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# OPERATOR INEQUALITIES IN ALGEBRAS

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Abstract. Non commutative $W^*$-algebras are characterized in terms of two orderings. These conditions improve the previous results. Some new operator inequalities are discussed. In particular, a decomposition of normal operators in terms of the Moore-Penrose inverse is established which in turn, is applied to prove a perturbation inequality. We obtain some inequalities for operators of the form $SR^{\frac{1}{2}}$

## 1. Introduction

Let $H$ be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators on $H$. A subset $X$ of $B(H)$ is a $W^*$-algebra on $H$ if $X$ is a $C^*$-algebra which is closed in the weak star topology or weak topology, for example see [2]. We note that a $W^*$-algebra is commutative if and ony if zero is the only nilpotent element of the algebra. We denote the set of self-adjoint operators in $X$ by $X_{SA}$, that is, if $T \in X_{SA}$ then $T = T^*$ where $T^*$ is the adjoint of $T$. Let$(X_{SA})^*$ be the dual space of $X_{SA}$.

We prove the following theorem since it will be used frequently in this paper.

**Theorem A.** *Let $X$ be a $C^*$-algebra. If there are some $A, B \in X$ with $\|A\|,\|B\| \leq 1$ such that*

$$\|A + B\| > 1 + \|AB\| \tag{I}$$

*then $X$ is noncommutative.*

*Proof.* Suppose $X$ is commutative, so $X = C_0(K)$ with $K$ a locally compact Hausdorff space. For any $A$,$B \in X$ with $\|A\|,\|B\| \leq 1$ we then have for all $x \in K$ that

$$|A(x) + B(x)| \leq 1 + |A(x)B(x)|$$

since $|A(x)|, |B(x)| \leq 1$ (namely if $r$,$s \in \mathbb{R}$ then $r + s \leq 1 + rs \Leftrightarrow r(1 - s) \leq (1 - s)$) The proof is complete by taking the sup on the right, and then on the left. □

We define two ordered relations on $X_{SA}$. The first ordering on $X_{SA}$ is defined as follows. For every $U$ in $X_{SA}$ there is a $U_+ \geq 0 \ni$: $U_+ - U \geq 0, U' \geq U_+$ for every $U' \in X_{SA}$ that satisfies $U' \geq 0$, $(U' - U) \geq 0$.

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The second ordering on $X_{SA}$ is defined as follows. Suppose $T_1$,$T_2$,$T_3$ are three positive operators in $X_{SA}$ such that $T_2 + T_3 \geq T_1$. Then there exist operators $R$ and $S$ in $X_{SA}$ with

$$R \leq T_2, S \leq T_3, R + S = T_1 \leq T_2 + T_3.$$

## 2. Non-Commutative $W^*$-Algebras

In this section we present a characterization for non-commutative $W^*$-algebras in terms of the first and second orderings on $X_{SA}$. The proof uses the ideas of Theorem A and the following Theorem B of this paper. This result is simpler than the Theorem 1 in [8].

**Theorem B.** *Suppose that $X$ is a $W^*$-algebra. Let $\lambda, \mu, \sigma$ be the real numbers with $\sigma > 0, \lambda > 0$. Then there are operators $T_1, T_2, T_3 \ni$: $\lambda T_1 + \mu T_2 + \sigma T_3 \geq 0 \Leftrightarrow \lambda\sigma \geq \mu^2$*

*Proof.* By Theorem A we have $\lambda T_1 + \mu T_2 + \sigma T_3 = \Delta = \begin{pmatrix} \lambda(SS^*) & \mu(\sqrt{SS^*})S \\ \mu(S^*)(\sqrt{SS^*}) & \sigma(S^*S) \end{pmatrix}$. Here $\|S\| = 1$. The theorem can be proved by reducing the matrix $\Delta$ into a product of three matrices. Let $L = \left(\begin{smallmatrix} I & W \\ O & I \end{smallmatrix}\right)$, $W = \mu(\sigma)^{-1}(\sqrt{SS^*})S(\sqrt{S^*S})$.

For $D = \begin{pmatrix} (\lambda - \frac{\mu^2}{\sigma})SS^* & O \\ O & \sigma(S^*S) \end{pmatrix}$ we have $\Delta = L^*DL$. This is possible by the Partial Communication relation $(\sqrt{SS^*})S = S(\sqrt{S^*S})$. Under the assumption about $\sigma$ and $S^*S$, the Sylvester type test applies. That is, $\Delta$ is a positive semi definite matrix if and only if $\sigma$ is positive and $(\lambda - \frac{\mu^2}{\sigma}) \geq 0$. □

**Theorem 2.1.** *The $W^*$-algebra $X$ is non-commutative if and only if the real dual space $(X_{SA})^*$ does not have the first ordering.*

*Proof.* Let $T$ be an operator such that $T$ is non-zero but its square is the zero operator. Suppose $N$ is the range of $T$ and $M$ is the orthogonal complement of $N$. Clearly, $H = N \otimes M$. If the algebra $X$ is not commutative and the dual space $(X_{SA})^*$ has the first ordering then an operator $D$ in $X_{SA}$ can be defined as a matrix

$$D = \begin{pmatrix} D_1 & D_2 \\ D_2^* & D_3 \end{pmatrix}.$$

A simple calculation shows that $D = D^*$. The positivity of $D_1$ and $D_2$ depends on whether the operator $D$ is positive. Now we construct the first ordering on $(X_{SA})^*$.

