ASYMPTOTICALLY $\mathcal{I}_2$-CESÀRO EQUIVALENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, we defined concept of asymptotically $\mathcal{I}_2$-Cesàro equivalence and investigate the relationships between the concepts of asymptotically strongly $\mathcal{I}_2$-Cesàro equivalence, asymptotically strongly $\mathcal{I}_2$-lacunary equivalence, asymptotically $p$-strongly $\mathcal{I}_2$-Cesàro equivalence and asymptotically $\mathcal{I}_2$-statistical equivalence of double sequences of sets.

1. INTRODUCTION

The concept of convergence of real number sequences has been extended to statistical convergence independently by Fast [8] and Schoenberg [23]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [12] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

Freedman et al. [9] established the connection between the strongly Cesàro summable sequences space and the strongly lacunary summable sequences space. Connor [4] gave the relationships between the concepts of statistical and strongly $p$-Cesàro convergence of sequences.

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1, 3, 10, 14, 27, 32]). The concepts of statistical convergence and lacunary statistical convergence of sequences of sets were studied in [14, 27]. Also, new convergence notions, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence, Wijsman $\mathcal{I}$-statistical convergence and Wijsman $\mathcal{I}$-Cesàro summability by using ideal were introduced in [10, 11, 30].

Nuray et al. [17] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationship between them. Also, Nuray et al. [15] studied the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2$-Cauchy double sequences of sets. Ulusu et al. [26] studied $\mathcal{I}_2$-Cesàro summability of double sequences of sets. Dündar et al. [7] investigated $\mathcal{I}_2$-lacunary statistical convergence of double sequences of sets.
Marouf [13] presented definitions for asymptotically equivalent and asymptotic regular matrices. This concept was investigated in [19, 21].

The concept of asymptotically equivalence of real numbers sequences which is defined by Marouf [13] has been extended by Ulusu and Nuray [28] to concepts of Wijsman asymptotically equivalence of set sequences. Moreover, natural inclusion theorems are presented. Kiş et al. [11] introduced the concepts of Wijsman asymptotically \( I \)-equivalence of sequences of sets. Ulusu [24] investigated asymptotically \( I \)-Cesàro equivalence of sequences of sets.

2. Definitions and Notations

Now, we recall the basic definitions and concepts (See [3, 5, 7, 12, 15, 17, 22, 25, 26, 29, 31]).

Let \( (X, \rho) \) be a metric space. For any point \( x \in X \) and any non-empty subset \( A \) of \( X \), we define the distance from \( x \) to \( A \) by

\[
d(x, A) = \inf_{a \in A} \rho(x, a).\]

Throughout the paper we take \( (X, \rho) \) be a separable metric space and \( A, A_{kj} \) be non-empty closed subsets of \( X \).

The double sequence \( \{A_{kj}\} \) is Wijsman convergent to \( A \) if

\[
P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \quad \text{or} \quad \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)
\]

for each \( x \in X \) and we write \( \text{W}_{2} - \lim A_{kj} = A \).

The double sequence \( \{A_{kj}\} \) is Wijsman statistically convergent to \( A \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),

\[
\lim_{m,n \to \infty} \frac{1}{mn} \left| \left\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \right| = 0,
\]

that is,

\[
|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a.} \ (k,j)
\]

and we write \( \text{st}_{2} - \lim_{I} A_{k} = A \).

Let \( X \neq \emptyset \). A class \( I \) of subsets of \( X \) is said to be an ideal in \( X \) provided:

\[ i) \emptyset \in I, \quad ii) A, B \in I \implies A \cup B \in I, \quad iii) A \in I, \quad B \subset A \implies B \in I. \]

\( I \) is called a non-trivial ideal if \( X \notin I \).

A non-trivial ideal \( I \) in \( X \) is called admissible if \( \{x\} \in I \) for each \( x \in X \). Throughout the paper we take \( I_{2} \) as an admissible ideal in \( \mathbb{N} \times \mathbb{N} \).

A non-trivial ideal \( I_{2} \) of \( \mathbb{N} \times \mathbb{N} \) is called strongly admissible if \( \{i\} \times \mathbb{N} \) and \( \mathbb{N} \times \{i\} \) belong to \( I_{2} \) for each \( i \in \mathbb{N} \).

