Abstract. Non commutative $W^*$-algebras are characterized in terms of two orderings. These conditions improve the previous results. Some new operator inequalities are discussed. In particular, a decomposition of normal operators in terms of the Moore-Penrose inverse is established which in turn, is applied to prove a perturbation inequality. We obtain some inequalities for operators of the form $SR^\frac{1}{2}$.

1. Introduction

Let $H$ be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. A subset $X$ of $B(H)$ is a $W^*$-algebra on $H$ if $X$ is a $C^*$-algebra which is closed in the weak star topology or weak topology, for example see [2]. We note that a $W^*$-algebra is commutative if and only if zero is the only nilpotent element of the algebra. We denote the set of self-adjoint operators in $X$ by $X_{SA}$, that is, if $T \in X_{SA}$ then $T = T^*$ where $T^*$ is the adjoint of $T$. Let $(X_{SA})^*$ be the dual space of $X_{SA}$.

We prove the following theorem since it will be used frequently in this paper.

**Theorem A.** Let $X$ be a $C^*$-algebra. If there are some $A, B \in X$ with $\|A\|, \|B\| \leq 1$ such that

$$\|A + B\| > 1 + \|AB\|$$

then $X$ is noncommutative.

**Proof.** Suppose $X$ is commutative, so $X = C_0(K)$ with $K$ a locally compact Hausdorff space. For any $A, B \in X$ with $\|A\|, \|B\| \leq 1$ we then have for all $x \in K$ that

$$|A(x) + B(x)| \leq 1 + |A(x)B(x)|$$

since $|A(x)|, |B(x)| \leq 1$ (namely if $r, s \in \mathbb{R}$ then $r + s \leq 1 + rs$).

The proof is complete by taking the sup on the right, and then on the left. \qed

We define two ordered relations on $X_{SA}$. The first ordering on $X_{SA}$ is defined as follows. For every $U$ in $X_{SA}$ there is a $U_+ \geq 0 \ni: U_+ - U \geq 0, U' \geq U_+$ for every $U' \in X_{SA}$ that satisfies $U' \geq 0$, $(U' - U) \geq 0$.

---

**2010 Mathematics Subject Classification.** 46H05; 47A10, 47A12, 47A30.

**Key words and phrases.** Numerical radius, operator inequality, star algebras, and Moore-Penrose inverse.

©2016 Ilirias Publications, Prishtinë, Kosovë.
Submitted June 12, 2016. Published September 5, 2016.

88
The second ordering on $X_{SA}$ is defined as follows. Suppose $T_1, T_2, T_3$ are three positive operators in $X_{SA}$ such that $T_2 + T_3 \geq T_1$. Then there exist operators $R$ and $S$ in $X_{SA}$ with

$$R \leq T_2, S \leq T_3, R + S = T_1 \leq T_2 + T_3.$$ 

2. **Non-Commutative $W^*$-Algebras**

In this section we present a characterization for non-commutative $W^*$-algebras in terms of the first and second orderings on $X_{SA}$. The proof uses the ideas of Theorem A and the following Theorem B of this paper. This result is simpler than the Theorem 1 in [8].

**Theorem B.** Suppose that $X$ is a $W^*$-algebra. Let $\lambda, \mu, \sigma$ be the real numbers with $\sigma > 0, \lambda > 0$. Then there are operators $T_1, T_2, T_3 \geq \lambda T_1 + \mu T_2 + \sigma T_3 \geq 0 \iff \lambda \sigma \geq \mu^2$

**Proof.** By Theorem A we have $\lambda T_1 + \mu T_2 + \sigma T_3 = \Delta = \begin{pmatrix} \lambda (SS^*) & \mu (\sqrt{SS^*}) \\ \mu (\sqrt{SS^*}) & \sigma (S^*S) \end{pmatrix}$. Here $\|S\| = 1$. The theorem can be proved by reducing the matrix $\Delta$ into the product of three matrices. Let $L = \begin{pmatrix} \beta & \gamma \\ \gamma & \delta \end{pmatrix}$, $W = \mu (\sigma)^{-1} (\sqrt{SS^*}) S (\sqrt{SS^*} S)$. Then for $D = \begin{pmatrix} (\lambda - \frac{\mu^2}{\sigma}) SS^* & O \\ O & \sigma (S^*S) \end{pmatrix}$ we have $\Delta = L^* DL$. This is possible by the Partial Communication relation $(\sqrt{SS^*}) S = S (\sqrt{SS^*} S)$. Under the assumption about $\sigma$ and $S^* S$, the Sylvester type test applies. That is, $\Delta$ is a positive semi definite matrix if and only if $\sigma$ is positive and $(\lambda - \frac{\mu^2}{\sigma}) \geq 0$.

**Theorem 2.1.** The $W^*$-algebra $X$ is non-commutative if and only if the real dual space $(X_{SA})^* \neq X_{SA}$ does not have the first ordering.