Suppose $u, v$, and $r$ are functionals in $(X_{SA})'$. For elements $a$ in $N$ in $b$ in $M$, let $(v - u)(D) = (D_3 b, b)$. Then $(v - u)(D) \geq 0$ because $v(D) = (D_1 a, a)$ is positive. Hence, $u \leq v \Rightarrow u_+ \leq v$.

To see that $u_+(T_3) = 0$, let $T_1 = \begin{pmatrix} SS^* & O \\ O & O \end{pmatrix}$ be an operator on $X$. Then

$u(T_1) \leq u_+(T_1) \leq v(T_1) \Rightarrow u_+(T_1) = (T_1 a, a)$.Further, $u_+(T_2) \leq v(T_2) \Rightarrow u_+(T_2) = 0$

if $T_2 = \begin{pmatrix} O & O \\ O & S^*S \end{pmatrix}$

Now, if $T_3 = \begin{pmatrix} O & (\sqrt{SS^*})S \\ S^*(\sqrt{SS^*}) & O \end{pmatrix}$ then

$$(v - u_+)(T_1 + T_3 + T_2) = -u_+(T_3), (v - u_+)(T_1 - T_3 + T_2) = u_+(T_3).$$

Hence, $u_+(T_3) = 0$ since $(-u_+(T_3)), u_+(T_3)$ are positive.

Next, we define the functional $r$ as $r(D) = (D(na+b),(na+b))$. To achieve a contradiction, we need to establish that when $n$ is greater than one, then $r$ is not greater than or equal to $u_+$.

Let us compute $(r-u)(D)$. Since $u(D) = v(D) - (D_3b, a)$, it follows that

$$\begin{aligned}(r-u)(D) &= v(D) + 2(D_3b,b) + 2n\mathrm{Re}(D_2b,a)\\ &= (D_1a,a) + 2(D_3b,b) + 2n\mathrm{Re}(D_2b,a) \qquad \text{(I)}\\ &= (D(a+nb),(a+nb))\end{aligned}$$

Clearly,$(r-u)(D) = r(D)$, if $n = 1$. Also, from (I) we have $r \geq u$. Let $T_1 + T_2 + T_3 = \Psi$.

Then $\Psi = \begin{pmatrix} SS^* & (\sqrt{SS^*})S \\ S^*(\sqrt{SS^*}) & S^*S \end{pmatrix}$ and because $u_+(T_2) = 0 = u_+(T_3)$ the following relation is true:

$$(r-u_+)(\Psi) = u_+(T_1) + (T_2b,b) + 2n\mathrm{Re}(T_3b,a) = ((T_1 + nT_3 + T_2)(a+b),(a+b))$$

.

Obviously, $n = 1 \Rightarrow (r-u_+)(\Psi) = (\Psi(a+b),(a+b))$. In order to have $(r-u_+)$ not greater than or equal to zero we must have by Theorem B, $(1-n^2)$ also not greater than or equal to zero. That is, $n$ must be greater than 1. Therefore, we can choose the values of $a$,$b$,and$n$ so that $r < u_+$. This proves that if $X$ is not commutative then the dual space $(X_{SA})^*$ does not have the first ordering. □

The other implication is trivial and the proof is omitted.

An application of Theorem B is given below and it provides a necessary and a sufficient condition for $(X_{SA})^*$ to possess the second ordering.

**Theorem 2.2.** *The second ordering exists in $X_{SA}$ if and only if $X$ is commutative.*

*Proof.* If $X$ is commutative then the second ordering exists on $X_{SA}$. Conversely, let us assume that $X$ is not commutative but $X_{SA}$ has the second ordering. If $X$ is not commutative then by Theorem A there exists a nonzero operator $T$ such that its square is the zero operator. The second ordering allows us to have operators $\Phi, \Gamma, \Omega$ defined in terms of the operators $T_1, T_2, T_3$ as follows:

$$\begin{aligned}T_3 &= \Omega - \Gamma - 6\Phi\\ T_2 &= -\Omega + 4\Phi + 2\Gamma\\ T_1 &= 2\Phi\end{aligned}$$

We note that $(\Gamma + \Omega - \Phi) \geq 0$ by Theorem B. Also, the order of these operators is important. To see this, consider $(\Gamma + \Omega - \Phi) = 3T_3 + \frac{9}{2}T_1 + 2T_2$. This order of operators does not satisfy Theorem B.

Our next goal in this construction is to decompose the operator $\Phi$ as a sum $\Phi = \Phi_1 + \Phi_2$ so that $\Phi_1 = 0$. We assume that $\Phi_1 \leq \Gamma$,$\Phi_2 \leq \Omega$. Using the matrix representation of $\Phi_1$ and by the condition that $\Phi \geq \Phi_1$, one can write $\Phi_1 = \begin{pmatrix} \theta & O \\ O & O \end{pmatrix}$ for some $\theta$.

If $a \in N$,$b \in M$ then we obtain the following inequality from theorem B.

$$\|\sqrt{SS^*}a + Sb\|^2 = (\Psi(a+b),(a+b)) \geq (\theta a, a) \qquad \text{(II)}$$

We remark that from Theorem B one can also have the following inequality for $\lambda T_1 + \mu T_2 + \sigma T_3 = \Delta$. That is,

$$\|(\Delta - \epsilon)h\|^2 = \|\Delta h\|^2 - 2\epsilon(\Delta h, h) + \epsilon^2\|h\|^2 \geq \epsilon^2\|h\|^2$$

By letting $\lambda = \mu = \sigma = 1$ the relation (II) holds.

