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# THE BEST QUADRATIC APPROXIMATION OF HYPERBOLA WITH ORDER FOUR

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Abstract. In this article, the best parametric polynomial approximation of degree 2 to the hyperbola is given. This approximation has order 4. The associated error function vanishes 4 times and equioscillates 5 times. For an arc of the hyperbola of length 4.16708, the error is bounded by $6 \times 10^{-2}$. Details of the derivation are presented to show how to apply the method. The method is simple; this encourages and motivates people working in CG and CAD to apply it in their works.

## 1. Introduction

Bézier curves become the basis for representing curves in all CAD systems. They have many interesting geometric, evaluation, and programming advantages, see [4, 1, 5]. Approximation using polynomials of low degree is favourable, and in many cases is a must issue. Many CAD systems are limited to using only parametric polynomial curves of low degree, and, therefore, polynomials of higher degrees have to be reduced to polynomials of low degree that the system allows. This causes two major disadvantages: high accumulated error, and slow and costly software. So, it is favourable to approximate using low degree polynomials.

Quadratic Bézier curves are commonly used in encoding and rendering of type fonts and HTML techniques by many companies. In these and other applications in CG and CAGD, conic sections are the most commonly used curves. Parabolas are represented exactly using parametric curves of degree 2. Approximation of circular arcs using quadratic Bézier curves with minimum error is considered in [9]; the approximation has order 4.

Motivated by the results in [2, 6, 7, 8, 3] that suggest approximating curves by polynomial curves of degree $n$ with order $2n$, we find approximation for the hyperbola of degree 2 and order 4. Bézier curve techniques are used to represent the hyperbola using parametric polynomials of degree 2 that has the least uniform error. The method is simple, needs to determine only 3 Bézier points, and therefore, makes a CAD system more efficient and minimizes the cost.

In this paper, a novel approach to represent the hyperbola with high accuracy is presented. The method leads to the solution that minimizes a variation of the

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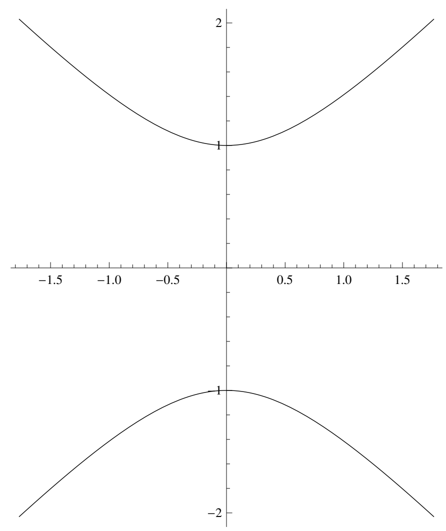


FIGURE 1. The Hyperbola.

Euclidean error and has order 4. We want to represent the longest arc of the hyperbola, and, at the same time, the resulting Bézier curve has to satisfy the Chebyshev error.

There are two general forms of hyperbolas; in each, there are two vaguely parabola shaped branches with a vertex . The first one, $\left(\frac{y-y_0}{a}\right)^2 - \left(\frac{x-x_0}{b}\right)^2 = 1$, opens up and down, and the second one, $\left(\frac{x-x_0}{a}\right)^2 - \left(\frac{y-y_0}{b}\right)^2 = 1$, opens right and left. The difference between the two forms is the minus sign. If the $x^2$ term has the minus sign then the hyperbola opens up and down. If the $y^2$ term has the minus sign then the hyperbola opens left and right. Using suitable translation and scaling, they can be written in the basic forms: $y^2 - x^2 = 1$ and $x^2 - y^2 = 1$. Every form has two branches. So, there are four branches. Geometrically, all of these branches are identical. Therefore, it is sufficient to represent one branch and the other three branches can be represented using rotation of this branch. So, we consider the upper branch of the hyperbola $y^2 - x^2 = 1$, see Fig. 1. It can be written in parametric form $c : t \mapsto (\cosh(t), \sinh(t))\,,\ t \in \Re$. We are interested in finding out the longest arc of the hyperbola that can be approximated and that the error function is the Chebyshev polynomial.