**Proof.** Let $T$ be an operator such that $T$ is non-zero but its square is the zero operator. Suppose $N$ is the range of $T$ and $M$ is the orthogonal complement of $N$. Clearly, $H = N \otimes M$. If the algebra $X$ is not commuting and the dual space $(X_{SA})^*$ has the first ordering then an operator $D$ in $X_{SA}$ can be defined as a matrix

$$D = \begin{pmatrix} D_1 & D_2 \\ D_2^* & D_3 \end{pmatrix}.$$ 

A simple calculation shows that $D = D^*$. The positivity of $D_1$ and $D_2$ depends on whether the operator $D$ is positive. Now we construct the first ordering on $(X_{SA})^*$.

Suppose $u, v$, and $w$ are functionals in $(X_{SA})'$. For elements $a$ in $N$ in $b$ in $M$, let $(v - u)(D) = (D_3 b, b)$. Then $(v - u)(D) \geq 0$ because $v(D) = (D_1 a, a)$ is positive. Hence, $u \leq v \Rightarrow u_+ \leq v$.

To see that $u_+(T_3) = 0$, let $T_1 = \begin{pmatrix} SS^* & O \\ O & O \end{pmatrix}$ be an operator on $X$. Then $u(T_1) \leq u_+(T_1) \leq v(T_1) \Rightarrow u_+(T_1) = (T_1 a, a)$. Further, $u_+(T_2) \leq v(T_2) \Rightarrow u_+(T_2) = 0$ if $T_2 = \begin{pmatrix} O & O \\ O & S^* S \end{pmatrix}$.

Now, if $T_3 = \begin{pmatrix} O \\ SS^* (\sqrt{SS^*}) S \end{pmatrix}$ then

$$(v - u_+) (T_1 + T_3 + T_2) = -u_+(T_3), (v - u_+) (T_1 - T_3 + T_2) = u_+(T_3).$$

Hence, $u_+(T_3) = 0$ since $(-u_+(T_3))$, $u_+(T_3)$ are positive.
Next, we define the functional $r$ as $r(D) = (D(ua + b), (ua + b))$. To achieve a contradiction, we need to establish that when $n$ is greater than one, then $r$ is not greater than or equal to $u_+.

Let us compute $(r - u_+)(D)$. Since $u(D) = v(D) - (D_3b, a)$, it follows that

$$
(r - u_+)(D) = v(D) + 2(D_3b, b) + 2n\text{Re}(D_2b, a)
$$

(1)

$$
= (D_1a, a) + 2(D_3b, b) + 2n\text{Re}(D_2b, a)
$$

$$
= (D(a + nb), (a + nb))
$$

Clearly, $(r - u_+)(D) = r(D)$, if $n = 1$. Also, from (1) we have $r \geq u$. Let $T_1 + T_2 + T_3 = \Psi.

Then $\Psi = \left( \begin{array}{cc}
SS^* & (\sqrt{SS^*})S \\
S^*(\sqrt{SS^*}) & SS^* 
\end{array} \right)$ and because $u_+(T_2) = 0 = u_+(T_3)$ the following relation is true:

$$(r - u_+)(\Psi) = u_+(T_1) + (T_2b, b) + 2n\text{Re}(T_3b, a) = ((T_1 + nT_3 + T_2)(a + b), (a + b)) .$$

Obviously, $n = 1 \Rightarrow (r - u_+)(\Psi) = (\Psi(a + b), (a + b))$. In order to have $(r - u_+)$ not greater than or equal to zero we must have by Theorem B, $(1 - n^2)$ also not greater than or equal to zero. That is, $n$ must be greater than 1. Therefore, we can choose the values of $a, b, a$, and $n$ so that $r < u_+$. This proves that if $X$ is not commutative then the dual space $(X_{SA})^*$ does not have the first ordering. □

The other implication is trivial and the proof is omitted.

An application of Theorem B is given below and it provides a necessary and a sufficient condition for $(X_{SA})^*$ to possess the second ordering.

**Theorem 2.2.** The second ordering exists in $X_{SA}$ if and only if $X$ is commutative.

**Proof.** If $X$ is commutative then the second ordering exists on $X_{SA}$. Conversely, let us assume that $X$ is not commutative but $X_{SA}$ has the second ordering. If $X$ is not commutative then by Theorem A there exists a nonzero operator $T$ such that its square is the zero operator. The second ordering allows us to have operators $\Phi, \Gamma, \Omega$ defined in terms of the operators $T_1, T_2, T_3$ as follows:

$$
T_3 = \Omega - \Gamma - 6\Phi
$$

$$
T_2 = -\Omega + 4\Phi + 2\Gamma
$$

$$
T_1 = 2\Phi
$$

We note that $(\Gamma + \Omega - \Phi) \geq 0$ by Theorem B. Also, the order of these operators is important. To see this, consider $(\Gamma + \Omega - \Phi) = 3T_3 + 2T_1 + 2T_2$. This order of operators does not satisfy Theorem B.