Now, for $b_n \in M$,$(\sqrt{SS^*}a + Sb_n) \to 0$ as $n \to \infty$. This means that $\Phi_1 = 0$. Accordingly, $\Phi = \Phi_2$ and $T_1 \leq 2\Omega \Rightarrow T_1 - 8T_1 - 4T_3 - 2T_2 \leq 0 \Rightarrow 7T_1 + 4T_3 + 2T_2 \geq 0 \Leftrightarrow (7)(2) - (4)^2 \geq 0$ (Theorem B) which is false and contrary to our assumption. Thus, $X$ has to be commutative. □

## 3. Some Norm Inequalities for Operators

This section contains some norm inequalities for operators on a Hilbert space $H$. These results were obtained during the course of the current research. For the sake of completeness we include them here.

**Theorem 3.1.** *Let $T$ and $S$ be the bounded linear operators on $H$. For a vector $h$ in $H$ where the norm of $h$ equals the number one and for a positive constant $m$ the following implication holds:* $\|T-S\| \leq m \Rightarrow \|(T^*T)^n + (S^*S)^n\| \leq 2w((S^*T)^n) + m^2$

*Proof.* Here, by $w(A)$ we mean the numerical radius of an operator $A$. For $h$ in $H$ and $\|T - S\| \leq m$ we have

$$\Big(\big((T^*T)^n + (S^*S)^n\big)h, h\Big) = (T^nh, T^nh) + (S^nh, S^nh) = \|T^nh\|^2 + \|S^nh\|^2 \leq 2\mathrm{Re}(T^nh, S^nh) + m^2, m > 0$$

(I) That is, $\Big(\big((T^*T)^n + (S^*S)^n\big)h, h\Big) \leq 2\mathrm{Re}(T^nh, S^nh) + m^2, m > 0$

The operator $((T^*T)^n + (S^*S)^n)$ is self-adjoint and hence its numerical radius equals its norm. Thus the theorem follows by taking the 'sup' of both sides of (I). It is obvious to observe that $\|(T^*T)^n + (S^*S)^n\| - 2w((S^*T)^n) \leq \|T^n - S^n\|^2$ since $\|T^nh\|^2 + \|S^nh\|^2 \geq 2\|T^nh\|^2\|S^nh\|^2 \geq 2|(T^nh, S^nh)|$.

Also, if $T$ is a normal operator and $\|T - T^*\| \leq m$ then the following inequalities are true.

$$\|T^*T + S^*S\| \leq 2w(T^2) + m^2;$$

$$\|TT^*\| = \|T^*T\| \leq w(T^2) + \frac{1}{2}m^2$$

□

The central idea in the proof of Theorem 3.2 is based on the concept of the generalized Moore-Penrose inverse of an operator on a Hilbert space and the "Moore-Penrose Equations. We briefly give an account of these facts to make the presentation coherent and smooth. For more details on these topics, we refer the reader to [6] and [7].

The Moore-Penrose inverse $E^\wedge$ of an operator $E$ is defined as the unique linear extension of $\bar{E}^{-1}$ to $D(E^\wedge) := R(E) + R(E)^\perp$ with

$$N(E^\wedge) = R(E)^\perp \tag{1}$$

Here, $\bar{E} = E|_{N(E)^\perp} : N(E)^\perp \to R(E)$. The inverse $E^\wedge$ is well defined:

Since $N(\bar{E}) = \{0\}$ and $R(E) = R(\bar{E})$, the inverse $\bar{E}^{-1}$ exists. Let $y \in D(E^\wedge)$. Then $y$ can be uniquely written as $y = y_1 + y_2$ for $y_1 \in R(E)$, $y_2 \in R(E)^\perp$. Further, the lineqrity of $E^\wedge$ and the existence of condition (1) above imply that

$$\bar{E}^{-1} y_1 = E^\wedge y$$

If $P$ and $Q$ are orthogonal projections onto $N(E)$ and $R(E)$ respectively then the "Moore-Penrose equations" hold true.

$$EE^\wedge E = E \tag{2}$$

$$E^\wedge E E^\wedge = E^\wedge \tag{3}$$

$$E^\wedge E = I - P \tag{4}$$

$$E^\wedge E = Q|_{D(E^\wedge)} \tag{5}$$

Also, $R(E^\wedge) = N(E)^\perp$.

The proof of these equalities is not difficult. For all $y \in D(E^\wedge), E^\wedge y = \bar{E}^{-1} Q y \Rightarrow E^\wedge y \in R(\bar{E}^{-1}) = N(E^\wedge)$. For all $x \in N(E^\wedge) : E^\wedge E x = \bar{E}^{-1} \bar{E} x = x$. This proves that $R(E^\wedge) = N(E)^\perp$.

Now for any $y \in D(E^\wedge)$ the conditions: $R(E^\wedge) = N(E)^\perp$ and $\bar{E}^{-1} Q y \in (E)^\perp$ imply that

$$EE^\wedge y = EE^\wedge Q y = E\bar{E}^{-1} Q y = \bar{E}\bar{E}^{-1} Q y = Q y,$$

Consequently, (5) holds. By the definition of "Moore-Penrose" inverse we have for all $x \in X : E^\wedge E x = \bar{E}^{-1} E(Px - (I-P)x) = (I-P)x$, thus the relation (2) is true. Conditions $R(E^\wedge) = N(E)^\perp$ and (5) together imply (3).

