. But the converse is not in general true. Here is an example that shows a mapping is generalized weakly relaxed monotone, but not weakly relaxed monotone.

Example 2. *We consider,* $T : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$, $Tv = \sin v,\ \eta(v,w) = v - w,\ \alpha(v,w) = -(\sin v - \sin w)^2$, *then* $\lim_{t\to 0} \alpha(tv+(1-t)w, w) = 0$ *and* $\lim_{t\to 0} \frac{d}{dt}\alpha(tv+(1-t)w, w) = 0$.

$$\langle Tv - Tw, \eta(v,w)\rangle = \langle \sin v - \sin w, v - w\rangle \geq -(\sin v - \sin w)^2 = \alpha(v,w).$$

So T is generalized weakly relaxed η–α monotone, but not weakly relaxed η–α monotone as $-(\sin v - \sin w)^2$ is not a function of $v - w$.

The following is a list of definitions and results that will be frequently used in the sequel.

Definition 2. *If $N : E\times E \to E^*$ and $\eta : K\times K \to E^*$, then N is η-hemicontinuous if $f(t) = \langle N(w+t(v-w)), \eta(v,w)\rangle$ is continuous at 0, where $f : [0,1] \to (-\infty, +\infty)$.*

Definition 3. *N and η are said to have 0-diagonally concave relation, if the function $\phi : K \times K \to R$ defined by*

$$\phi(w,v) = \langle N(w,v), \eta(w,v)\rangle$$

is 0-diagonally concave in v, that is, for any finite set $\{v_1, \cdots, v_m\} \subset K$ and for any convex combination of v_i, $\sum_{i=1}^{m} \lambda_i \phi(w, v_i) \leq 0$. N and η are said to have 0-diagonally convex relation on K if $-N$ and η have 0-diagonally concave relation.

Definition 4. *N is η-monotone with respect to first argument if*

$$\langle N(w_1, v) - N(w_2, v), \eta(w,v)\rangle \geq 0,\ \forall w, v \in K.$$

Definition 5. *N is η-antimonotone with respect to second argument if*

$$\langle N(w, v_1) - N(w, v_2), \eta(w,v)\rangle \leq 0,\ \forall w, v \in K.$$

Definition 6. *A mapping $F : K \to E^*$ is KKM mapping if, for any $\{x_1, \cdots, x_n\} \subset K$, $co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$, where $co\{x_1, \cdots, x_n\}$ denotes the convex hull of $x_1, x_2 \cdots, x_n$.*

Definition 7. *If $S = \{x_1, \cdots, x_n\}$, $co\{x_1, \cdots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \sum_{i=1}^{n} \alpha_i = 1, \forall \alpha_i \geq 0 \right\}$.*

Definition 8. *A function $f : K \to (-\infty, +\infty]$ is lower semicontinuous at x_0 if $f(x_o) \leq lim\ inf_{x\to x_0} f(x)$.*

Definition 9. *An operator $T : K \to E^*$ is Lipschitz continuous if there exists $\alpha > 0$, such that $\|Tx - Ty\| \leq \alpha\|x - y\|$, $\forall\ x, y \in K$.*

Definition 10. *T is η-coercive with respect to a proper function $f : K \to (-\infty, +\infty]$ if there exists $x_0 \in K$ such that*

$$\frac{\langle Tx - Tx_0, \eta(x, x_0)\rangle + f(x) - f(x_0)}{\|\eta(x, x_0)\|} \to \infty.$$

Now, we cite two Lemmas, which are essential for establishing the existence results for the MVLIP (1.1) and the auxiliary variational inequality problem (4.2) respectively.

Lemma 2.1 ([6]). *If M is a nonempty subset of a Hausdorff topological vector space X, $F : M \to 2^X$ is a KKM mapping, $F(x)$ is closed in X, $\forall x \in K$ and compact for some $x \in K$, then $\bigcap_{x\in M} F(x) \neq \phi$.*

Lemma 2.2 ([4]). *Let K be a nonempty convex subset of a topological vector space and let $\phi : K \times K \to R$ be such that:*

(i) for each $x \in K, y \to \phi(x, y)$ is lower semicontinuous on each nonempty compact subset of K,

(ii) for each nonempty finite set $\{x_1, \cdots, x_m\} \subset K$ and for each $y = \sum_{i=1}^{m} \lambda_i x_i (\lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1)$, $min_{1\leq i\leq m}\phi(x_i, y) \leq 0$,

(iii) there exists nonempty compact convex subset X_0 of K and a nonempty compact subset D of K such that for each $y \in K\backslash D$, there is an $x \in co(X_0 \cup \{y\})$ with $\phi(x, y) > 0$.