Our next goal in this construction is to decompose the operator $\Phi$ as a sum $\Phi = \Phi_1 + \Phi_2$ so that $\Phi_1 = 0$. We assume that $\Phi_1 \leq \Gamma, \Phi_2 \leq \Omega$. Using the matrix representation of $\Phi_1$ and by the condition that $\Phi \geq \Phi_1$, one can write $\Phi_1 = \left( \begin{array}{c}
\theta \\
O \\
O
\end{array} \right)$ for some $\theta$.

If $a \in N, b \in M$ then we obtain the following inequality from theorem B.

$$
||\sqrt{SS^*}a + Sb||^2 = (\Psi(a + b), (a + b)) \geq (\theta a, a)
$$

(II)
We remark that from Theorem B one can also have the following inequality for \( \lambda T_1 + \mu T_2 + \sigma T_3 = \Delta \). That is,
\[
\| (\Delta - \epsilon) h \|^2 = \| \Delta h \|^2 - 2 \epsilon (\Delta h, h) + \epsilon^2 \|h\|^2 \geq \epsilon^2 \|h\|^2
\]
By letting \( \lambda = \mu = \sigma = 1 \) the relation (II) holds.
Now, for \( b_n \in M_c(\sqrt{SS^*}a + Sb_n) \to 0 \) as \( n \to \infty \). This means that \( \Phi_1 = 0 \). Accordingly, \( \Phi = \Phi_2 \) and \( T_1 \leq 2 \Omega \Rightarrow T_1 - 8T_1 - 4T_3 - 2T_2 \leq 0 \Rightarrow 7T_1 + 4T_3 + 2T_2 \geq 0 \Leftrightarrow (7)(2) \geq 0 \) (Theorem B) which is false and contrary to our assumption. Thus, \( X \) has to be commutative.

3. SOME NORM INEQUALITIES FOR OPERATORS
This section contains some norm inequalities for operators on a Hilbert space \( H \). These results were obtained during the course of the current research. For the sake of completeness we include them here.

**Theorem 3.1.** Let \( T \) and \( S \) be the bounded linear operators on \( H \). For a vector \( h \) in \( H \) where the norm of \( h \) equals the number one and for a positive constant \( m \) the following implication holds:
\[
\|T - S\| \leq m \Rightarrow \| (T^*T)^n + (S^*S)^n \| \leq 2 \omega ((S^*T)^n) + m^2
\]

**Proof.** Here, by \( w(A) \) we mean the numerical radius of an operator \( A \). For \( h \) in \( H \) and \( \|T - S\| \leq m \) we have
\[
\left( (T^*T)^n + (S^*S)^n \right) h, h \right) = (T^n h, T^n h) + (S^n h, S^n h) = \|T^n h\|^2 + \|S^n h\|^2 \leq 2 \text{Re}(T^n h, S^n h) + m^2, m > 0
\]
(I)
That is, 
\[
\left( (T^*T)^n + (S^*S)^n \right) h, h \right) \leq 2 \text{Re}(T^n h, S^n h) + m^2, m > 0
\]
The operator \((T^*T)^n + (S^*S)^n\) is self-adjoint and hence its numerical radius equals its norm. Thus the theorem follows by taking the ‘sup’ of both sides of (I). It is obvious to observe that 
\[
\| (T^*T)^n + (S^*S)^n \| - 2 \omega ((S^*T)^n) \leq \|T^n - S^n\|^2
\]
since \( \|T^n h\|^2 + \|S^n h\|^2 \geq 2 \|T^n h\|^2 \|S^n h\|^2 \geq 2 \|T^n h, S^n h\| \).

Also, if \( T \) is a normal operator and \( \|T - T^*\| \leq m \) then the following inequalities are true.
\[
\|T^*T + S^*S\| \leq 2w(T^2) + m^2;
\]
\[
\|TT^*\| = \|T^*T\| \leq w(T^2) + \frac{1}{2} m^2
\]

The central idea in the proof of Theorem 3.2 is based on the concept of the generalized Moore-Penrose inverse of an operator on a Hilbert space and the “Moore-Penrose Equations. We briefly give an account of these facts to make the presentation coherent and smooth. For more details on these topics, we refer the reader to [6] and [7].

The Moore-Penrose inverse \( E^\dagger \) of an operator \( E \) is defined as the unique linear extension of \( E^{-1} \) to \( D(E^\dagger) := R(E) + R(E)^\perp \) with
\[
N(E^\dagger) = R(E)^\perp
\]
Here, \( \bar{E} = E|_{N(E)^\perp} : N(E)^\perp \to R(E) \). The inverse \( E^\dagger \) is well defined:
Since $N(E) = \{0\}$ and $R(E) = R(\bar{E})$, the inverse $\bar{E}^{-1}$ exists. Let $y \in D(\bar{E}^\perp)$. Then $y$ can be uniquely written as $y = y_1 + y_2$ for $y_1 \in R(E)$, $y_2 \in R(E)^\perp$. Further, the linearity of $\bar{E}^\perp$ and the existence of condition (1) above imply that 

$$\bar{E}^{-1}y_1 = \bar{E}^\perp y$$

If $P$ and $Q$ are orthogonal projections onto $N(E)$ and $R(E)$ respectively then the “Moore-Penrose equations” hold true.