It is not possible to exactly represent a hyperbola with a polynomial curve. It can be represented exactly using rational Bézier curves, a polynomial parametric form is preferred in many applications. The ability to represent a primitive hyperbola is a must issue especially in computer graphics and data and image processing. Thus, there is a demand to find a parametrically defined polynomial curve $p_n : t \mapsto (x_n(t), y_n(t))\,,\ 0 \le t \le 1$, where $x_n(t), y_n(t)$ are polynomials of degree $n$. The degree of $p_n$ has to be as small as possible, and $p_n$ has to approximate $c$ within tolerable error. Having the degree $n$ low makes the software very fast, convenient, obviates complications of high degree, and reduces the cost. In this paper, degree 2 parametric curves are considered, and it is shown that it works well and produces results that are as good as the results of higher degrees. This makes the method competitive.

A possible function to measure the error between $p_2$ and $c$ is the Euclidean error function:

$$E(t) := \sqrt{y_2^2(t) - x_2^2(t)} - 1. \tag{1.1}$$

The square root complicates the analysis. Thus, to avoid radicals, we find the square of the $p_2$ components of the hyperbola. So, $E(t)$ is replaced by the following error function

$$e(t) := y_2^2(t) - x_2^2(t) - 1. \tag{1.2}$$

Note that this choice makes sense because it allows the components of $p_2$ to satisfy the equation of the hyperbola. Also both $e(t)$ and $E(t)$ attain their roots and extrema at the same parameters. In this paper, we are interested in finding the quadratic best uniform approximation that has the highest order of approximation and the minimum error. This research is motivated by the conjecture in [6] which states that it is possible to approximate a curve by a polynomial curve of degree $n$ with order $2n$, rather than the classical order $n+1$. In quadratic case, the associated error function has to equioscillate five times. Consequently, the approximation problem can be formulated as follows.

**The approximation problem** in this paper is to find $p_2 : t \mapsto (x_2(t), y_2(t))$ , $0 \leq t \leq 1$, where $x_2(t), y_2(t)$ are polynomials of degree 2, that approximates $c$ by satisfying the following three conditions:

(1) $p_2$ minimizes $\max_{t\in[0,1]} \; |e(t)|$,
(2) $p_2$ approximates $c$ with order four,
(3) $e(t)$ equioscillates five times over $[0, 1]$.

The solution to this problem is shown in section 3 to be as follows:

$$x(t) \;=\; \sqrt{\frac{3}{\sqrt{2}} + 1} \;\; (2t-1), \qquad y(t) \;=\; \left(\frac{3}{2\sqrt{2}} + 1\right) \;-\; 4\;(t - t^2), \quad t \in [0,1].$$

This solution is presented in Fig. 3; the corresponding error is shown in Fig. 4, and the solution has arc length of 4.16708 units.

This paper is organized as follows. Section 2 introduces some preliminaries and defines the Bézier points for the best solution (the Bézier curve). The main result is given in Theorem 1 in section 3. In section 4, the properties of the best solution are presented. Section 5 states all other possible solutions. In section 6, conclusions and suggested open problems are given.

## 2. Preliminaries

Let $p_2(t) = (x_2(t), y_2(t))$ be quadratic polynomial parametric representation of the curve $c$. In CAGD, curves are presented using the Bézier form, see Fig. 2. The Bézier curve $p_2(t)$ is presented as follows:

$$p_2(t) = \sum_{i=0}^{2} p_i B_i^2(t) =: \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}, \quad 0 \leq t \leq 1, \tag{2.1}$$