Let \( X \neq \emptyset \). A non-empty class \( F \) of subsets of \( X \) is said to be a filter in \( X \) provided:

\[ i) \emptyset \notin F, \quad ii) A, B \in F \implies A \cap B \in F, \quad iii) A \in F, \quad A \subset B \implies B \in F. \]

If \( I \) is a non-trivial ideal in \( X \), \( X \neq \emptyset \), then the class

\[
F(I) = \{ M \subset X : (\exists A \in I)(M = X \setminus A) \}
\]

is a filter on \( X \), called the filter associated with \( I \).

The double sequence \( \{A_{kj}\} \) is \( I_{W_{2}} \)-convergent to \( A \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),

\[
\{ (k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \} \in I_{2}
\]

and we write \( I_{W_{2}} - \lim A_{kj} = A \).
The double sequence \( \{A_{k,j}\} \) is Wijsman \( \mathcal{I}_2 \)-Cesàro summable to \( A \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),
\[
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{k,j}) - d(x, A) \right| \geq \varepsilon \right\} \in \mathcal{I}_2
\]
and we write \( A_{k,j} \xrightarrow{C_{1}[\mathcal{I}_2]} A \).

The double sequence \( \{A_{k,j}\} \) is Wijsman strongly \( \mathcal{I}_2 \)-Cesàro summable to \( A \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),
\[
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{k,j}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2
\]
and we write \( A_{k,j} \xrightarrow{C_{1}[\mathcal{I}_2]} A \).

The double sequences \( \{A_{k,j}\} \) is Wijsman \( p \)-strongly \( \mathcal{I}_2 \)-Cesàro summable to \( A \) if for every \( \varepsilon > 0 \), for each \( p \) positive real number and for each \( x \in X \),
\[
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{k,j}) - d(x, A)|^p \geq \varepsilon \right\} \in \mathcal{I}_2
\]
and we write \( A_{k,j} \xrightarrow{C_{p}[\mathcal{I}_2]} A \).

The double sequence \( \{A_{k,j}\} \) is Wijsman \( \mathcal{I}_2 \)-statistical convergent to \( A \) or \( S(\mathcal{I}_2) \)-convergent to \( A \) if for every \( \varepsilon > 0, \delta > 0 \) and for each \( x \in X \),
\[
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left\{ k \leq m, j \leq n : |d(x, A_{k,j}) - d(x, A)| \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}_2
\]
and we write \( A_{k,j} \rightarrow A(\mathcal{I}_2) \).

The double sequence \( \theta = \{(k_r, j_u)\} \) is called double lacunary sequence if there exist two increasing sequence of integers such that
\[
k_0 = 0, \quad h_r = k_r - k_{r-1} \rightarrow \infty \quad \text{as} \quad r \rightarrow \infty
\]
and
\[
j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \rightarrow \infty \quad \text{as} \quad u \rightarrow \infty.
\]

We use the following notations in the sequel:
\[
k_{(r,u)} = k_r j_u, \quad h_{(r,u)} = h_r \bar{h}_u, \quad I_{(r,u)} = \{(k,j) : k_{j-1} < k \leq k_r \quad \text{and} \quad j_{u-1} < j \leq j_u\},
\]
\[
q_r = \frac{k_r}{k_{r-1}} \quad \text{and} \quad q_u = \frac{j_u}{j_{u-1}}.
\]

The double sequence \( \{A_{k,j}\} \) is said to be Wijsman strongly \( \mathcal{I}_2 \)-lacunary convergent to \( A \) or \( N_0[\mathcal{I}_2] \)-convergent to \( A \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),
\[
A(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{(r,u)}} |d(x, A_{k,j}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2
\]
and we write \( A_{k,j} \rightarrow A(N_0[\mathcal{I}_2]) \).