The proof in the next theorem uses the above construction of operators which are decomposed in terms of the "Moore-Penrose" inverse.

**Theorem 3.2.** *Let $A_1$,$A_2$,$B_1$,$B_2$ be commuting operators in $(X_{SA})$ such that for all $x$ in $X$, $(\|A_1 x\|^2 + \|A_2 x\|^2) < \infty$, $(\|B_1 x\|^2 + \|B_2 x\|^2) < \infty$. Then the following inequality holds: $\|A_1 B_1 + A_2 B_2\| \leq \|AB\|$, where $A^2 = (A_1^* A_1 + A_2^* A_2)$ and $B^2 = (B_1^* B_1 + B_2^* B_2)$.*

*Proof.* Let $C^\perp$ denote the orthogonal projection operator on the closure of the range of an operator $C$. Suppose that $\{y_n\}$ is a sequence of elements in $X$ and $x$ is in $X$ such that

$$\lim_{n\to\infty} A y_n = A^\perp x \tag{V}$$

Now the limits as $n \to \infty$ of $A_1 y_n$ and $A_2 y_n$ exist and they are independent of the selected sequence. In other words, if $\{y_m\}$ is any other sequence of elements of $X$ then by (V) the following relations hold:

$\|A_1 y_n - A_1 y_m\| \leq \|A y_n - A y_m\| \to 0$ and $\|A_2 y_n - A_2 y_m\| \leq \|A y_n - A y_m\| \to 0$ as $n, m \to \infty$

Thus for an arbitrarily chosen sequence $\{z_n\}$ the inequalities

$\|A_1 y_n - A_1 z_n\| \leq \|A y_n - A z_n\| \to 0$ and $\|A_2 y_n - A_2 z_n\| \leq \|A y_n - A z_n\| \to 0$ are true.

The existence of the above limits helps us to define the limit operators $T$ and $S$. We will claim that the operators $T$ and $S$ are commuting contractions such that $TA = A_1 = AT$ and $SA = A_2 = AS$ with $A^\perp = T^* T + S^* S$. In fact our goal is to write $T = cl(A_1 A^\wedge) = cl(A^\wedge A_1)$ and $S = cl(A_2 A^\wedge) = cl(A^\wedge A_2)$. Here, $cl$ means

the closure of a set and $A^\wedge$ is the "Moore-Penrose" generalized inverse of $A$. This should not be difficult to see and the construction below will help us to achieve the desired forms of the operators $T$ and $S$.

Let $Tx = \lim_{n\to\infty} A_1 y_n$ and $Sx = \lim_{n\to\infty} A_2 y_n$ if (V) holds. Accordingly, $TA^\perp = T$ and $SA^\perp = S$. Moreover, $TA = A_1 = AT$ and $SA = A_2 = AS$. We also note that $A^\perp = T^*T + S^*S$. Since $A^\perp$ is an orthogonal projection. It is easy to verify that the equality $A^\perp A^2 = A^2 \Rightarrow T^*TA^\perp + S^*SA^\perp = A^\perp$, which together with the fact that $T(I - A^\perp) = 0 = S(I - A^\perp)$ establishes the "Moore-Penrose" inverse. As a matter of fact, these operatons are commutative on the range space as well as on the null space of $A^2$. This proves our claim. Thus the operators $T$ and $S$ are normal contractions and commute with the operators $A_1$ and $A_2$. The decomposition of $A$ is complete and established. One can also have a commuting family $\{T_n\}$ of contractions which are self adjoint such that $T_nA = AT_n = A_n, \sum_n T_n^*T_n = A^\wedge$. Clearly, this family of operators commutes with the family $\{A_n\}$ and $T_n = cl(A_nA^\wedge) = cl(A^\wedge A_n)$. On the same lines we can factor the operator $B$. The consequent of the theorem follows from Theorem 2.5.1 in [3] and Theorem 1.19 of [9]. □

**Theorem 3.3.** *For operators $A$ and $B$ in $X_{SA}$ the following inequality is true for all $p > 2$ :* $\left\|\left(\frac{1+A^*A}{2}\right)^{\frac{1}{2}}\left(\frac{1+B^*B}{2}\right)^{\frac{1}{2}}\right\| \leq \left\|\left(\frac{1+(A^*A)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}}\left(\frac{1+(B^*B)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}}\right\|$

*Proof.* Let $A$,$B$, and $I$ be commuting operators. If $p > 2$ then the function $G(\alpha) = \alpha^{\frac{2}{p}}$ is a concave mapping. Basically for $p > 2$, the correspondence $\alpha \to \left(\alpha^2\right)^{\frac{1}{p}}$ is an operator monotonic mapping. Hence the following inequalities

$$\left(\frac{1+(A^*A)^{\frac{p}{2}}}{2}\right)^{\frac{2}{p}} \geq \left(\frac{1+A^*A}{2}\right) \text{ and } \left(\frac{1+(B^*B)^{\frac{p}{2}}}{2}\right)^{\frac{2}{p}} \geq \left(\frac{1+B^*B}{2}\right) \text{ imply}$$

$$\left\|\left(\frac{1+(A^*A)}{2}\right)^{\frac{1}{2}}\left(\frac{2}{1+(A^*A)}\right)^{\frac{1}{p}}\right\| \leq 1 \text{ and } \left\|\left(\frac{1+(B^*B)}{2}\right)^{\frac{1}{2}}\left(\frac{2}{1+(B^*B)}\right)^{\frac{1}{p}}\right\| \leq 1.$$