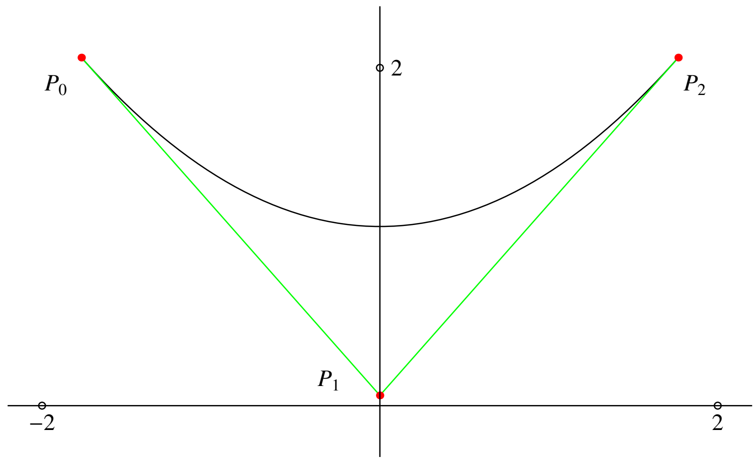


FIGURE 2. Possible Bézier points of the hyperbola.

where $p_0, p_1, p_2$ are the Bézier points, and the Bernstein polynomial basis of degree 2 is given by:

$$B_i^2(t) = \binom{2}{i}(1-t)^{2-i}t^i, \quad i = 0, 1, 2, \quad t \in [0, 1].$$

As explained in the previous section, it is sufficient to consider the upper branch of the hyperbola, the lower branch can be given using symmetry.
The simplicity of this method should encourage people working in the fields of Computer Graphics, Image Processing, CAD, and Data Processing to adopt it in their design and applications.

The symmetry in the hyperbola is used to better locate the Bézier points. We begin by letting $p_0 = (-\alpha,\ \beta)$. The hyperbola is symmetric around the $y$-axis, so, to obey this symmetry, the point $p_2$ should have the form $p_2 = (\alpha,\ \beta)$. There is one remaining point; if this point lies in either halfs of the plane around the $y$-axis, then the symmetry of the hyperbola is kicked. Thus, the point $p_1$ must lie on the $y$-axis and has the form $p_1 = (0,\ \gamma)$. It is clear from the choice of the points that

$$\alpha,\ \beta,\ \gamma > 0. \tag{2.2}$$

Therefore, the proper choice for the Bézier points is

$$p_0 = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \quad p_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2.3}$$

The Bézier polynomial curve $p_2(t)$ in (3) is given in the form

$$p_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \alpha\left(B_2^2(t) - B_0^2(t)\right) \\ \beta\left(B_0^2(t) + B_2^2(t)\right) + \gamma B_1^2(t) \end{pmatrix}, \qquad 0 \le t \le 1. \tag{2.4}$$

The free three parameters $\alpha,\ \beta,\ \gamma$ are used to have the polynomial approximation $p_2$ comply with the conditions of the approximation problem; this is done in the following section.

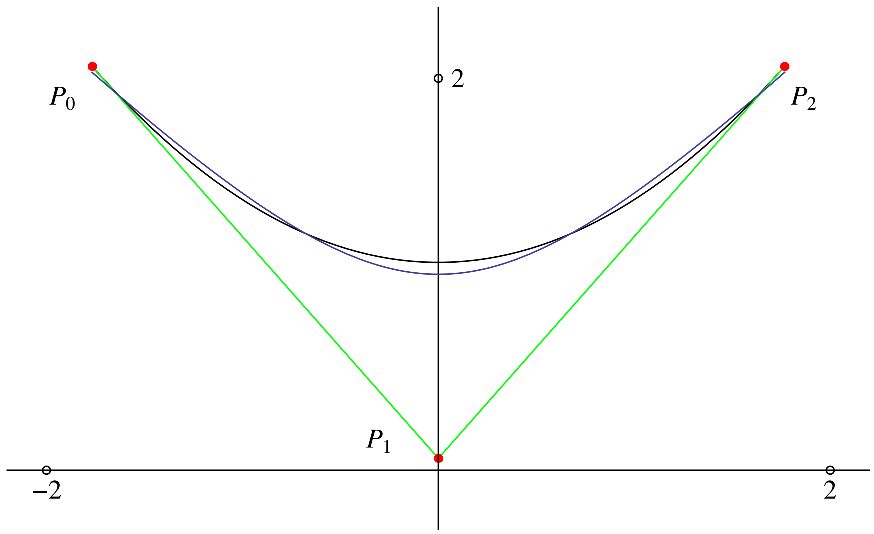


FIGURE 3. The Hyperbola and the quadratic approximating Bézier curve.