We define \( d(x; A_{k,j}, B_{k,j}) \) as follows:
\[
d(x; A_{k,j}, B_{k,j}) = \begin{cases} 
\frac{d(x, A_{k,j})}{d(x, B_{k,j})}, & x \notin A_{k,j} \cup B_{k,j} \\
L, & x \in A_{k,j} \cup B_{k,j}.
\end{cases}
\]
The double sequences \( \{A_{k_j}\} \) and \( \{B_{k_j}\} \) are Wijsman asymptotically equivalent of multiple \( L \) if for each \( x \in X \),
\[
\lim_{k,j \to \infty} d(x; A_{k_j}, B_{k_j}) = L
\]
and we write \( A_{k_j} \sim^{W_L} B_{k_j} \) and simply Wijsman asymptotically equivalent if \( L = 1 \).

The double sequences \( \{A_{k_j}\} \) and \( \{B_{k_j}\} \) are Wijsman asymptotically \( \mathcal{I}_2 \)-equivalent of multiple \( L \) if for every \( \varepsilon > 0 \) and each \( x \in X \)
\[
\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon\} \in \mathcal{I}_2
\]
and we write \( A_{k_j} \sim^{\mathcal{I}_2} B_{k_j} \) and simply Wijsman asymptotically \( \mathcal{I}_2 \)-equivalent if \( L = 1 \).

The double sequences \( \{A_{k_j}\} \) and \( \{B_{k_j}\} \) are Wijsman asymptotically \( \mathcal{I}_2 \)-statistical equivalent of multiple \( L \) if for every \( \varepsilon, \delta > 0 \) and for each \( x \in X \),
\[
\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{(k,j) \in I_{r,u}} |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{I}_2
\]
and we write \( A_{k_j} \sim^{S,\mathcal{I}_2} B_{k_j} \) and simply Wijsman asymptotically \( \mathcal{I}_2 \)-statistical equivalent if \( L = 1 \).

Let \( \theta \) be a double lacunary sequence. The double sequences \( \{A_{k_j}\} \) and \( \{B_{k_j}\} \) are said to be Wijsman asymptotically strongly \( \mathcal{I}_2 \)-lacunary equivalent of multiple \( L \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),
\[
\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{hrhu} \sum_{(k,j) \in I_{r,u}} |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\} \in \mathcal{I}_2
\]
and we write \( A_{k_j} \sim^{\mathcal{I}_2,h} B_{k_j} \) and simply Wijsman asymptotically strongly \( \mathcal{I}_2 \)-lacunary equivalent if \( L = 1 \).

\[X \subset \mathbb{R}, \ f, g : X \to \mathbb{R} \text{ functions and a point } a \in X' \text{ are given. If } f(x) = \alpha(x)g(x) \text{ for } \forall x \in \overset{\circ}{U}_\delta(a) \cap X, \text{ then for } x \in X \text{ we write } f = \mathcal{O}(g) \text{ as } x \to a, \text{ where for any } \delta > 0, \alpha : X \to \mathbb{R} \text{ is bounded function on } \overset{\circ}{U}_\delta(a) \cap X. \text{ In this case, if there exists a } c \geq 0 \text{ such that } |f(x)| \leq c|g(x)| \text{ for } \forall x \in \overset{\circ}{U}_\delta(a) \cap X, \text{ then for } x \in X, f = \mathcal{O}(g) \text{ as } x \to a.\]

3. Main Results

In this section, we defined concepts of asymptotically \( \mathcal{I}_2 \)-Cesàro equivalence, asymptotically strongly \( \mathcal{I}_2 \)-Cesàro equivalence and asymptotically \( p \)-strongly \( \mathcal{I}_2 \)-Cesàro equivalence of double sequences of sets. Also, we investigate the relationship between the concepts of asymptotically strongly \( \mathcal{I}_2 \)-Cesàro equivalence, asymptotically strongly \( \mathcal{I}_2 \)-lacunary equivalence, asymptotically \( p \)-strongly \( \mathcal{I}_2 \)-Cesàro equivalence and asymptotically \( \mathcal{I}_2 \)-statistical equivalence of double sequences of sets.

**Definition 3.1.** The double sequence \( \{A_{k_j}\} \) and \( \{B_{k_j}\} \) are asymptotically \( \mathcal{I}_2 \)-Cesàro equivalence of multiple \( L \) if for every \( \varepsilon > 0 \) and for each \( x \in X \),
\[
\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x; A_{k_j}, B_{k_j}) - L \geq \varepsilon \right\} \in \mathcal{I}_2.
\]
In this case, we write $A_{kj} \overset{I_2}{\sim} B_{kj}$ and simply asymptotically $I_2$-Cesàro equivalent if $L = 1$.