(2) $\quad EE^\perp E = E$

(3) $\quad E^\perp EE^\perp = E^\perp$

(4) $\quad E^\perp E = I - P$

(5) $\quad E^\perp E = Q|_{D(\bar{E}^\perp)}$

Also, $R(\bar{E}^\perp) = N(E)^\perp$.

The proof of these equalities is not difficult. For all $y \in D(\bar{E}^\perp)$, $\bar{E}^\perp y = \bar{E}^{-1}Qy \Rightarrow \bar{E}^\perp y \in R(\bar{E}^{-1}) = N(\bar{E}^\perp)$. For all $x \in N(\bar{E}^\perp): \bar{E}^\perp Ex = \bar{E}^{-1}E x = x$. This proves that $R(\bar{E}^\perp) = N(\bar{E}^\perp)$.

Now for any $y \in D(\bar{E}^\perp)$ the conditions: $R(\bar{E}^\perp) = N(\bar{E}^\perp)$ and $\bar{E}^{-1}Qy \in (E)^\perp$ imply that 

$$EE^\perp y = EE^\perp Qy = E \bar{E}^{-1}Qy = \bar{E}E^{-1}Qy = Qy.$$

Consequently, (5) holds. By the definition of “Moore-Penrose” inverse we have for all $x \in X: E^\perp Ex = \bar{E}^{-1}E(Px - (I - P)x) = (I - P)x$, thus the relation (2) is true. Conditions $R(\bar{E}^\perp) = N(\bar{E}^\perp)$ and (5) together imply (3).

The proof in the next theorem uses the above construction of operators which are decomposed in terms of the “Moore-Penrose” inverse.

**Theorem 3.2.** Let $A_1, A_2, B_1, B_2$ be commuting operators in $(X_{\mathbb{C}A})$ such that for all $x \in X$, \(|A_1x|^2 + |A_2x|^2\) $< \infty$, \(|B_1x|^2 + |B_2x|^2\) $< \infty$. Then the following inequality holds: \(|A_1B_1 + A_2B_2| \leq |AB|\) where $A^\ast = (A_1^* A_1 + A_2^* A_2)$ and $B^2 = (B_1^* B_1 + B_2^* B_2)$.

**Proof.** Let $C^\perp$ denote the orthogonal projection operator on the closure of the range of an operator $C$. Suppose that $\{y_n\}$ is a sequence of elements in $X$ and $x$ is in $X$ such that

(V) \[ \lim_{n \to \infty} A y_n = A^\perp x \]

Now the limits as $n \to \infty$ of $A_1y_n$ and $A_2y_n$ exist and they are independent of the selected sequence. In other words, if $\{y_m\}$ is any other sequence of elements of $X$ then by (V) the following relations hold:

\[ |A_1y_n - A_1y_m| \leq |Ay_n - Ay_m| \to 0 \text{ and } |A_2y_n - A_2y_m| \leq |Ay_n - Ay_m| \to 0 \text{ as } n, m \to \infty \]

Thus for an arbitrarily chosen sequence \(\{z_n\}\) the inequalities

\[ |A_1z_n - A_1z_m| \leq |Az_n - Az_m| \to 0 \text{ and } |A_2z_n - A_2z_m| \leq |Az_n - Az_m| \to 0 \]

are true.

The existence of the above limits helps us to define the limit operators $T$ and $S$. We will claim that the operators $T$ and $S$ are commuting contractions such that $TA = A_1 = AT$ and $SA = A_2 = AS$ with $A^\perp = TA^*T + S^*S$. In fact our goal is to write $T = cl(A_1 A^\perp) = cl(A^\perp A_1)$ and $S = cl(A_2 A^\perp) = cl(A^\perp A_2)$. Here, $cl$ means
the closure of a set and $A^\perp$ is the “Moore-Penrose” generalized inverse of $A$. This should not be difficult to see and the construction below will help us to achieve the desired forms of the operators $T$ and $S$.