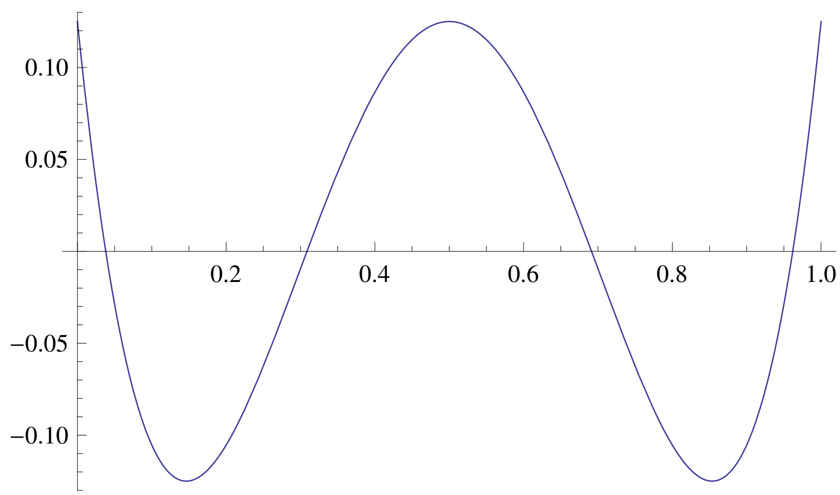


FIGURE 4. The error of the quadratic approximating Bézier curve.

## 3. THE BEST QUADRATIC UNIFORM APPROXIMATION

The values of $\alpha,\ \beta,\ \gamma$ that minimize the uniform error and satisfy the conditions of the approximation problem are given in the following theorem.

**Theorem 1:** The Bézier curve in (6) with the Bézier points in (5), where

$$\alpha = \alpha^* := \sqrt{1+\frac{3}{\sqrt{2}}}, \quad \beta = \beta^* := 1+\frac{3}{2\sqrt{2}}, \quad \gamma = \gamma^* := \frac{3}{2\sqrt{2}} - 1, \tag{3.1}$$

satisfies the following three conditions: $p_2$ minimizes the uniform error $\max_{t\in[0,1]}|e(t)|$ and approximates $c$ with order four, and the error function $e(t)$ equioscillates five times in $[0,1]$. More precisely, the error functions satisfy:

$$-\frac{1}{2^3} \leq e(t) \leq \frac{1}{2^3}, \qquad -\frac{1}{2^3(2-\epsilon)} \leq E(t) \leq \frac{1}{2^3(2+\epsilon)}, \quad \text{where} \quad \epsilon = \max_{0\leq t\leq 1}|E(t)|. \tag{3.2}$$

**Proof:** Substituting the components of $p_2(t)$ into equation (2) for the error function $e(t)$ gives

$$\begin{aligned} e(t) &= \left(4\beta^2 - 8\beta\gamma + 4\gamma^2\right)t^4 + \left(-8\beta^2 + 16\beta\gamma - 8\gamma^2\right)t^3 + \left(-4\alpha^2 + 8\beta^2 - 12\beta\gamma + 4\gamma^2\right)t^2 + \\ &+ \left(4\alpha^2 - 4\beta^2 + 4\beta\gamma\right)t + \left(\beta^2 - \alpha^2 - 1\right). \end{aligned} \tag{3.3}$$

