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NEW STOCHASTIC INEQUALITIES INVOLVING THE F AND GAMMA DISTRIBUTIONS

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Abstract. We derive a new inequality involving either the F or Gamma distribution and its quantiles and call it either the F-Inequality or G-Inequality. The stochastic representation of the new inequality involves $\alpha, p \in (0,1)$ such that if $p > \alpha$ and $k > 1$ with W being a random variable with an $F(\nu_1, \nu_2)$ or $Gamma(\tau, \theta)$ distribution, then

$$\frac{1}{p} P\left(W < \frac{W_p}{k}\right) > \frac{1}{\alpha} P\left(W < \frac{W_\alpha}{k}\right),$$

where for any γ between 0 and 1, W_γ is defined by $\gamma = P(W < W_\gamma)$. The inequality reverses for $k \in [0,1)$; it becomes equality for $k = 1$ and, trivially, for $k = \infty$. This inequality seems to hold for a larger class of distributions that include for example the F, Gamma, Cauchy, and special cases of the Beta distribution and perhaps others. However, we provide rigorous proofs only for the F and Gamma distributions.

1. Introduction

One of these inequalities, the F-inequality, arose while proving that for the generalized split-plot model the F-test for the whole plot variance being zero has larger power than either the likelihood ratio test (LRT) or the restricted likelihood ratio test (RLRT) when $\alpha_* > 1 - p_*$. In here α_* is the level of the test and p_* is the probability that the LRT or the RLRT statistic is zero (see [1] for more details). Besides their use in power comparisons for testing variance parameters on the boundaries of the parameter space, these inequalities promise value in developing stochastic lower and upper bounds in many fields. Moreover, they present a striking example of a case in which the unjustified inequality $a/c > b/d$ for $a > b > 0$ and $c > d > 0$ holds (see [2] for more details about dividing inequalities). Section 2 presents background material needed for the proofs. Section 3 gives the F-inequality, and Section 4 establishes the G-inequality.

2. Hypergeometric Functions

We now present two lemmas related to special hypergeometric functions that we need to prove the inequalities.

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Definition 2.1. *For real numbers α, β, and γ with $\gamma \neq 0, -1, -2, \ldots$, the Gauss hypergeometric function, commonly written as ${}_2F_1(\alpha, \beta, \gamma, x)$ but for clarity written here as $H(\alpha, \beta, \gamma, x)$, is defined [3] as*

$$\begin{aligned} H(\alpha, \beta, \gamma, x) &= 1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \ldots, \end{aligned} \tag{2.1}$$

for $|x| < 1$.

Lemma 2.2. *Let $k > 0$, let F_p be the pth quantile of the $F(\nu_1, \nu_2)$ distribution such that $p = Pr(F < F_p)$, and define the function $s(p|\nu_1, \nu_2, k)$ in terms of Gauss's hypergeometric function as*

$$s(p|\nu_1, \nu_2, k) = H\left(\frac{\nu_1+\nu_2}{2}, 1, \frac{\nu_1}{2}+1, \tilde{F}_{p,1}\right) - H\left(\frac{\nu_1+\nu_2}{2}, 1, \frac{\nu_1}{2}+1, \tilde{F}_{p,k}\right),$$

where

$$\tilde{F}_{p,k} \equiv \frac{F_p}{F_p + \frac{k\nu_2}{\nu_1}}. \tag{2.2}$$

Then the following statements hold.
Case I: If $k \in (0, 1)$, then $s(p|\nu_1, \nu_2, k) < 0$.
Case II: If $k = 1$, then $s(p|\nu_1, \nu_2, k) = 0$.
Case III: If $k > 1$, then $s(p|\nu_1, \nu_2, k) > 0$.