The theorem follows by combining the above inequalities. □

**Theorem 3.4.** *For operators $T$ and $S$ in $X_{SA}$ the inequality $\|T+S\| \leq 2^{1-\frac{1}{p}}\left\|(T^*T)^{\frac{p}{2}} + (S^*S)^{\frac{p}{2}}\right\|^{\frac{1}{p}}$ holds for all real $p \geq 2$.*

*Proof.* From Theorem 3.3 the inequality $\left\|\frac{T+S}{2}\right\| \leq \frac{1}{m}\left\|\left(\frac{1+m^p(T^*T)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}}\left(\frac{1+m^p(S^*S)^{\frac{p}{2}}}{2}\right)^{\frac{1}{p}}\right\|$ exists for all $m > 0$. This implies that $\left\|\frac{T+S}{2}\right\| \leq \frac{1}{m}\left\|\left(\frac{1+m^p(T^*T)^{\frac{p}{2}}}{2}\right)^{\frac{1}{2}}\left(\frac{1+m^p(S^*S)^{\frac{p}{2}}}{2}\right)^{\frac{1}{2}}\right\|^{\frac{2}{p}}$

Hence an application of the geometric-arithmetic mean inequality yields the following inequality

$$\left\|\frac{T+S}{2}\right\| \leq \frac{1}{2m}\left\|\left(\frac{1+m^p(T^*T)^{\frac{p}{2}}}{2}\right) + \left(\frac{1+m^p(S^*S)^{\frac{p}{2}}}{2}\right)\right\|^{\frac{2}{p}}$$

The proof is complete since the minimum on the right side of the above inequality is attained at $m = \left\| \frac{2}{(T^*T)^{\frac{p}{2}} + (S^*S)^{\frac{p}{2}}} \right\|^{\frac{1}{p}}$. □

*Remark 3.5.* By Theorem 3.2 we can have an elementary inequality for normal operators $T$ and $S$. In other words, $\|T+S\| \leq \left\|(I+T^*T)^{\frac{1}{2}}(I+S^*S)^{\frac{1}{2}}\right\|$. By Theorem 3.4 we have $\|T+S\| \leq \sqrt{2}\|T^*T+S^*S\|^{\frac{1}{2}}$. The operator $(T^*T+S^*S)$ is self-adjoint so its numerical radius equals its norm and hence, $\|T+S\|^2 \leq 2w(T^*T+S^*S)$.

The above inequality can be compared with Theorem 3.1. Lastly, if $T$ and $S$ are self-adjoint and idempotent operators then $\|T+S\| \leq \||I+T|\dot{|}I+S|\|$.

The next theorem proves a pertubation inequaliy for self-adjoint operators on $X$. The theorem is useful in comparing a class of derivations on $X$ which are induced by a pair of self-adjoint operators.

**Theorem 3.6.** *For $T$ and $S$ in $X_{SA}$ and for all $\theta \in (0,1)$ we have*

$$\left\|(T^*T)^{\frac{\theta}{2}} - (S^*S)^{\frac{\theta}{2}}\right\| \leq 2^{2-\theta}\|T-S\|^{\theta}$$

*Proof.* If $T$ and $S$ belong to $X_{SA}$ then for unitary operators $U$ and $V$ these operators can be written as : $T = U(T^*T)^{\frac{1}{2}}$,$S = (S^*S)^{\frac{1}{2}}$. In this case,

$$\begin{aligned}\left\|(T^*T)^{\frac{\theta}{2}} - (S^*S)^{\frac{\theta}{2}}\right\| &= \left\|U\,(T^*T)^{\frac{\theta}{2}} - U\,(T^*T)^{\frac{\theta}{4}}\,V\,(S^*S)^{\frac{\theta}{4}} + U\,(T^*T)^{\frac{\theta}{4}}\,V\,(S^*S)^{\frac{\theta}{4}} - V\,(S^*S)^{\frac{\theta}{2}}\right\| \\ &= \left\|U\,(T^*T)^{\frac{\theta}{4}}\left(U\,(T^*T)^{\frac{\theta}{4}} - V\,(S^*S)^{\frac{\theta}{4}}\right) + \left(U\,(T^*T)^{\frac{\theta}{4}} - V\,(S^*S)^{\frac{\theta}{4}}\right)V\,(S^*S)^{\frac{\theta}{4}}\right\|\end{aligned}$$

Let $\left[U\,(T^*T)^{\frac{\theta}{4}} - V\,(S^*S)^{\frac{\theta}{4}}\right] = R$. Then $\left\|(T^*T)^{\frac{\theta}{2}} - (S^*S)^{\frac{\theta}{2}}\right\| = \left\|U\,(T^*T)^{\frac{\theta}{4}}\,R + RV\,(S^*S)^{\frac{\theta}{4}}\right\|$. Since $T$ and $S$ are self-adjoint and $R$ is any operator on $X$ the following inequality is justified from Theorem 3.4. That is,

$$\text{(VI)} \qquad \left\|(T^*T)^{\frac{\theta}{2}} - (S^*S)^{\frac{\theta}{2}}\right\| \leq 2\|R\|^{\frac{1-\theta}{1-\frac{\theta}{2}}}\left\|(T^*T)^{\frac{1-\frac{\theta}{2}}{\theta}}\,R + R\,(S^*S)^{\frac{1-\frac{\theta}{2}}{\theta}}\right\|^{\frac{\frac{\theta}{2}}{1-\frac{\theta}{2}}}$$