The last equality for $e(t)$ is a polynomial of degree 4. To prove the theorem, we substitute the values of $\alpha = \alpha^*, \quad \beta = \beta^*, \quad \gamma = \gamma^*$ from (7) and simplify the resulting equation to get

$$e(t) = 16t^4 - 32t^3 + 20t^2 - 4t + \frac{1}{8}, \quad t \in [0,1].$$

The last polynomial is the shifted monic Chebyshev polynomial of degree 4, $\tilde{T}_4(2t-1),\ t \in [0,1]$, which is the unique polynomial of degree 4 that minimizes $|e(t)|$ over $[0,1]$ and equioscillates five times between $\pm\frac{1}{2^3}$ for all $t \in [0,1]$, see [10]. Consequently, $p_2$ has fourth order of contact with $c$. The error function $e(t)$ minimized is related to the Euclidean error $E(t)$ by the following formula

$$e(t) = y^2(t) - x^2(t) - 1 \;=\; (\sqrt{y^2(t) - x^2(t)} - 1)\,(\sqrt{y^2(t) - x^2(t)} + 1) \;=\; E(t)\;(2 + E(t))\,.$$

Note that, since $y^2 = 1 + x^2$ and the uniform error is bounded by $1/8$ then $7/8 \leq y_2^2(t) - x_2^2(t) \leq 9/8$, which means that the square root is well-defined. Rewriting the last equation for $E(t)$, we get

$$E(t) = \frac{e(t)}{2 + E(t)}.$$

Substituting the bounds of $e(t)$ gives

$$-\frac{1}{2^3(2-\epsilon)} \leq E(t) \leq \frac{1}{2^3(2+\epsilon)}, \quad \text{where} \quad \epsilon = \max_{0\leq t\leq 1}|E(t)|.$$

This proves Theorem 1. □

Conditions (2) and (3) given in Theorem 1 are consequences of the fact that the conditions were imposed on the error function to make it coincide with the monic quartic Chebyshev polynomial. In particular, condition (2) assures the improvement of the order of approximation over the standard order (from 3 to 4). The function of condition (3) is to assure that the approximation is the best uniform approximation which is clear from Fig. 4.

Fig. 3 shows the hyperbola and the approximating Bézier curve, Fig. 4 shows the corresponding error, and Fig. 5 shows the Euclidean error.

**Remarks:**

(1) The Bézier curve in Theorem 1 is the longest parametric curve that can satisfy the Chebyshev error. It is 4.16708 units length.
(2) The Bézier curve with $\alpha = \alpha^*, \quad \beta = \beta^*, \quad \gamma = \gamma^*$ represents the upper branch of the hyperbola, see Fig. 3.

(3) For programming purposes, it is faster to compute the parameters $\alpha^*,\ \beta^*,\ \gamma^*$ by defining $c$ as follows:

$$c = \frac{3}{2\sqrt{2}}, \quad \alpha^* = \sqrt{1+2c}, \quad \beta^* = c+1, \quad \gamma^* = c-1. \tag{3.4}$$

In the following section, the properties of the approximating quadratic Bézier curve are given.

## 4. Properties of approximating quadratic Bézier curve

The approximating quadratic Bézier curve has the following properties; they have the same proof as in [9].

**Proposition I:** The roots of the error functions $e(t)$ and $E(t)$ are:

$$t_1 = \frac{1}{2}(1+\cos(\frac{\pi}{8})) = 0.96194, \qquad t_2 = \frac{1}{2}(1+\sin(\frac{\pi}{8})) = 0.691342$$

$$t_3 = \frac{1}{2}(1-\sin(\frac{\pi}{8})) = 0.308658, \qquad t_4 = \frac{1}{2}(1-\cos(\frac{\pi}{8})) = 0.03806.$$

Because of symmetry, they also satisfy $t_1 + t_4 = 1, \quad t_2 + t_3 = 1.$

**Proposition II:** The extreme values of e(t) and $E(t)$ occur at

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}(1+\frac{1}{\sqrt{2}}) = 0.853553, \quad \tilde{t}_2 = \frac{1}{2}, \quad \tilde{t}_3 = \frac{1}{2}(1-\frac{1}{\sqrt{2}}) = 0.146447, \ \tilde{t}_4 = 0.$$