Then there exists an $\hat{y} \in D$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in K$.

3. Results

In order to prove the existence of solution to the MVLIP (1.1), we first proceed to show that the problems (3.1) and (3.2) are equivalent. Next, using KKM technique and Lemma 2.1 solvability for the MVLIP is established. Solvability is also established in the case, where K is unbounded with an additional condition of η-coercivity of the mapping N, in Theorem 5.

3.1. Existence result.

Theorem 3. *Let $N : E \times E \to E^*$ be η-hemicontinuous and generalized weakly relaxed η–α monotone and $b : E\times E \to \mathbb{R}$ be a convex lower semicontinuous function in second argument, such that,*

(i) $\eta(w, w) = 0, \forall w \in K$,

(ii) $v \to \langle N(w, y), \eta(v, w)\rangle$ is convex for any $w, y \in K$,

Then the following problems are equivalent:

$$w \in K, \langle N(w, y), \eta(v, w)\rangle + b(w, v) - b(w, w) \geq 0, \forall v \in K, \tag{3.1}$$

$$w \in K, \langle N(v, y), \eta(v, w)\rangle + b(w, v) - b(w, w) \geq \alpha(v, w), \forall v \in K. \tag{3.2}$$

Proof. Let $w \in K$ be a solution of (3.1). As N is generalized weakly relaxed η–α monotone, we have,

$$\begin{aligned}\langle N(v, y), \eta(v, w)\rangle + b(w, v) - b(w, w) &\geq \langle N(w, y), \eta(v, w)\rangle + \alpha(w - v) + b(w, v) - b(w, w)\\ &\geq \alpha(v, w), \forall v \in K.\end{aligned}$$

Hence w is a solution of (3.2) and (3.1) $\Longrightarrow$ (3.2). Conversely let $w \in K$ be a solution of (3.2). Let $v_t = (1 - t)w + tv, t \in (0, 1)$. So $v_t \in K$. As $w \in K$ is a solution of (3.2), we have,

$$\langle N(v_t, y), \eta(v_t, w)\rangle + b(w, v_t) - b(w, w) \geq \alpha(v_t, w), \text{ and}$$
$$b(w, v_t) - b(w, w) \leq t(b(w, v) - b(w, w)).$$

Using these results, we get,

$$\langle N(w+t(v-w),y),\eta(v,w)\rangle + b(w,v) - b(w,w) \geq \frac{\alpha(v_t,w)}{t}, \forall v \in K.$$

Since N is η-hemicontinuous and $\lim_{t\to 0} \frac{d}{dt}\alpha(tv+(1-t)w,w) = 0$, letting $t \to 0$ and applying L'Hospital's rule, we get,

$$\langle N(w,y),\eta(v,w)\rangle + b(w,v) - b(w,w) \geq 0, \forall v \in K.$$

This completes the proof. □

Theorem 4. *If $N : E \times E \to E^*$ is η–hemicontinuous and generalized weakly relaxed η–α monotone and $b : E \times E \to \mathbb{R}$ is a convex lower semicontinuous function in second argument and the following assumptions*

(i) $\eta(v,v) = 0,\ \forall v \in K$,
(ii) $v \to \langle N(w,y),\eta(v,w)\rangle$ *is convex and lower semicontinuous for any* $w, y \in K$,
(iii) For any v_β, v_β *converging to* v, $\alpha(v) \leq lim\ inf\ \alpha(v_\beta)$,

hold, then problem (1.1) *is solvable.*

Proof. Let $F, G : K \to 2^E$ be defined by,

$$F(v) = \{w \in K, \langle N(w,y),\eta(v,w)\rangle + b(w,v) - b(w,w) \geq 0\}, \forall v \in K,$$

$$G(v) = \{w \in K, \langle N(v,y),\eta(v,w)\rangle + b(w,v) - b(w,w) \geq \alpha(v,w)\}, \forall v \in K.$$