Proof. By Definition 2.1, $H(\alpha, \beta, \gamma, x)$ is increasing in x for positive α, β, and γ. In our case, $\alpha := \frac{\nu_1+\nu_2}{2} > 0$, $\beta := 1 > 0$, and $\gamma := \frac{\nu_1}{2} + 1 > 0$. We have the following.
Case I: If $k \in (0, 1)$, then $\tilde{F}_{p,1} < \tilde{F}_{p,k}$ and consequently

$$s(p|\nu_1, \nu_2, k) = H\left(\alpha, \beta, \gamma, \tilde{F}_{p,1}\right) - H\left(\alpha, \beta, \gamma, \tilde{F}_{p,k}\right) < 0.$$

Case II: If $k = 1$, then $\tilde{F}_{p,1} = \tilde{F}_{p,k}$ and consequently

$$s(p|\nu_1, \nu_2, k) = H\left(\alpha, \beta, \gamma, \tilde{F}_{p,1}\right) - H\left(\alpha, \beta, \gamma, \tilde{F}_{p,k}\right) = 0.$$

Case III: If $k > 1$, then $\tilde{F}_{p,1} > \tilde{F}_{p,k}$ and consequently

$$s(p|\nu_1, \nu_2, k) = H\left(\alpha, \beta, \gamma, \tilde{F}_{p,1}\right) - H\left(\alpha, \beta, \gamma, \tilde{F}_{p,k}\right) > 0.$$

□

Definition 2.3. *For real numbers α and β with $\beta \neq 0, -1, -2, \ldots$, the Kummer confluent hypergeometric function, commonly written as ${}_1F_1(\alpha, \beta, x)$ but for clarity written here as $M(\alpha, \beta, x)$, is defined [3] by*

$$\begin{aligned} M(\alpha, \beta, x) &= 1 + \frac{\alpha}{\beta}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^3}{3!} + \ldots, \end{aligned}$$

for $-\infty < x < \infty$.

Lemma 2.4. *Let $k > 0$, let G_p be the pth quantile of the gamma distribution with positive shape parameter τ and scale parameter θ such that $p = P(G < G_p)$, and define the function $S(p|\tau, \theta, k)$ in terms of Kummer's confluent hypergeometric function such that*

$$S(p|\tau,\theta,k) = M\left(1, \tau+1, \frac{G_p}{\theta}\right) - M\left(1, \tau+1, \frac{G_p}{k\theta}\right).$$

Then the following assertions hold.
Case I: If $k \in [0,1)$, then $S(p|\tau,\theta,k) < 0$.
Case II: If $k = 1$, then $S(p|\tau,\theta,k) = 0$.
Case III: If $k > 1$, then $S(p|\tau,\theta,k) > 0$.

Proof. By Definition 2.3, $M(\alpha,\beta,x)$ is increasing in x for positive α and β. In our case, $\alpha := 1 > 0$ and $\beta := \tau + 1 > 0$. Thus
Case I: If $k \in (0,1)$, then $G_p/\theta < G_p/k\theta$ so that $M(1,\tau+1,G_p) - M\left(1,\tau+1,\frac{G_p}{k}\right) < 0$ and hence $S(p|\tau,\theta,k) < 0$.
Case II: If $k = 1$, then $G_p/\theta = G_p/k\theta$ so that $M(1,\tau+1,G_p) - M\left(1,\tau+1,\frac{G_p}{k}\right) = 0$ and hence $S(p|\tau,\theta,k) = 0$.
Case III: If $k > 1$, then $G_p/\theta > G_p/k\theta$ so that $M(1,\tau+1,G_p) - M\left(1,\tau+1,\frac{G_p}{k}\right) > 0$ and hence $S(p|\tau,\theta,k) > 0$. □

3. The F-Inequality

The F-inequality exploits the well-known relationship between the F distribution and the beta distribution, namely that if $F \sim F(\nu_1,\nu_2)$ then $\frac{F}{F+\frac{\nu_2}{\nu_1}} \sim Beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)$.

Definition 3.1. *For $a, b > 0$, the regularized incomplete beta function $I_x(a,b)$ is defined in terms of the incomplete beta function $B(x;a,b)$ and the complete beta function $B(a,b)$ as [4]*

$$I_x(a,b) = \frac{B(x;a,b)}{B(a,b)},$$

where

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$$

for $x \in [0,1]$ and

$$B(a,b) := B(1;a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

Note that $I_x(a,b)$ is the cumulative distribution function (CDF) of a $Beta(a,b)$ distribution.