**Definition 3.2.** The double sequence \{A_{kj}\} and \{B_{kj}\} are asymptotically strongly $I_2$-Cesàro equivalence of multiple $L$ if for every $\varepsilon > 0$ and for each $x \in X$,

\[
\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon \right\} \in I_2.
\]

In this case, we write $A_{kj} \overset{I_2}{\sim} B_{kj}$ and simply asymptotically strongly $I_2$-Cesàro equivalent if $L = 1$.

**Theorem 3.3.** Let $\theta$ be a double lacunary sequence. If $\lim \inf_r q_r > 1$, $\lim \inf_u q_u > 1$, then

\[
A_{kj} \overset{I_2}{\sim} B_{kj} \Rightarrow A_{kj} \overset{N_2}{\sim} B_{kj}.
\]

**Proof.** Let $\lim \inf_r q_r > 1$ and $\lim \inf_u q_u > 1$. Then, there exist $\lambda, \mu > 0$ such that $q_r \geq 1 + \lambda$ and $q_u \geq 1 + \mu$ for all $r, u \geq 1$, which implies that

\[
\frac{k_r j_u}{h_r h_u} \leq \frac{(1 + \lambda)(1 + \mu)}{\lambda \mu} \quad \text{and} \quad \frac{k_{r-1} j_{u-1}}{h_r h_u} \leq \frac{1}{\lambda \mu}.
\]

Let $\varepsilon > 0$ and for each $x \in X$ we define the set

\[
S = \left\{(k_r, j_u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r j_u} \sum_{i,s=1}^{k_r j_u} |d(x; A_{is}, B_{is}) - L| < \varepsilon \right\}.
\]

We can easily say that $S \in F(I_2)$, which is a filter of the ideal $I_2$, so we have

\[
\frac{1}{h_r h_u} \sum_{(i,s) \in I_r} |d(x; A_{is}, B_{is}) - L|
\]

\[
= \frac{1}{h_r h_u} \sum_{i,s=1}^{k_r j_u} |d(x; A_{is}, B_{is}) - L|
\]

\[
- \frac{1}{h_r h_u} \sum_{i,s=1}^{k_{r-1} j_{u-1}} |d(x; A_{is}, B_{is}) - L|
\]

\[
= \frac{k_r j_u}{h_r h_u} \left( \frac{1}{k_r j_u} \sum_{i,s=1}^{k_r j_u} |d(x; A_{is}, B_{is}) - L| \right)
\]

\[
- \frac{k_{r-1} j_{u-1}}{h_r h_u} \left( \frac{1}{k_{r-1} j_{u-1}} \sum_{i,s=1}^{k_{r-1} j_{u-1}} |d(x; A_{is}, B_{is}) - L| \right)
\]

\[
\leq \left( \frac{(1 + \lambda)(1 + \mu)}{\lambda \mu} \right) \varepsilon - \left( \frac{1}{\lambda \mu} \right) \varepsilon'
\]
for every \( \varepsilon' > 0 \), for each \( x \in X \) and \( (k_r, j_u) \in S \). Choose \( \eta = \left( \frac{(1+\lambda)(1+\mu)}{\lambda \mu} \right) \varepsilon' + \left( \frac{1}{\lambda \mu} \right) \varepsilon'. \) Therefore,

\[
\left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r h_u} \sum_{(k,j) \in I_{ru}} |d(x; A_{k,j}, B_{k,j}) - L| < \eta \right\} \in \mathcal{F}(\mathcal{I}_2)
\]

and it completes the proof. \( \square \)

**Theorem 3.4.** Let \( \theta \) be a double lacunary sequence. If \( \lim \sup_q r_q < \infty, \lim \sup_u q_u < \infty \), then

\[
A_{k,j}^{N_r [I_{\theta}]} B_{k,j} \Rightarrow A_{k,j}^{C^+ [I_{\theta}]} B_{k,j}.
\]