We claim that F is a KKM mapping, if not, then there exists $\{v_1, \cdots, v_n\} \subset K$ and $t_i > 0, i = 1, 2, \cdots, n$, such that, $\sum_{i=1}^{n} t_i = 1,\ v = \sum_{i=1}^{n} t_i v_i \notin \bigcup_{i=1}^{n} F(v_i)$.
Then by definition of F, $\langle N(v,y),\eta(v_i,v)\rangle + b(w,v_i) - b(w,v) < 0$, for $i = 1, 2 \cdots, n$. By our assumption,

$$\begin{aligned} 0 &= \langle N(v,y),\eta(v,v)\rangle \\ \implies \langle N(v,y),\eta(\sum_{i=1}^{n} t_i v_i, v)\rangle &\leq \sum_{i=1}^{n} t_i \langle N(v,y),\eta(v_i,v)\rangle \\ &< \sum_{i=1}^{n} t_i (b(w,v) - b(w,v_i)) \\ &= b(w,v) - \sum_{i=1}^{n} t_i b(w,v_i) \\ &\leq b(w,v) - b(w,v) = 0. \end{aligned}$$

This is a contradiction. So F is a KKM mapping. We now claim that $F(v) \subset G(v), \forall v \in K$. Let $w \in F(v)$, then

$$\langle N(w,y),\eta(v,w)\rangle + b(w,v) - b(w,w) \geq 0, \text{ for any } v \in K.$$

Using generalized weakly relaxed η–α monotonicity of N, we have $w \in G(v)$. So $F(v) \subset G(v), \forall v \in K$. So G is a KKM mapping. As α is weakly lower semicontinuous, $G(v)$ is weakly closed. $v \to \langle N(w,y),\eta(v,w)\rangle$ and b are convex and lower semicontinuous and hence weakly lower semicontinuous. As K is bounded closed and convex in the reflexive Banach space E, it is weakly compact. Since $G(v)$ is weakly closed for all $v \in K$, it is weakly compact and hence the family $\{G(v)\}$ has

finite intersection property, that is, $\bigcap_{v\in K} G(v) \neq \Phi$. So conditions of Lemma 2.1 are satisfied, hence we have,

$$\bigcap_{v\in K} F(v) = \bigcap_{v\in K} G(v) \neq \Phi.$$

So there exists $w \in K$ such that

$$\langle N(w,y), \eta(v,w)\rangle + b(w,v) - b(w,w) \geq 0, \forall v \in K.$$

□

Next we prove the existence result for an unbounded set K.

Theorem 5. *If $N : E \times E \to E^*$ is η-hemicontinuous and generalized weakly relaxed η–α monotone, $b : E \times E \to \mathbb{R}$ is a convex lower semicontinuous functional in second argument, K is a nonempty closed convex unbounded subset of E and the following assumptions*

(i) N is η-coercive with respect to b in the second argument, that is, for $w_0 \in K$, $[\langle N(w,y) - N(w_0,y), \eta(v,w_0)\rangle + b(w,v) - b(w_0,v)]/\|\eta(v,w_0)\| \to \infty$, as $\|v\| \to \infty$.

(ii) $\eta(w,v) + \eta(v,w) = 0$, $\forall w, v \in K$,

(iii) $v \to \langle N(w,y), \eta(v,w)\rangle$ is convex and lower semicontinuous for any $w, y \in K$,

(iv) for any v_β, v_β converging to v, $\alpha(v) \leq \liminf \alpha(v_\beta)$,

hold, then problem (1.1) is solvable.

Proof. Consider the problem, find $w_r \in K \cap B_r$ such that

$$\langle N(w_r,y), \eta(v,w_r)\rangle + b(w_r,v) - b(w_r,w_r) \geq 0, \ \forall v \in K \cap B_r \tag{3.3}$$

where $B_r = \{v \in E : \|v\| \leq r\}$.