Let $Tx = \lim_{n \to \infty} A_1 y_n$ and $Sx = \lim_{n \to \infty} A_2 y_n$ if (V) holds. Accordingly, $TA^\perp = T$ and $SA^\perp = S$. Moreover, $TA = A_1 = AT$ and $SA = A_2 = AS$. We also note that $A^\perp = T^* T + S^* S$. Since $A^\perp$ is an orthogonal projection. It is easy to verify that the equality $A^\perp A^2 = A^2 \Rightarrow T^* T A^\perp + S^* S A^\perp = A^\perp$, which together with the fact that $T(I - A^\perp) = 0 = S(I - A^\perp)$ establishes the “Moore-Penrose” inverse. As a matter of fact, these operators are commutative on the range space as well as on the null space of $A^2$. This proves our claim. Thus the operators $T$ and $S$ are normal contractions and commute with the operators $A_1$ and $A_2$. The decomposition of $A$ is complete and established. One can also have a commuting family $\{T_n\}$ of contractions which are self adjoint such that $T_n A = A T_n = A_n, \sum_n T_n T_n = A^\perp$. Clearly, this family of operators commutes with the family $\{A_n\}$ and $T_n = c(l(A_n A^\perp)) = c(l(A^\perp A_n))$. On the same lines we can factor the operator $B$. The consequent of the theorem follows from Theorem 2.5.1 in [3] and Theorem 1.19 of [9]. □

**Theorem 3.3.** For operators $A$ and $B$ in $X_{SA}$ the following inequality is true for all $p > 2$:

$$\left\| \left(\frac{1 + A^* A}{2}\right)^{\frac{p}{2}} \left(\frac{1 + B^* B}{2}\right)^{\frac{p}{2}} \right\| \leq \left\| \left(\frac{1 + (A^* A)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \left(\frac{1 + (B^* B)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \right\|$$

**Proof.** Let $A, B$, and $I$ be commuting operators. If $p > 2$ then the function $G(\alpha) = \alpha^{\frac{p}{2}}$ is a concave mapping. Basically for $p > 2$, the correspondence $\alpha \to (\alpha^2)^{\frac{p}{2}}$ is an operator monotonic mapping. Hence the following inequalities

$$\left(\frac{1 + A^* A}{2}\right)^{\frac{p}{2}} \geq \left(\frac{1 + A^* A}{2}\right)$$

and

$$\left(\frac{1 + (B^* B)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \geq \left(\frac{1 + B^* B}{2}\right)$$

imply

$$\left\| \left(\frac{1 + (A^* A)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \left(\frac{2}{1 + (A^* A)^{\frac{p}{2}}}\right)^{\frac{p}{2}} \right\| \leq 1 \text{ and } \left\| \left(\frac{1 + (B^* B)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \left(\frac{2}{1 + (B^* B)^{\frac{p}{2}}}\right)^{\frac{p}{2}} \right\| \leq 1.$$  

The theorem follows by combining the above inequalities. □

**Theorem 3.4.** For operators $T$ and $S$ in $X_{SA}$ the inequality $\|T + S\| \leq 2^{1 - \frac{1}{p}} \left\| (T^* T)^{\frac{p}{2}} + (S^* S)^{\frac{p}{2}} \right\|^{\frac{1}{p}}$ holds for all real $p \geq 2$.

**Proof.** From Theorem 3.3 the inequality $\left\| \frac{T + S}{2} \right\| \leq \frac{1}{m} \left\| \left(\frac{1 + m p (T^* T)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \left(\frac{1 + m p (S^* S)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \right\|$ exists for all $m > 0$. This implies that $\left\| \frac{T + S}{2} \right\| \leq \frac{1}{m} \left\| \left(\frac{1 + m p (T^* T)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \left(\frac{1 + m p (S^* S)^{\frac{p}{2}}}{2}\right)^{\frac{p}{2}} \right\|^\frac{1}{p}$

Hence an application of the geometric-arithmetic mean inequality yields the following inequality

$$\left\| \frac{T + S}{2} \right\| \leq \frac{1}{2m} \left\| \left(\frac{1 + m p (T^* T)^{\frac{p}{2}}}{2}\right) + \left(\frac{1 + m p (S^* S)^{\frac{p}{2}}}{2}\right) \right\|.$$
The proof is complete since the minimum on the right side of the above inequality is attained at \( m = \left\| \frac{2}{(T^*T)^{\frac{1}{2}} + (S^*S)^{\frac{1}{2}}} \right\|^{\frac{1}{2}}. \)

**Remark 3.5.** By Theorem 3.2 we can have an elementary inequality for normal operators \( T \) and \( S \). In other words, \( \| T + S \| \leq \left\| (I + T^*T)^{\frac{1}{2}} (I + S^*S)^{\frac{1}{2}} \right\| \). By Theorem 3.4 we have \( \| T + S \| \leq \sqrt{2}\| T^*T + S^*S \|^{\frac{1}{2}} \). The operator \( T^*T + S^*S \) is self-adjoint so its numerical radius equals its norm and hence, \( \| T + S \|^2 \leq 2w(T^*T + S^*S) \).

The above inequality can be compared with Theorem 3.1. Lastly, if \( T \) and \( S \) are self-adjoint and idempotent operators then \( \| T + S \| \leq \| I + T \| \| I + S \| \).