Because of symmetry, they also satisfy $\tilde{t}_0 + \tilde{t}_4 = 1, \quad \tilde{t}_1 + \tilde{t}_3 = 1, \quad 2\tilde{t}_2 = 1.$ Moreover, the values of the error functions $E(t)$ and $e(t)$ at $\tilde{t}_i$ are given by

$$e(\tilde{t}_0) = e(\tilde{t}_2) = e(\tilde{t}_4) = \frac{1}{8}, \quad e(\tilde{t}_1) = e(\tilde{t}_3) = \frac{-1}{8},$$

$$E(\tilde{t}_0) = E(\tilde{t}_2) = E(\tilde{t}_4) = 6.1 \times 10^{-2}, \quad E(\tilde{t}_1) = E(\tilde{t}_3) = -6.5 \times 10^{-2}.$$

Thus

$$\frac{-1}{8} \le e(t) \le \frac{1}{8}, \qquad -6.5 \times 10^{-2} \le E(t) \le 6.1 \times 10^{-2},\ \forall t \in [0,1].$$

To get the solution in Theorem 1, some conditions were imposed on $\alpha,\ \beta,\ \gamma$ in (4). However, if the conditions on $\alpha,\ \beta,\ \gamma$ are removed, there will be other possible solutions. In the following section, all the possible real solutions are listed.

## 5. All quadratic Bézier curves

If the conditions imposed on $\alpha,\ \beta,\ \gamma$ in (4) are removed, then the other solutions are given in the following theorem.

**Theorem 2:** Removing the conditions on $\alpha, \beta, \gamma$ in (4), then the approximation problem has eight solutions; four of these solutions are complex, and the other four are real. The real solutions are sign multiple of the solution in Theorem 1 and are summarized in the following table:

| Solution | Sign $\alpha$ | Sign $\gamma$ | Sign $\beta$ | curve in quadrants | generated |
|---|---|---|---|---|---|
| 1st | $+$ | $+$ | $+$ | 1st and 2nd | counter clockwise |
| 2nd | $-$ | $+$ | $+$ | 1st and 2nd | clockwise |
| 3rd | $-$ | $-$ | $-$ | 3rd and 4th | counter clockwise |
| 4th | $+$ | $-$ | $-$ | 3rd and 4th | clockwise |

Table 1: All real solutions to the approximation problem

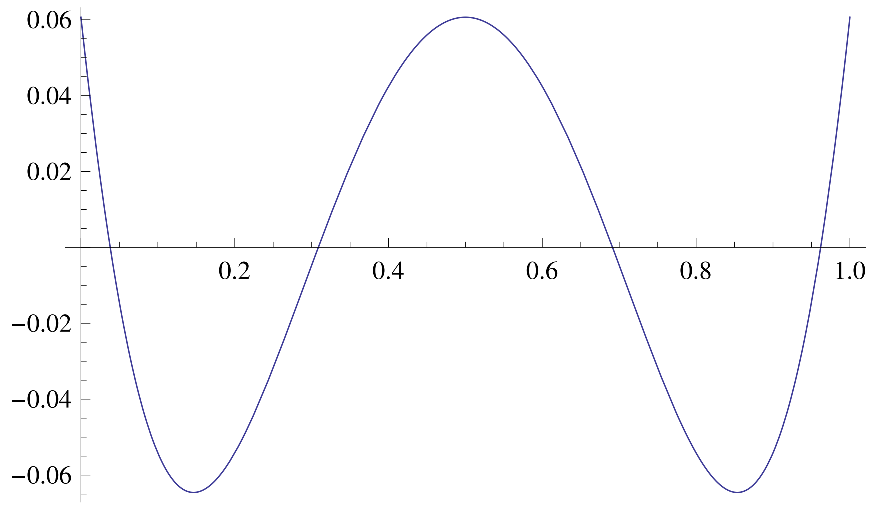


FIGURE 5. The Euclidean error of the quadratic Bézier curve.