By Theorem 4, (3.3) has a solution $w_r \in K \cap B_r$. Choosing $\|w_0\| < r$, we can put w_0 in place of v in (3.3), and hence,

$$\langle N(w_r,y), \eta(w_0,w_r)\rangle + b(w_r,w_0) - b(w_r,w_r) \geq 0.$$

Now,

$$\begin{aligned}
&\langle N(w_r,y), \eta(w_0,w_r)\rangle + b(w_r,w_0) - b(w_r,w_r)\\
&= -\langle N(w_r,y), \eta(w_r,w_0)\rangle + \langle N(w_0,y), \eta(w_0,w_r)\rangle + \langle N(w_0,y), \eta(w_r,w_0)\rangle\\
&+ b(w_r,w_0) - b(w_r,w_r)\\
&= -\langle N(w_r,y) - N(w_0,y), \eta(w_r,w_0)\rangle + b(w_r,w_0) - b(w_r,w_r) + \langle N(w_0,y), \eta(w_0,w_r)\rangle\\
&\leq \|\eta(w_r,w_0)\|\left(\frac{-\langle N(w_r,y) - N(w_0,y), \eta(w_r,w_0)\rangle + b(w_r,w_r) - b(w_r,w_0)}{\|\eta(w_r,w_0)\|} + \|N(w_0,y)\|\right).
\end{aligned}$$

If $\|w_r\| = r$ and $r \to \infty$, then by η-coercivity of N with respect to b in the second argument, the above inequality reduces to

$$\langle N(w_r,y), \eta(w_0,w_r)\rangle + b(w_r,w_0) - b(w_r,w_r) < 0.$$

This is a contradiction as $\langle N(w_r,y), \eta(w_0,w_r)\rangle + b(w_r,w_0) - b(w_r,w_r) \geq 0$. So $\|w_r\| < r$. Now for any $v \in K$, we choose $0 < \epsilon < 1$, such that,

$$w_r + \epsilon(v - w_r) \in K \cap B_r.$$

By assumption (iii) and convexity of b, we have from equation (3.3),

$$\begin{aligned}0 &\leq \langle N(w_r, y), \eta(w_r + \epsilon(v - w_r), w_r)\rangle + b(w_r + \epsilon(v - w_r)) - b(w_r, w_r)\\ &\leq (1-\epsilon)\langle N(w_r, y), \eta(w_r, w_r)\rangle + \epsilon\langle N(w_r, y), \eta(v, w_r)\rangle + (1-\epsilon)b(w_r, w_r) + \epsilon b(v, w_r)\\ &- b(w_r, w_r)\\ &= \epsilon\langle N(w_r, y), \eta(v, w_r)\rangle + \epsilon b(w_r, v) - \epsilon b(w_r, w_r).\end{aligned}$$

So,

$$\langle N(w_r, y), \eta(v, w_r)\rangle + b(w_r, v) - b(w_r, w_r) \geq 0, \forall y \in K$$

and $w_r \in K$ is a solution of problem (1.1). This completes the proof. □

4. Iterative algorithm and Convergence analysis

In this section, using auxiliary principle technique we formulate an iterative algorithm that generates approximate solutions to the nonlinear mixed variational-like inequality problem (1.1). For this purpose, we first formulate an auxiliary minimizing problem and then characterize it by an auxiliary variational inequality problem. Existence of solution to the latter is established in Theorem 6. Formulation of the iterative algorithm is mainly based on this existence result.

The differentiable convex functional $\alpha : E \to \mathbb{R}$, involved in the formulation of the auxiliary minimizing problem is considered as an auxiliary differentiable convex functional. The auxiliary minimizing problem is defined as follows:

$$\min_{w\in K}\{\alpha(w) + \langle \rho N(v, y), \eta(w, v)\rangle - \langle \alpha'(v), w\rangle + \rho b(v, w)\}, \tag{4.1}$$

where $w \in E, v \in K$ and ρ is a positive constant. If $w \mapsto \langle N(v, y), \eta(w, v)\rangle$ is convex, then (4.1) is equivalent to following auxiliary variational inequality problem in the sense that solution to both the problems are the same. The auxiliary variational inequality problem is given by,

$$\langle \alpha'(w) - \alpha'(v), u - w\rangle \geq -\rho\langle N(v, y), \eta(u, w)\rangle + \rho b(v, w) - \rho b(v, u), \text{ for all } u \in K. \tag{4.2}$$

Note: If $w = v$, then v is a solution of (1.1).