The next theorem proves a perturbation inequality for self-adjoint operators on \( X \). The theorem is useful in comparing a class of derivations on \( X \) which are induced by a pair of self-adjoint operators.

**Theorem 3.6.** For \( T \) and \( S \) in \( X_{SA} \) and for all \( \theta \in (0, 1) \) we have

\[
\left\| (T^*T)^{\frac{1}{2}} - (S^*S)^{\frac{1}{2}} \right\| \leq 2^{2-\theta} \| T - S \|^{\theta}
\]

**Proof.** If \( T \) and \( S \) belong to \( X_{SA} \) then for unitary operators \( U \) and \( V \) these operators can be written as : \( T = U(T^*T)^{\frac{1}{2}}, S = (S^*S)^{\frac{1}{2}} \). In this case,

\[
\left\| (T^*T)^{\frac{1}{2}} - (S^*S)^{\frac{1}{2}} \right\| = \left\| U (T^*T)^{\frac{1}{2}} - U (T^*T)^{\frac{1}{2}} V (S^*S)^{\frac{1}{2}} + U (T^*T)^{\frac{1}{2}} V (S^*S)^{\frac{1}{2}} - V (S^*S)^{\frac{1}{2}} \right\|
\]

\[
= \left\| U (T^*T)^{\frac{1}{2}} (U (T^*T)^{\frac{1}{2}} - V (S^*S)^{\frac{1}{2}}) + (U (T^*T)^{\frac{1}{2}} - V (S^*S)^{\frac{1}{2}}) V (S^*S)^{\frac{1}{2}} \right\|
\]

Let \( U (T^*T)^{\frac{1}{2}} - V (S^*S)^{\frac{1}{2}} = R. \) Then \( \left\| (T^*T)^{\frac{1}{2}} - (S^*S)^{\frac{1}{2}} \right\| = \left\| U (T^*T)^{\frac{1}{2}} R + RV (S^*S)^{\frac{1}{2}} \right\|. \)

Since \( T \) and \( S \) are self-adjoint and \( R \) is any operator on \( X \) the following inequality is justified from Theorem 3.4. That is,

\[
(T^*T)^{\frac{1}{2}} - (S^*S)^{\frac{1}{2}} \leq 2 \| R \|^{\frac{1-\theta}{2}} \left\| (T^*T)^{\frac{1-\theta}{2}} R + R (S^*S)^{\frac{1-\theta}{2}} \right\|^{\frac{\theta}{1-\theta}} \]

\[
\text{(VI)} \quad \left\| (T^*T)^{\frac{1}{2}} - (S^*S)^{\frac{1}{2}} \right\| \leq 2 \| R \|^{\frac{1-\theta}{2}} \left\| (T^*T)^{\frac{1-\theta}{2}} R + R (S^*S)^{\frac{1-\theta}{2}} \right\|^{\frac{\theta}{1-\theta}} \]

For \( \theta \in (0, 1) \), the following inequalities are also true.

\[
\| R \| \leq 2^{1-\theta} \| T - S \|^{\frac{\theta}{2}}, \quad \| R \|^{\frac{1-\theta}{2}} \leq 2^{1-\theta} \| T - S \|^{\frac{\theta}{2}}, \quad \left\| (T^*T)^{\frac{1-\theta}{2}} R + R (S^*S)^{\frac{1-\theta}{2}} \right\| \leq 2 \| T - S \|.
\]

The inequality in (VI) now yields the theorem.

**Remark 3.7.** The conclusion of Theorem 3.6 can also be written in the following form. For self-adjoint operators \( A \) and \( B \) if \( \| A - B \| \leq m, \) \( m > 0 \) and \( \theta \in (0, 1) \) then there exists a positive constant \( M \) so that \( \| |A|^\theta - |B|^\theta \| \leq M. \) Also, if \( B = I \) (Identity operator) then \( \| |A|^\theta - I \| \leq M. \)

The above remark has some value. As a matter of fact Theorem 3.1 and Remark 3.7 produce the following bound for \( \| |A|^\theta + |B|^\theta \|. \) In particular, we present the following theorem.
Theorem 3.8. If for $m > 0, \|A - B\| \leq m$ then

$$\|\|A\|^\theta + |B|\|^\theta\| \leq \delta w (|B|\|^\theta) w (|A|\|^\theta) + m^2, \forall \theta \in (0, 1)$$

The positive valuation of $\delta$ depends on whether the operators are commutative.

Proof. Recall the operators $A$ and $B$ as in Remark 3.7. Since $(|A|\|^\theta)^* (|A|\|^\theta) + (|B|\|^\theta)^* (|B|\|^\theta) = (|A|\|^\theta + |B|\|^\theta)$ for all $\theta \in (0, 1)$, by Theorem 3.1 we have $\|\|A\|^\theta + |B|\|^\theta\| \leq 2w (|B|\|^\theta|A|\|^\theta) + m^2$.