**Proof.** Let \( \lim \sup_q q_r < \infty \) and \( \lim \sup_u q_u < \infty \). Then, there exist \( M, N > 0 \) such that

\[
q_r < M \quad \text{and} \quad q_u < N \quad \text{for all} \quad r, u \geq 1.
\]

Let \( A_{k,j}^{N_r [I_{\theta}]} B_{k,j} \) and for \( \varepsilon_1, \varepsilon_2 > 0 \) define the sets \( T \) and \( R \) such that

\[
T = \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r h_u} \sum_{(k,j) \in I_{ru}} |d(x; A_{k,j}, B_{k,j}) - L| < \varepsilon_1 \right\}
\]

and

\[
R = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{k,j}, B_{k,j}) - L| < \varepsilon_2 \right\},
\]

for each \( x \in X \). Let

\[
a_{t,v} = \frac{1}{h_t h_v} \sum_{(i,s) \in I_{tv}} |d(x; A_{i,s}, B_{i,s}) - L| < \varepsilon_1,
\]

for each \( x \in X \) and for all \( (t, v) \in T \). It is obvious that \( T \in \mathcal{F}(\mathcal{I}_2) \).

Choose \( m, n \) is any integer with \( k_{r-1} < m < k_r \) and \( j_{u-1} < n < j_u \), where \( (r, u) \in T \). Then, for each \( x \in X \) we have

\[
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{k,j}, B_{k,j}) - L| \leq \frac{1}{k_{r-1} j_{u-1}} \sum_{k,j=1,1}^{k_r j_u} |d(x; A_{k,j}, B_{k,j}) - L|
\]

\[
= \frac{1}{k_{r-1} j_{u-1}} \left( \sum_{(k,j) \in I_{t_1}} |d(x; A_{k,j}, B_{k,j}) - L| + \sum_{(k,j) \in I_{t_2}} |d(x; A_{k,j}, B_{k,j}) - L| + \sum_{(k,j) \in I_{t_3}} |d(x; A_{k,j}, B_{k,j}) - L| + \cdots + \sum_{(k,j) \in I_{t_u}} |d(x; A_{k,j}, B_{k,j}) - L| \right)
\]
\[
\begin{align*}
&= \frac{k_{ij}}{k_{r-1}a_{11}} \left( \frac{1}{h_1h_2} \sum_{(k,j) \in I_{11}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
&\quad + \frac{k_1(j_2-j_1)}{k_{r-1}a_{12}} \left( \frac{1}{h_1h_2} \sum_{(k,j) \in I_{12}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
&\quad + \frac{(k_2-k_1)j_1}{k_{r-1}a_{21}} \left( \frac{1}{h_1h_2} \sum_{(k,j) \in I_{21}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
&\quad + \frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}a_{22}} \left( \frac{1}{h_1h_2} \sum_{(k,j) \in I_{22}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
&\quad + \cdots + \frac{(k_r-k_{r-1})(j_{u-1})}{k_{r-1}a_{ru}} \left( \frac{1}{h_1h_2} \sum_{(k,j) \in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
&= \frac{k_{ij}}{k_{r-1}a_{11}} a_{11} + \frac{k_1(j_2-j_1)}{k_{r-1}a_{12}} a_{12} + \frac{(k_2-k_1)j_1}{k_{r-1}a_{21}} a_{21} \\
&\quad + \frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}a_{22}} a_{22} + \cdots + \frac{(k_r-k_{r-1})(j_{u-1})}{k_{r-1}a_{ru}} a_{ru} \\
&\leq \left( \sup_{(t,v) \in T} a_{tv} \right) \frac{k_{r-1}}{k_{r-1}a_{11}} \\
&\quad < \varepsilon_1 \cdot M \cdot N.
\end{align*}
\]

Choose \( \varepsilon_2 = \frac{\varepsilon_1}{M \cdot N} \) and in view of the fact that
\[
\bigcup_{(r,u) \in T} \{(m,n) : k_{r-1} < m < k_r, j_{u-1} < n < j_u \} \subset R,
\]
where \( T \in \mathcal{F}(I_2) \), it follows from our assumption on \( \theta \) that the set \( R \) also belongs to \( \mathcal{F}(I_2) \) and this completes the proof of the theorem.