Keeping in view these results we propose an iterative algorithm as follows:

(i) Let v_0 be the initial approximation for $n = 0$.
(ii) At the nth step solve the auxiliary minimizing or auxiliary variational inequality problem with $v^* = v_n$. Let v_{n+1} be the solution.
(iii) If $\|v_{n+1} - v_n\| \leq \epsilon;\ \epsilon > 0$, stop, otherwise repeat (ii).

Next we prove the result that guarantees the existence of solution of (4.1) or (4.2).

Theorem 6. *Let $N : E \times E \to E^*,\ b : E \times E \to \mathbb{R}$ be η-hemicontinuous, convex lower semicontinuous functional in second argument, linear in the first argument, bounded and $b(u, v) - b(u, w) \leq b(u, v - w)$ and $\alpha : E \to \mathbb{R}$ be a differentiable convex functional, which is weakly lower semicontinuous such that*

(i) $v \to \langle N(w, y), \eta(v, w)\rangle$ is convex and lower semicontinuous for any $w, y \in K$,
(ii) N is strongly η-monotone, generalized weakly relaxed η–α monotone and η-convex with respect to first argument for any $w, y \in K$,
(iii) N is η-antimonotone and generalized weakly relaxed monotone in second argument and Lipschitz continuous in both the arguments,
(iv) η is antisymmetric and Lipschitz continuous and $\eta(v, w) = \eta(v, u) + \eta(u, w)$, for any $u, v, w \in K$,

(v) N *and* η *have* 0*-diagonally convex relation with respect to first argument,*
(vi) $w \to \alpha'(w)$ *is continuous from weak to strong topology and* α' *is strongly monotone,*
(vii) $\alpha \neq 2\mu, (\sigma_1 + \sigma_2)\delta + \mu > 0$ *and* $0 < \rho < \frac{\rho(\sigma_1+\sigma_2)\delta}{\alpha - 2\mu}$,

then there exists a solution $w \in K$ *of the problem* (1.1) *and for each* $\rho > 0$, *there exists a solution* $w_{n+1} \in K$ *of problems* (4.1) *or* (4.2) *and the approximate solutions converge strongly to the exact solution.*

Proof. As N is η-convex with respect to first argument for any $w, y \in K$, it is easy to check that condition (ii) of Theorem 4 is satisfied. By antisymmetry of η, $\eta(v, v) = 0$. Hence condition (i) of Theorem 4 is also satisfied. By our assumption α is weakly lower semicontinuous. Hence solution to problem (1.1) exists. Now to prove the second part of the conclusion, we have to show that all the conditions of Lemma 2.2 are satisfied. For this purpose we define $\phi : K \times K \to R$, by,

$$\phi(u, w) = \langle \alpha'(v_n) - \alpha'(w), u - w\rangle - \rho\langle N(v_n, y_n), \eta(u, w)\rangle + \rho b(v_n, w) - \rho b(v_n, u).$$

As $w \to \alpha'(w)$ is continuous from weak to strong topology, the function $w \to \langle \alpha'(w), w\rangle$ is weak continuous on K. So $w \to \phi(u, w)$ is weakly lower semicontinuous. Thus, condition (i) of Lemma 2.2 is satisfied. To prove the second condition we assume the contrary. So there exists $\{u_1, \cdots, u_n\} \subset K$ and w which is a convex combination of u_i, such that $\phi(u_i, w) > 0$. From this we get,

$$\sum_{i=1}^{n} \lambda_i \langle \alpha'(v_n) - \alpha'(w), u_i - w\rangle - \rho\langle N(v_n, y_n), \eta(u_i, w)\rangle + \rho b(v_n, w) - \rho \sum_{i=1}^{n} \lambda_i b(v_n, u_i) > 0.$$