For $\theta \in (0,1)$, the following inequalities are also true.

$$\|R\| \leq 2^{1-\frac{\theta}{2}}\|T-S\|^{\frac{\theta}{2}},\ \|R\|^{\frac{1-\theta}{1-\frac{\theta}{2}}} \leq 2^{1-\theta}\|T-S\|^{\frac{\theta}{2}\left(\frac{1-\theta}{1-\frac{\theta}{2}}\right)} = 2^{1-\theta}\|T-S\|^{\frac{\theta(1-\theta)}{2-\theta}},$$

and $\left\|(T^*T)^{\frac{1-\frac{\theta}{2}}{\theta}}R + R(S^*S)^{\frac{1-\frac{\theta}{2}}{\theta}}\right\| \leq 2\|T-S\|$. The inequality in (VI) now yields the theorem. □

*Remark 3.7.* The conclusion of Theorem 3.6 can also be written in the following form. For self-adjoint operators $A$ and $B$ if $\|A-B\| \leq m$, $m > 0$ and $\theta \in (0,1)$ then there exists a positive constant $M$ so that $\left\||A|^{\theta} - |B|^{\theta}\right\| \leq M$. Also, if $B = I$ (Identity operator) then $\left\||A|^{\theta} - I^{\theta}\right\| \leq M$.

The above remark has some value. As a matter of fact Theorem 3.1 and Remark 3.7 produce the following bound for $\left\||A|^{\theta} + |B|^{\theta}\right\|$. In paticular, we present the following theorem.

**Theorem 3.8.** *If for $m > 0, \|A - B\| \leq m$ then*

$$\left\| |A|^\theta + |B|^\theta \right\| \leq \delta w\left(|B|^\theta\right) w\left(|A|^\theta\right) + m^2, \forall \theta \in (0,1)$$

*The positive valuation of $\delta$ depends on whether the operators are commutative.*

*Proof.* Recall the operators A and B as in Remark 3.7. Since $\left(|A|^\theta\right)^* \left(|A|^\theta\right) + \left(|B|^\theta\right)^* \left(|B|^\theta\right) = \left(|A|^\theta + |B|^\theta\right)$ for all $\theta \in (0,1)$, by Theorem 3.1 we have $\left\| |A|^\theta + |B|^\theta \right\| \leq 2w\left(|B|^\theta |A|^\theta\right) + m^2$.

We consider two cases. First, if the operators are not commutative then by [4, page 37] the inequality $w(TS) \leq 4w(T)w(S)$ implies that $\delta = 8$. Second, if the operators are commutative we have $\delta = 4$ because $w(TS) \leq 2w(T)w(S)$ by [4, page 37]. We further remark that the self-adjoint property of the operator $\left(|A|^\theta + |B|^\theta\right)$ gives the following inequality:

$$w\left(|A|^\theta + |B|^\theta\right) = \left\| |A|^\theta + |B|^\theta \right\| \leq \delta w\left(|A|^\theta\right) w\left(|B|^\theta\right) + m^2, \forall \theta \in (0,1), m > 0$$

□

**Notation.** *Let $S^{-1}$ be the inverse of the operator $S$ such that $\|S^{-1}\| = \beta \neq 0$. We write $\|T\| = \alpha$, $w(T^2) = \alpha_0$, $\lambda = w(S^*T)$, and $z = w(T)$.*

With these notations we present some applications of previous results of the current paper, especially Theorem 3.1.

**Theorem3.9.** *Let $T$ and $S$ be bounded linear operators on $H$. For $m > 0$, we have $\|T - S\| \leq m \Rightarrow \alpha\beta^{-1} \leq \lambda + 2^{-1}m^2$.*

*Proof.* From Theorem 3.1 the inequality $\alpha^2 + \beta^{-2} \leq 2\lambda + m^2$ is valid. Thus the proof follows from the fact that $\alpha^2 + \beta^{-2} \leq 2\alpha\beta^{-1}$. □

**Corollary 3.10.** *For $0 < m \leq 1$ the inequality $\alpha^2 - \alpha_0 \leq \left(2 - \sqrt{1 - m^2}\right) z$ holds.*

*Proof.* If $S = I$ then $\|T - I\| \leq m$. Now the corollary follows from Theorem 3.9. □

**Corollary 3.11.** *Let $T$ and $S$ be bounded and linear operators on $H$ such that $S^{-1}$ exists. The inequality $\alpha^2 \leq \lambda^2 + 2\lambda\left[1 - \beta^{-1}\left(\sqrt{1 - \beta^2}\right)\right]$ holds for $2^{\frac{-1}{2}} \leq \beta < 1$ and $\|T - S\| \leq 1$.*

*Proof.* If $h$ belongs to $H$ and $\|h\| = 1$ then by Theorem 3.9 we have

$$\|Th\|^2 + \beta^{-2} \leq 2|(Th, Sh)| + 1 \tag{VII}$$

By the hypothesis $|(Th, Sh)| > 0$ and if $\gamma = |(Th, Sh)|$ then (VII) reduces to the following inequality

$$\gamma^{-1}\|Th\|^2 - \gamma \leq (2 - \gamma) + \gamma^{-1}(1 - \beta^{-2}) \leq 2\left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right) \tag{VIII}$$