We consider two cases. First, if the operators are not commutative then by [4, page 37] the inequality $w(TS) \leq 4w(T)w(S)$ implies that $\delta = 8$. Second, if the operators are commutative we have $\delta = 4$ because $w(TS) \leq 2w(T)w(S)$ by [4, page 37]. We further remark that the self-adjoint property of the operator $(|A|\|^\theta + |B|\|^\theta)$ gives the following inequality:

$$w (|A|\|^\theta + |B|\|^\theta) = \|\|A\|^\theta + |B|\|^\theta\| \leq \delta w (|A|\|^\theta) w (|B|\|^\theta) + m^2, \forall \theta \in (0, 1), m > 0$$

Notation. Let $S^{-1}$ be the inverse of the operator $S$ such that $\|S^{-1}\| = \beta \neq 0$. We write $\|T\| = \alpha, w(T^2) = \alpha_0, \lambda = w(S^*T)$, and $z = w(T)$.

With these notations we present some applications of previous results of the current paper, especially Theorem 3.1.

Theorem 3.9. Let $T$ and $S$ be bounded linear operators on $H$. For $m > 0$, we have $\|T - S\| \leq m \Rightarrow \alpha \beta^{-1} \leq \lambda + 2^{-1}m^2$.

Proof. From Theorem 3.1 the inequality $\alpha^2 + \beta^{-2} \leq 2\lambda + m^2$ is valid. Thus the proof follows from the fact that $\alpha^2 + \beta^{-2} \leq 2\alpha\beta^{-1}$.

Corollary 3.10. For $0 < m \leq 1$ the inequality $\alpha^2 - \alpha_0 \leq (2 - \sqrt{1 - m^2})z$ holds.

Proof. If $S = I$ then $\|T - I\| \leq m$. Now the corollary follows from Theorem 3.9.

Corollary 3.11. Let $T$ and $S$ be bounded and linear operators on $H$ such that $S^{-1}$ exists. The inequality $\alpha^2 \leq \lambda^2 + 2(1 - \beta^{-1})z$ holds for $2^{-1} \leq \beta < 1$ and $\|T - S\| \leq 1$.

Proof. If $h$ belongs to $H$ and $\|h\| = 1$ then by Theorem 3.9 we have

$$(VII) \quad \|Th\|^2 + \beta^{-2} \leq 2|(Th, Sh)| + 1$$

By the hypothesis $|(Th, Sh)| > 0$ and if $\gamma = |(Th, Sh)|$ then (VII) reduces to the following inequality

$$(VIII) \quad \gamma^{-1}\|Th\|^2 - \gamma \leq (2 - \gamma) + \gamma^{-1}(1 - \beta^{-2}) \leq 2 \left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right)$$

Note that the factor $\left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right) \geq 0$ and by taking the ‘sup’ of both sides in (VIII) over $h$ we have the corollary. Observe that $\text{sup}(\gamma) = \lambda$.

Remark 3.12. In Corollary 3.10 if $S = T^*$ and the inverse of the operator $T$ exists then

$$\alpha^2 \leq \alpha_0^2 + 2\alpha_0 \left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right)$$
Theorem 3.13. If $T$ is a normal operator with $\|(T - T^*)h\| \leq 1 \leq \|T^*h\|, h \in H, \|h\| = 1$ then $\|T\|^4 \leq \|T\|^2 + (\omega(T^2))^2$.

Proof. By the Schwarz inequality we obtain $\|Th\|^2|T^*h\|^2 \leq |T^*h|^2 + \|(Th, T^*h)\|^2$. That is, $\|T\|^4 \leq \|T^*h\|^2 + \|(Th, T^*h)\|^2$ because $T$ is normal. Now the theorem follows by taking the 'sup' of both sides over $h$. \qed

4. Spectral Radius Inequalities For Operators of the Form $T = SR^{\frac{1}{2}}$

Let $H$ be a Hilbert space with a bounded inner-product $(x, y)$ where the norm of a vector $x$ is defined by $\|x\|_H = \sqrt{(x, x)}, \forall x \in H$. Since the inner-product is bounded it is a known fact that there exists a positive operator $R$ on $H$ such that $(x, y) = (Rx, y), \forall y \in H$. We also note that $\|x\| = \left\|R^{\frac{1}{2}}x\right\|_H, \forall x \in H$. For more on these operators we refer the reader to [1]. A general formula for spectral radius is given in [5] and we state the formula here. We will prove some results for the sum and the difference of these operators.

Formula A. If $A$ and $B$ belong to $B(H)$ then

$$r(A + B) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4Min(\|AB\|, \|BA\|)}\right)$$

In fact, one can have the following inequality. For $A, B, C$, and $D$ in $B(H)$

$$r(AB + CD) \leq \frac{1}{2}\left(\|BA\| + \|DC\| + \sqrt{(\|BA\| - \|DC\|)^2 + 4(\|BC\|\|DA\|)}\right)$$

We present the following inequality for operators of the form $T = SR^{\frac{1}{2}}$.