As b is convex in the second argument we have

$$\sum_{i=1}^{n} \lambda_i \langle \alpha'(v_n) - \alpha'(w), u_i - w\rangle - \rho\langle N(v_n, y_n), \eta(u_i, w)\rangle > 0.$$

This contradicts condition (v). Thus, condition (2) of Lemma 2.2 holds. Now considering a set $D = \{v \in K : \|v - u^*\| \leq \theta\}$, where $\theta = \frac{1}{\alpha}[\mu\|u^*\|] + \delta\|N(u^*, y)\|$ and using condition (ii)–(v) and conditions on b, condition (iii) is proved. Hence all the conditions of Lemma 2.2 are satisfied. So there exists $w_0 \in K$ such that

$$\langle \alpha'(w_0) - \alpha'(v_n), u - w_0\rangle \geq -\rho\langle N(v_n, y_n), \eta(u, w_0)\rangle + \rho b(v_n, w_0) - \rho b(v_n, u) \quad (4.3)$$

for all $u \in K$. This shows that there exists a solution to the auxiliary variational inequality problem.
Now for convergence analysis we consider the following functional $\Gamma : K \to (-\infty, +\infty]$, defined by,

$$\Gamma(v) = \alpha(v_0) - \alpha(v) - \langle \alpha'(v), v_0 - v\rangle,$$

where v_0 is assumed to be the unique solution of problem (1.1). By strong monotonicity of α', we have,

$$\Gamma(v) = \alpha(v_0) - \alpha(v) - \langle \alpha'(v), v_0 - v\rangle \geq \frac{\sigma}{2}\|v - v_0\|^2.$$

Putting $w_0 = v_{n+1}, u = v_0$ in (4.2) and by antisymmetricity of η, strong monotonicity of α', we get,

$$\begin{aligned}\Gamma(v_n) - \Gamma(v_{n+1}) &\geq \frac{\sigma}{2}\|v_n - v_{n+1}\|^2 + \rho\langle N(v_n, y_n), \eta(v_{n+1}, v_0)\rangle + \rho b(v_n, v_{n+1}) - \rho b(v_n, v_0)\\ &= \frac{\sigma}{2}\|v_n - v_{n+1}\|^2 + \rho\langle N(v_n, y_n) - N(v_0, y_0), \eta(v_{n+1}, v_0)\rangle\\ &\quad + \rho\langle N(v_0, y_0), \eta(v_{n+1}, v_0)\rangle + \rho b(v_n, v_{n+1}) - \rho b(v_n, v_0).\end{aligned}$$

v_0 being a solution of (1.1) it follows that,

$$\begin{aligned}\Gamma(v_n) - \Gamma(v_{n+1}) &\geq \frac{\sigma}{2}\|v_n - v_{n+1}\|^2 + \rho\langle N(v_n, y_n) - N(v_0, y_0), \eta(v_{n+1}, v_0)\rangle\\ &\quad + \rho[b(v_0, v_0) - b(v_0, v_{n+1}) + b(v_n, v_{n+1}) - b(v_n, v_0)]\\ &= \frac{\sigma}{2}\|v_n - v_{n+1}\|^2 + M.\end{aligned}$$

Now using the conditions on b and the conditions $(ii) - (iv)$, we get,

$$\begin{aligned}M =& \rho\langle N(v_n, y_n) - N(v_0, y_0), \eta(v_{n+1}, v_0)\rangle\\ &- \rho[b(v_n - v_0, v_0) - b(v_n - v_0, v_{n+1}) + b(v_n - v_0, v_n) - b(v_n - v_0, v_n)]\\ &\geq \rho[\langle N(v_n, y_n) - N(v_0, y_0), \eta(v_{n+1}, v_n)\rangle + \langle N(v_n, y_n) - N(v_0, y_0), \eta(v_n, v_0)\rangle]\\ &- \rho[b(v_n - v_0, v_0 - v_n) + b(v_n - v_0, v_n - v_{n+1})]\\ &\geq \rho[\langle N(v_n, y_n) - N(v_0, y_n), \eta(v_n, v_0)\rangle\\ &+ \langle N(v_0, y_n) - N(v_0, y_0), \eta(v_n, v_0)\rangle\\ &+ \langle N(v_n, y_n) - N(v_0, y_n), \eta(v_{n+1}, v_n)\rangle\\ &+ \langle N(v_0, y_n) - N(v_0, y_0), \eta(v_{n+1}, v_n)\rangle]\\ &- \rho\mu[\|v_n - v_0\|^2 + \|v_n - v_0\|\|v_n - v_{n+1}\|]\\ &\geq \rho\alpha\|v_n - v_0\|^2 - \rho\sigma_1\delta\|v_n - v_0\|\|v_{n+1} - v_n\|\\ &- \rho\sigma_2\delta\|v_n - v_0\|\|v_{n+1} - v_n\| - \rho\mu[\|v_n - v_0\|^2 + \|v_n - v_0\|\|v_n - v_{n+1}\|].\end{aligned}$$