Note that the factor $\left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right) \geq 0$ and by taking the 'sup' of both sides in (VIII) over $h$ we have the corollary. Observe that $\sup(\gamma) = \lambda$. □

*Remark 3.12.* In Corollary 3.10 if $S = T^*$ and the inverse of the operator $T$ exists then

$$\alpha^2 \leq \alpha_0^2 + 2\alpha_0\left(1 - \beta^{-1}\sqrt{1 - \beta^2}\right)$$

**Theorem 3.13.** *If $T$ is a normal operator with $\|(T-T^*)h\| \leq 1 \leq \|T^*h\|, h \in H, \|h\| = 1$ then $\|T\|^4 \leq \|T\|^2 + (w(T^2))^2$.*

*Proof.* By the Schwarz inequality we obtain $\|Th\|^2\|T^*h\|^2 \leq \|T^*h\|^2 + |(Th, T^*h)|^2$. That is, $\|Th\|^4 \leq \|T^*h\|^2 + |(Th, T^*h)|^2$ because $T$ is normal. Now the theorem follows by taking the 'sup' of both sides over $h$ □

## 4. Spectral Radius Inequalities For Operators of the Form $T = SR^{\frac{1}{2}}$

Let $H$ be a Hilbert space with a bounded inner-product $(x, y)$ where the norm of a vector $x$ is defined by $\|x\|_H = \sqrt{(x,x)}, \forall x \in H$. Since the inner-product is bounded it is a known fact that there exists a positive operator $R$ on $H$ such that $(x, y) = (Rx, y), \forall, y \in H$. We also note that $\|x\| = \left\|R^{\frac{1}{2}}x\right\|_H$, $\forall x \in H$. For more on these operators we refer the reader to [1]. A general formula for spectral radius is given in [5] and we state the formula here. We will prove some results for the sum and the difference of these operators.

**Formula A.** *If $A$ and $B$ belong to $B(H)$ then*

$$r(A+B) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4Min(\|AB\|, \|BA\|)}\right)$$

In fact, one can have the following inequality. For $A$,$B$,$C$, and $D$ in $B(H)$

$$r(AB+CD) \leq \frac{1}{2}\left(\|BA\| + \|DC\| + \sqrt{(\|BA\| - \|DC\|)^2 + 4(\|BC\|\|DA\|)}\right)$$

We present the following inequality for operators of the form $T = SR^{\frac{1}{2}}$.

**Theorem 4.1.** *Let $A$,$B$,$S$, and $T$ be operators in $B(H)$ such that $A = SR^{\frac{1}{2}}$, $B = TR^{\frac{1}{2}}$. Then $r(A+B) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\|\|T\|)}\right)$.*

*Proof.* From Formula A:

$$r(A+B) = r\left(SR^{\frac{1}{2}} + TR^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\|SR^{\frac{1}{2}}\| + \|TR^{\frac{1}{2}}\| + \sqrt{\left(\|SR^{\frac{1}{2}}\| - \|TR^{\frac{1}{2}}\|\right)^2 + 4(\|SR^{\frac{1}{2}}\|\|TR^{\frac{1}{2}}\|)}\right)$$

We note the following observation. For any operator $G$ in $B(H)$

$$\|GR^{\frac{1}{2}}x\|^2 = \left(GR^{\frac{1}{2}}x, GR^{\frac{1}{2}}x\right) = (x,x) = \|x\|^2 \Rightarrow \|GR^{\frac{1}{2}}\| = \|G\|$$

Hence, $r(A+B) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\|\|T\|)}\right)$ □

We omit the proofs of corollaries 4.2 and 4.3 as they are straightforward. We denote the spectrum of an operator $P$ by $\sigma(P)$.

**Corollary 4.2.** *Let $r(R^{\frac{1}{2}}) = 1$. Then*
$r(A+B) \leq r(S+T) \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4(\|S\|\|T\|)}\right)$

**Corollary 4.3.** *If $\sigma(A+B) \subseteq \sigma(S+T)$ then the conclusion of Corollary 4.2 holds.*

**Corollary 4.4.** *(Invariance Inequality):*
$r\left(R^{\frac{1}{2}}A + BR^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4Min(\|AB\|, \|BA\|)}\right)$

*Proof.* By Formula A we have the following inequality:

$$r\left(R^{\frac{1}{2}}A+BR^{\frac{1}{2}}\right)\leq\frac{1}{2}\left(\left\|R^{\frac{1}{2}}A\right\|+\left\|BR^{\frac{1}{2}}\right\|+\sqrt{\left(\left\|R^{\frac{1}{2}}A\right\|-\left\|BR^{\frac{1}{2}}\right\|\right)^{2}+4(\|AB\|\|R\|)}\right)$$

By [1] it follows that $\|R\|=1$. Hence,

$$r\left(R^{\frac{1}{2}}A+BR^{\frac{1}{2}}\right)\leq\frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^{2}+4(\|BRA\|)}\right)$$

Also, by symmetry $r\left(BR^{\frac{1}{2}}+R^{\frac{1}{2}}A\right)\leq\frac{1}{2}\left(\|B\|+\|A\|+\sqrt{(\|B\|-\|A\|)^{2}+4(\|ARB\|)}\right)$

Since $\|BRA\|=\|B\|\|A\|$, $\|ARB\|=\|A\|\|B\|1$, therefore,

$$r\left(R^{\frac{1}{2}}A+BR^{\frac{1}{2}}\right)\leq\frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^{2}+4Min(\|AB\|,\|BA\|)}\right)$$