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

**Theorem 3.5.** Let \( \theta \) be a double lacunary sequence. If \( 1 < \lim \inf_r q_r < \lim \sup_r q_r < \infty \) and \( 1 < \lim \inf_u q_u < \lim \sup_u q_u < \infty \), then
\[ A_{kj} \sim [I_{W_2}] B_{kj} \Leftrightarrow A_{kj} \sim [I_{W_2}] B_{kj}. \]

**Definition 3.6.** The double sequences \( \{A_{kj}\} \) and \( \{B_{kj}\} \) are asymptotically \( p \)-strongly \( I_2 \)-Cesàro equivalence of multiple \( L \) if for every \( \varepsilon > 0 \), for each \( p \) positive real number and for each \( x \in X \),
\[
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj}) - L|^p \geq \varepsilon \right\} \in \mathcal{I}_2.
\]

In this case, we write \( A_{kj} \sim [I_{W_2}] B_{kj} \) and simply asymptotically \( p \)-strongly \( I_2 \)-Cesàro equivalent if \( L = 1 \).

**Theorem 3.7.** If the sequences \( \{A_{kj}\} \) and \( \{B_{kj}\} \) are asymptotically \( p \)-strongly \( I_2 \)-Cesàro equivalence of multiple \( L \), then \( \{A_{kj}\} \) and \( \{B_{kj}\} \) are asymptotically \( I_2 \)-statistical equivalence of multiple \( L \).
Proof. Let $A_{k_j} \stackrel{L_p}{\sim} W_{2}$ $B_{k_j}$ and $\varepsilon > 0$ given. Then, for each $x \in X$ we have
\[
\sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p \geq \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p
\]
and so
\[
\frac{1}{\varepsilon^{p mn}} \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p \geq \frac{1}{mn} \left\{ k \leq m, j \leq n : |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\}.
\]
So for a given $\delta > 0$ and for each $x \in X$
\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left\{ k \leq m, j \leq n : |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\} \geq \delta \right\}
\leq \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p \geq \varepsilon^p \cdot \delta \right\} \in I_2.
\]
Therefore, $A_{k_j} \stackrel{S(I_{W_2})}{\sim} B_{k_j}$. \hfill \Box

**Theorem 3.8.** Let $d(x, A_{k_j}) = O(d(x, B_{k_j}))$. If $\{A_{k_j}\}$ and $\{B_{k_j}\}$ are asymptotically $I_2$-statistical equivalence of multiple $L$, then $\{A_{k_j}\}$ and $\{B_{k_j}\}$ are asymptotically $p$-strongly $I_2$-Cesàro equivalence of multiple $L$.

Proof. Suppose that $d(x, A_{k_j}) = O(d(x, B_{k_j}))$ and $A_{k_j} \stackrel{S(I_{W_2})}{\sim} B_{k_j}$. Then, there is an $M > 0$ such that
\[
|d(x; A_{k_j}, B_{k_j}) - L| \leq M,
\]
for all $k, j$ and for each $x \in X$. Given $\varepsilon > 0$ and for each $x \in X$, we have
\[
\frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p
= \frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p
+ \frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{k_j}, B_{k_j}) - L|^p
\leq \frac{1}{mn} M^p \cdot \left\{ k \leq m, j \leq n : |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\}
+ \frac{1}{mn} \varepsilon^p \cdot \left\{ k \leq m, j \leq n : |d(x; A_{k_j}, B_{k_j}) - L| < \varepsilon \right\}
\leq \frac{M^p}{mn} \cdot \left\{ k \leq m, j \leq n : |d(x; A_{k_j}, B_{k_j}) - L| \geq \varepsilon \right\} + \varepsilon^p.
Then, for any $\delta > 0$ and for each $x \in X$,
\[
\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1}^{m,n} |d(x; A_{k,j}, B_{k,j}) - L|^p \geq \delta \right\}
\subseteq \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{k \leq m, j \leq n : |d(x; A_{k,j}, B_{k,j}) - L| \geq \varepsilon \} \right| \geq \frac{\delta^p}{M^p} \right\} \in \mathcal{I}_2.
\]
Therefore, $A_{k,j} \overset{c_{I_2}[Iw_2]}{\longrightarrow} B_{k,j}$. □

References


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