From this we have,

$$\Gamma(v_n) - \Gamma(v_{n+1}) \geq \rho[\alpha - (2\mu + (\sigma_1 + \sigma_2)\delta)]\|v_n - v_0\|^2.$$

Condition (vii) implies that $\{\Gamma(v_n)\}$ is a strictly decreasing sequence and it is nonnegative by the strong monotonicity property and hence converges. So $\{v_n\}$ converges to v_0 strongly as $n \to \infty$. This completes the proof. □

5. Application to Equilibrium Problem

Let $\phi : K \times K \times K \to R$ be denoted as $\phi(y, v, w) = \langle N(v, y), \eta(w, v)\rangle$, then the mixed equilibrium problem corresponding to MVLIP is defined as,

$$\text{find } \bar{w} \in K : \phi(y, v, \bar{w}) + b(\bar{w}, v) - b(\bar{w}, \bar{w}) \geq 0, \forall v \in K. \tag{5.1}$$

If ϕ is a bifunction from $K \times K$ to $\mathbb{R}$, it reduces to the classical equilibrium problem, introduced by Blum and Oettili [3], given by,

$$\text{find } \bar{w} \in K : \ \phi(\bar{w}, v) \leq 0, \text{ for all } v \in K.$$

This was later studied by various authors ([12], [13]).

Definition 11. *ϕ is relaxed α–monotone if $\exists \alpha : E \to R$, with $\alpha(tz) = t^p \alpha(z), \forall t > 0, p > 1$ and $z \in E$, such that*

$$\phi(y, v, w) + \phi(y, w, v) \leq \alpha(v, w).$$

Remark: If $\alpha \equiv 0$, ϕ is monotone, i.e., $\phi(y, v, w) + \phi(y, w, v) \leq 0, \forall y, v, w \in K$.

Definition 12. *ϕ is weakly relaxed α–monotone if*

$$\phi(y, v, w) + \phi(y, w, v) \leq \alpha(v, w),$$

with $\lim_{t \to 0} \alpha(tz) = 0$, $\lim_{t \to 0} \frac{d}{dt} \alpha(tz) = 0$.

Definition 13. *ϕ is generalized weakly relaxed α–monotone if*

$$\phi(y, v, w) + \phi(y, w, v) \leq \alpha(v, w),$$

with $\lim_{t \to 0} \frac{d}{dt} \alpha(tv + (1-t)w, w) = 0$.

Remark: It is clear from the above definitions that monotonicity $\Longrightarrow$ relaxed α–monotonicity $\Longrightarrow$ weakly α– monotonicity $\Longrightarrow$ generalized weakly α– monotonicity.

The results regarding MVLIP discussed in previous section can be extended to the class of equilibrium problem defined above. So we state the results without proof that follow the similar pattern.