Proposition 3.2. *Let F be a random variable with an $F(\nu_1,\nu_2)$ distribution. Define the FR-function as $h(p|k) = \frac{1}{p}B\left(\tilde{F}_{p,k}, \frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$ where $p \in (0,1)$. Then $h(p|k)$ is a monotonic function for $k \geq 0$ such that the following statements hold.*
Case I: If $0 \leq k < 1$, then $h(p|k)$ is strictly decreasing in p.
Case II: If $k = 1$, then $h(p|k) = B(\nu_1/2, \nu_2/2)$ (constant).
Case III: If $k > 1$, then $h(p|k)$ is strictly increasing in p.

Proof. The idea is to establish that $\frac{\partial}{\partial p}h(p|k) = 0$ if and only if $k = 1$. Since $\frac{\partial}{\partial p}h(p|k)$ is continuous in k, this means that for $k \in [0,1)$, $h(p|k)$ is monotonic and for $k > 1$, $h(p|k)$ is also monotonic. Moreover, for $k \in [0,1)$, we show that $h(p|k)$ is monotonically decreasing in p and for $k > 1$, that $h(p|k)$ is monotonically increasing in p.

Case II: The CDF of the F-distribution with ν_1 and ν_2 degrees of freedom is

$$F(x;\nu_1,\nu_2) := I_{\frac{x}{x+\frac{\nu_2}{\nu_1}}}\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right) = I_{\frac{\nu_1 x}{\nu_1 x+\nu_2}}\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right),$$

where I is the beta distribution CDF. Thus, incorporating equation (2.2),

$$\begin{aligned} & p = P\left(F \le F_p\right) = \frac{B\left(\frac{F_p}{F_p+\frac{\nu_2}{\nu_1}},\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}{B(\nu_1/2,\nu_2/2)} \\ \iff \quad & \frac{1}{p}B\left(\tilde{F}_{p,1},\frac{\nu_1}{2},\frac{\nu_2}{2}\right) = B(\nu_1/2,\nu_2/2). \end{aligned} \tag{3.1}$$

The left-hand side of this equality is precisely $h(p|k=1)$, completing the proof for this case.

The result of Case II tells us that $\frac{\partial}{\partial p}h(p|k=1) = 0$ for all p. If we let f be the density function of the F random variable we have $\frac{\partial}{\partial p}F_p = \frac{1}{f(F_p)}$ so that

$$\frac{\partial}{\partial p}h(p|k) = \frac{-1}{p^2}B\left(\tilde{F}_{p,k},\frac{\nu_1}{2},\frac{\nu_2}{2}\right)+\frac{1}{p}\left(\tilde{F}_{p,k}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,k}\right)^{\frac{\nu_2}{2}-1}\frac{\nu_2 k}{\nu_1 f(F_p)\left(F_p+\frac{\nu_2 k}{\nu_1}\right)^2}, \tag{3.2}$$

and for $k = 1$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial p}h(p|k=1) \\ &= -\frac{1}{p^2}B\left(\tilde{F}_{p,1},\frac{\nu_1}{2},\frac{\nu_2}{2}\right) \\ &\quad +\frac{1}{p}\left(\tilde{F}_{p,1}\right)^{\nu_1/2-1}\times\left(1-\tilde{F}_{p,1}\right)^{\nu_2/2-1}\frac{\nu_2}{\nu_1 f(F_p)\left(F_p+\frac{\nu_2}{\nu_1}\right)^2}, \end{aligned}$$

and thus

$$\frac{\nu_2}{\nu_1 f(F_p)} = \frac{B\left(\tilde{F}_{p,1},\frac{\nu_1}{2},\frac{\nu_2}{2}\right)\left(F_p+\frac{\nu_2}{\nu_1}\right)^2}{p\left(\tilde{F}_{p,1}\right)^{\nu_1/2-1}\left(1-\tilde{F}_{p,1}\right)^{\nu_2/2-1}}. \tag{3.3}$$

One can verify (3.3) as an exercise by using (3.1) and the substitution of F_p in the density function of the F distribution.