Theorem 4.1. Let $A, B, S$, and $T$ be operators in $B(H)$ such that $A = SR^{\frac{1}{2}}, B = TR^{\frac{1}{2}}$. Then $r(A + B) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\||\|T\|)}\right)$.

Proof. From Formula A:

$$r(A + B) = r\left(SR^{\frac{1}{2}} + TR^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\|SR^{\frac{1}{2}}\| + \|TR^{\frac{1}{2}}\| + \sqrt{(\|SR^{\frac{1}{2}}\| - \|TR^{\frac{1}{2}}\|)^2 + 4(\|SR^{\frac{1}{2}}\||\|TR^{\frac{1}{2}}\|)}\right)$$

We note the following observation. For any operator $G$ in $B(H)$

$$\|GR^{\frac{1}{2}}x\|^2 = (GR^{\frac{1}{2}}x, GR^{\frac{1}{2}}x) = (x, x) = \|x\|^2 \Rightarrow \|GR^{\frac{1}{2}}\| = \|G\|$$

Hence, $r(A + B) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\||\|T\|)}\right)$ \qed

We omit the proofs of corollaries 4.2 and 4.3 as they are straightforward. We denote the spectrum of an operator $P$ by $\sigma(P)$.

Corollary 4.2. Let $r(R^{\frac{1}{2}}) = 1$. Then

$$r(A + B) \leq r(S + T) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\||\|T\|)}\right)$$

Corollary 4.3. If $\sigma(A + B) \subseteq \sigma(S + T)$ then the conclusion of Corollary 4.2 holds.

Corollary 4.4. (Invariance Inequality):

$$r\left(R^{\frac{1}{2}}A + BR^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4Min(\|AB\|, \|BA\|)}\right)$$
Lemma 4.6. Let 

Proof. By Formula A we have the following inequality:

By [1] it follows that 

Also, by symmetry it follows that 

Theorem 4.7. If 

Then for such an operator 

Theorem 4.5. If 

The spectral mapping theorem asserts that for every positive integer 

The following result extends Corollary 4 of [5].

Lemma 4.6. Let 

Proof. Since the corresponding inner-product is bounded and 

The next result is a special case of Corollary 4 of [5].

Theorem 4.7. Let 

Proof. Let 

Theorem 4.5. If 

Lemma 4.8. Let the operator 

Proof. It is known that for operators 

Hence, 

□
Remark 4.9. If $B = AR$ and $C = DR$ then \[
\begin{bmatrix}
O & AR \\
DR & O
\end{bmatrix}
= \sqrt{r(ARDR)}.
\] If $R = I$, the identity operator in $B(H)$, then Lemma 4.8 is a special case of this remark. Theorem 4.7 and Lemma 4.8 produce a special case of Theorem 2 in [5]. In fact, we have the following inequality.

**Theorem 4.10.** Using the conditions of Theorem 4.7, we have

\[
2\sqrt{r(ARDR)} \leq \|T\|_H + \sqrt{\|T^2\|_H}.
\]

**Proof.** From Theorem 4.7 and Remark 4.9, we have

\[
2\sqrt{r(ARDR)} = 2r \begin{bmatrix}
O & AR \\
DR & O
\end{bmatrix} = r \begin{bmatrix}
O & -2AR \\
2DR & O
\end{bmatrix} = r(TU - UT) \leq \|T\|_H + \sqrt{\|T^2\|_H}
\]

and this completes the proof. \[\Box\]

**Remark 4.11.** Last but not least we remark that the above results present some pinch inequalities in form of matrices. Let $\nu = \begin{bmatrix} BR^2 & O \\
O & CR^2
\end{bmatrix}$. Then $\nu^t = \nu$ where $\nu^t$ is the transpose matrix of $\nu$. Note that if $\nu = \begin{bmatrix} BR^2 & B \\
C & CR^2
\end{bmatrix}$ then $\nu^t \neq \nu$.

With these notations the pinch inequalities can be written in the form given below. $\|\nu^t\|_H = \|\nu\|_H \leq \|\nu\|_H$. Further, if $t_E$ and $t_0$ denote the even and odd transposes respectively then $\|\nu^t\|_H \leq \|\nu\|_H$ and $\|\nu\|_H \leq \|\nu^{t_0}\|_H$ but $\|\nu\|_H$ is not less than or equal to $\|\nu^{t_0}\|_H$.

We also remark that the above inequalities are true for bounded extensions of (In the sense of [1]) operators. According to [1], if $T = SR^2$ where $S$ in $B(H)$, then $T$ has an extension $\bar{T} \in B(\bar{H})$. Here, $\bar{H}$ denotes the completion of $H$. For details on these operators, see [1].

**REFERENCES**


Department of Mathematics, Duquesne University, Pittsburgh, Pennsylvania 15282

E-mail address: gauraduq.edu