Corollary 1. *Let K be a nonempty compact convex subset of a real reflexive Banach space E. Let $\phi : K \times K \times K \to \mathbb{R}$ be generalized weakly relaxed α–monotone and hemicontinuous in the second argument and $b : E \times E \to \mathbb{R}$ be a convex lower semicontinuous function in second argument, such that,*

(i) $\phi(y, w, w) \geq 0$, *for all* $y \in K$,
(ii) $\phi(y, w, \cdot)$ *is convex, that is,*

$$\phi(y, w, tv_1 + (1-t)v_2) \leq t\phi(y, w, v_1) + (1-t)\phi(y, w, v_2), \ t \in (0, 1).$$

Then the following problems are equivalent:

$$w \in K, \phi(y, w, v) + b(w, v) - b(w, w) \geq 0, \text{ for all } y, v \in K, \tag{5.2}$$

$$w \in K, \phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v, w), \text{ for all } y, v \in K. \tag{5.3}$$

Corollary 2. *If the equilibrium function ϕ is generalized weakly relaxed α–monotone and hemicontinuous in second argument and the nonlinear bifuction b is convex and lower semicontinuous in second argument, then the equilibrium problem* (5.1) *has a solution under the following conditions:*

(1) $\phi(v, v, v) = 0$,
(2) $w \to \phi(y, v, w)$ *is convex and lower semicontinuous for fixed* $y, v \in K$,
(3) $\alpha : E \to R$ *is weakly upper semicontinuous.*

The next results deal with numerical aspects of the mixed equilibrium problem, that is, formulation of iterative algorithm and study of convergence analysis.
For this purpose, we first construct an auxiliary variational inequality as follows:

For a given $w \in K$, we consider the problem of finding $u \in K$, such that,

$$\langle \alpha'(u) - \alpha'(w), v - u \rangle \geq -\rho\phi(y, w, v) + \rho\phi(y, w, u) + \rho(b(w, u) - b(w, v)), \tag{5.4}$$

where $y, v \in K$, ρ is a positive constant and $\alpha : K \to (-\infty, +\infty)$ is a differentiable convex functional. Here α is considered as an auxiliary differentiable convex functional. If $u = w$, then w becomes the solution of the mixed equilibrium problem (5.1).

Keeping in view these results, we propose an iterative algorithm as follows:

(i) Let w_0 be the initial approximation for $n = 0$.
(ii) At the nth step solve auxiliary variational inequality problem with $w^* = w_n$. Let w_{n+1} be the solution.
(iii) If $\|w_{n+1} - w_n\| \leq \epsilon$; $\epsilon > 0$, stop, otherwise go to (ii).

Next we prove the result that guarantees the existence of solution of the auxiliary variational inequality problem.

Corollary 3. *Let E be a reflexive Banach space with dual space E^* and the equilibrium trifunction $\phi : K \times K \times K \to \mathbb{R}$ be generalized weakly relaxed α–monotone and hemicontinuous in the second argument. $b : E \times E \to \mathbb{R}$ is a convex lower semicontinuous function in second argument and $\alpha : E \to \mathbb{R}$ is a differentiable convex functional, such that*

(1) $u \to \alpha'(u)$ is continuous from weak to strong topology and α is strongly convex,
(2) ϕ is λ–Lipschitz continuous and strongly monotone with respect to the second and third arguments, respectively and $\phi(y, w, v) = -\phi(w, y, v)$,
(3) $b : E \times E \to \mathbb{R}$ satisfies following conditions,
  (i) $b(\cdot, v)$ is linear,
  (ii) $b(u, v)$ is bounded, that is there exists a positive constant μ, such that,

$$b(u, v) \leq \mu\|u\|\|v\|, \text{ for all } u, v \in K,$$

  (iii) $b(u, v) - b(u, w) \leq b(u, v - w)$, for all $u, v, w \in K$,
(4) $\frac{\sigma}{2\rho} \neq \lambda - \alpha$ and $0 < \rho < \frac{2(\sigma+2\rho\alpha-2\rho\lambda)(\alpha-\lambda-\mu)}{\rho\mu^2}$.

Then there exists a solution $w_{n+1} \in K$ to the auxiliary variational inequality problem for each $\rho > 0$ and the approximants converge strongly to the exact solution.

6. Concluding Remarks

In this work, we have studied the existence of the solution of nonlinear mixed variational-like inequality with respect to generalized weakly realaxed η–α monotone mapping in case of both bounded and unbounded sets. We have obtained an iterative algorithm using auxiliary principle technique and we have shown that the iterates approximate to the exact solution strongly. The results obtained are extended to corresponding equilibrium problem. Further we are trying to obtain the convergence rate and also to frame this problem in nonconvex setting using hemivariational inequality concept.

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