**Proof:** In Theorem 1, the first case was proved. The other cases can be shown by following the same steps in Theorem 1. □

**Remarks:**

(1) Table 1 lists all the (real) possible solutions to the approximation problem; fortunately, four out of the eight solutions are real, make sense, satisfy the three approximation conditions, and are related in being reflections to each other around the $y$-axis. The second solution coincides with the first solution, but generated clockwise. The third and fourth solutions are reflections of the first and second solutions around the $x$-axis, generated counter clockwise and clockwise, respectively.
(2) The sign of $\beta$ is the same as the sign of $\gamma$. If the sign of $\beta$ is positive, then the curve lies in the second and first quadrants, and if it is negative then the curve lies in the third and fourth quadrants.
(3) The roots of the error functions $e(t)$ and $E(t)$ for all of the solutions in Table 1 are the same as in Proposition I.
(4) The extreme values of e(t) and $E(t)$ for all of the solutions in Table 1 occur at the same parameters that are given in Proposition II.
(5) The third and the fourth solutions are reflections of the first and second solutions around the $x$-axis, respectively.
(6) The first solution is chosen because it is generated in the same direction as the hyperbola is generated.

As a consequence of Theorems 1 and 2, we have the following proposition regarding the error at any $t \in [0, 1]$.

**Proposition III:** For every $t \in [0, 1]$, the errors of approximating the hyperbola using the Bézier curves in Theorems 1 and 2 are given by:

$$e(t) = 16t^4 - 32t^3 + 20t^2 - 4t + \frac{1}{8}, \qquad E(t) \simeq 8t^4 - 16t^3 + 10t^2 - 2t + \frac{1}{16}, \quad \forall t \in [0, 1].$$

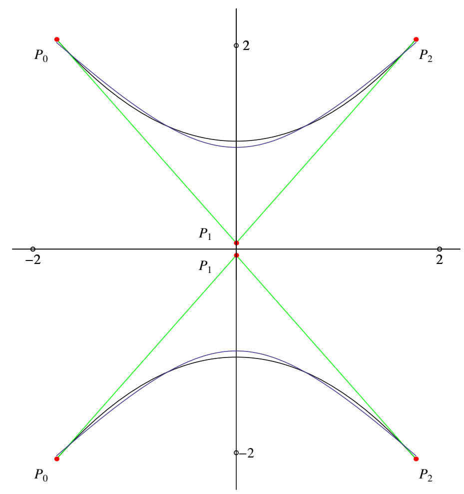


FIGURE 6. Both branches of the Hyperbola using two Bézier curves.

## 6. Conclusions

Reflecting the upper branch of the hyperbola around the $x$-axis gives the lower branch. The Bézier points for the lower branch are:

$$p_0 = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}, \quad p_2 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}. \tag{6.1}$$

The hyperbola is shown in Fig. 6.

It is a challenging issue and is still an open problem to find the best quadratic uniform approximation of a function with the following properties: the error function equioscillates four times, the approximation order is three, and the curve and the approximation intersect three times.

Despite these challenges, we are able to find in this article the best quadratic uniform approximation of the hyperbola with parametrically defined polynomial curve in explicit form. Fortunately, we did get better results than expected: the error function equioscillates five times (rather than four times); the approximation order is four (rather than three); the curve and the approximation intersect four times (rather than three times). The method is simple and efficient. This paper contributes to the answer of the question stated in [9], which is still valid to find polynomials of best uniform approximation for other curves.

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Fig. 1. A circular arc ( quarter, $\theta = \frac{\pi}{4}$)

Fig. 2. Possible Bézier points of circular arc

Fig. 3. The circular arc and the quadratic approximating Bézier curve

Fig. 4. Euclidean Error of the quadratic approximating Bézier curve

Fig. 5. The figure of the full circle using 4 Bézier curves

Fig. 6. The error of one out of four quarters of the full circle

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