The following result extends Corollary 4 of [5]. □

**Theorem 4.5.** *If $A$ and $U$ belong to $B(H)$ such that $A=SR^{\frac{1}{2}}$ and $U$ is a unitary operator, then $r(AU\pm UA)\leq\|S\|_{H}+\|S^{2}\|_{H}^{\frac{1}{2}}$. This conclusion also holds if $A=SR$.*

*Proof.* The proof follows from above results and definitions involved. □

**Lemma 4.6.** *Let $A=BR$ where $B^{*}=B$. Then for such an operator $A$,$(Aa,b)=(a,Ab)$, $a,b\in A$.*

*Proof.* Since the corresponding inner-product is bounded and $R$ is a positive operator, we have for all $a,b\in H$,$(Ra,b)=(a,Rb)$. Utilizing this, one can get $(Aa,b)=(bRa,b)=(R(BRa),b)=((RB)Ra,b)=(Ra,B^{*}Rb)=(Ra,BRb)=(a,BRb)=(a,Ab)$ □

The next result is a special case of Corollary 4 of [5].

**Theorem 4.7.** *Let $A$,$B$,$C$,$D\in B(H)$. If $T=\begin{bmatrix}O & B\\ C & O\end{bmatrix}$, $B=AR, C=DR$, and$U$ is unitary then $r(TU-UT)\leq\|T\|_{H}+\sqrt{\|T^{2}\|_{H}}$*

*Proof.* Let $U=\begin{bmatrix}I & O\\ O & -I\end{bmatrix}$ be a unitary operator on $H\oplus H$. Then $TU=\begin{bmatrix}O & -AR\\ DR & O\end{bmatrix}$ and $UT=\begin{bmatrix}O & AR\\ -DR & O\end{bmatrix}$. We note that $TU+UT=0$ and hence the Corollary 4 in [5] is a triviality in this case. Moreover, $TU-UT=\begin{bmatrix}O & -2AR\\ 2DR & O\end{bmatrix}$ and the proof is complete by Theorem 4.5. □

The spectral mapping theorem asserts that for every positive integer $n$, $(r(A))^{n}=r(A^{n})$. Suppose that $r\left(\begin{bmatrix}O & B\\ C & O\end{bmatrix}\right)=r_{0}$. Then $r_{0}^{2}=\left(r\begin{bmatrix}O & B\\ C & O\end{bmatrix}\right)^{2}$

**Lemma 4.8.** *Let the operator $T$ be as in Theorem 4.7. Then $r(BC)=r_{0}^{2}$.*

*Proof.* It is known that for operators $P$,$Q\in B(H)$,$r(PQ)=r(QP)$ if the operators are commutative. Hence, $r(BC)=r\left(\begin{bmatrix}BC & O\\ O & CB\end{bmatrix}\right)=r\left(\begin{bmatrix}O & B\\ C & O\end{bmatrix}^{2}\right)=\left(r\begin{bmatrix}O & B\\ C & O\end{bmatrix}\right)^{2}=r_{0}^{2}$. □

*Remark 4.9.* If $B = AR$ and $C = DR$ then $\left(\begin{bmatrix} O & AR \\ DR & O \end{bmatrix}\right) = \sqrt{r(ARDR)}$. If $R = I$, the identity operator in $B(H)$, then Lemma 4.8 is a special case of this remark. Theorem 4.7 and Lemma 4.8 produce a special case of Theorem 2 in [5]. In fact, we have the following inequality.

**Theorem 4.10.** *Using the conditions of Theorem 4.7, we have*

$$2\sqrt{r(ARDR)} \leq \|T\|_H + \sqrt{\|T^2\|_H}.$$

*Proof.* From Theorem 4.7 and Remark 4.9, we have

$$2\sqrt{r(ARDR)} = 2r\left(\begin{bmatrix} O & AR \\ DR & O \end{bmatrix}\right) = r\left(\begin{bmatrix} O & -2AR \\ 2DR & O \end{bmatrix}\right) = r(TU-UT) \leq \|T\|_H + \sqrt{\|T^2\|_H}$$

and this completes the proof. □

*Remark 4.11.* Last but not least we remark that the above results present some pinch inequalities in form of matricies. Let $\nu = \begin{bmatrix} BR^{\frac{1}{2}} & O \\ O & CR^{\frac{1}{2}} \end{bmatrix}$. Then $\nu^t = \nu$ where $\nu^t$ is the transpose matrix of $\nu$. Note that if $\nu = \begin{bmatrix} BR^{\frac{1}{2}} & B \\ C & CR^{\frac{1}{2}} \end{bmatrix}$ then $\nu^t \neq \nu$.

With these notations the pinch inequalities can be written in the form given below. $\|\nu^t\|_H = \|\nu\|_H \leq \|\nu\|_H$. Further, if $t_E$ and $t_0$ denote the even and odd transposes respectively then $\|\nu^t\|_H \leq \|\nu\|_H$ and $\|\nu\|_H \leq \|\nu^{t_E}\|_H$ but $\|\nu\|_H$ is not less than or equal to $\|\nu^{t_0}\|_H$.

We also remark that the above inequalities are true for bounded extensions of (In the sense of [1]) operators. According to [1], if $T = SR^{\frac{1}{2}}$ where $S$ in $B(H)$, then $T$ has an extension $\bar{T} \in B(\bar{H})$. Here, $\bar{H}$ denotes the completion of $H$. For details on these operators, see [1].

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