Before proving cases I and III, we provide an expression for $\frac{\partial}{\partial p}h(p|k)$ as follows. Letting $a := \frac{\nu_2}{\nu_1}$ and substituting (3.3) in (3.2), we obtain

$$\frac{\partial}{\partial p}h(p|k) = \frac{-1}{p^2}B\left(\tilde{F}_{p,k},\frac{\nu_1}{2},\frac{\nu_2}{2}\right)$$

$$+\frac{k\left(F_p+a\right)^2\left(\tilde{F}_{p,k}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,k}\right)^{\frac{\nu_2}{2}-1}B\left(\tilde{F}_{p,1},\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}{p^2\left(F_p+ka\right)^2\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}-1}}$$

$$= \frac{-1}{p^2}B_k+\frac{k}{p^2}\frac{\left(\tilde{F}_{p,k}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,k}\right)^{\frac{\nu_2}{2}-1}}{\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}-1}}\frac{(F_p+a)^2}{(F_p+ka)^2}B_1,$$

where

$$B_k \equiv B\left(\tilde{F}_{p,k},\frac{\nu_1}{2},\frac{\nu_2}{2}\right).$$

Observe that from (2.2),

$$\frac{\left(\tilde{F}_{p,k}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,k}\right)^{\frac{\nu_2}{2}-1}}{\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}-1}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}-1}}=\frac{(F_p+a)^{\frac{\nu_1}{2}+\frac{\nu_2}{2}-2}k^{\frac{\nu_2}{2}-1}}{(F_p+ka)^{\frac{\nu_1}{2}+\frac{\nu_2}{2}-2}}.$$

So

$$\begin{aligned}\frac{\partial}{\partial p}h(p|k) &= \frac{-1}{p^2}B_k+\frac{k^{\frac{\nu_2}{2}}}{p^2}\frac{(F_p+a)^{\frac{\nu_1+\nu_2}{2}-2}(F_p+a)^2}{(F_p+ka)^{\frac{\nu_1+\nu_2}{2}-2}(F_p+ka)^2}B_1\\ &= \frac{-1}{p^2}B_k+\frac{k^{\frac{\nu_2}{2}}}{p^2}\frac{(F_p+a)^{\frac{\nu_1+\nu_2}{2}}}{(F_p+ka)^{\frac{\nu_1+\nu_2}{2}}}B_1\\ &= \frac{-1}{p^2}B_k+\frac{1}{p^2}k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}}B_1. \qquad (3.4)\end{aligned}$$

As shown in [5], the hypergeometric representation of the incomplete beta function is $B(x;a,b)=\frac{x^a(1-x)^b}{a}H(a+b,1;a+1;x)$. Use of this representation reduces (3.4) to

$$\begin{aligned}\frac{\partial}{\partial p}h(p|k) &= \frac{-2}{\nu_1 p^2}\left(\tilde{F}_{p,k}\right)^{\frac{\nu_1}{2}}\left(1-\tilde{F}_{p,k}\right)^{\frac{\nu_2}{2}}H\left(\frac{\nu_1+\nu_2}{2},1,\frac{\nu_1}{2}+1,\tilde{F}_{p,k}\right)\\ &\quad+\frac{2}{\nu_1 p^2}k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}}\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}}\\ &\quad\times H\left(\frac{\nu_1+\nu_2}{2},1,\frac{\nu_1}{2}+1,\tilde{F}_{p,1}\right)\\ &= \frac{-2}{\nu_1 p^2}\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}}\left[-\left(\frac{\tilde{F}_{p,k}}{\tilde{F}_{p,1}}\right)^{\frac{\nu_1}{2}}\left(\frac{1-\tilde{F}_{p,k}}{1-\tilde{F}_{p,1}}\right)^{\frac{\nu_2}{2}}H_k\right.\\ &\quad\left.+k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}}H_1\right],\end{aligned}$$

where

$$H_k \equiv H\left(\frac{\nu_1+\nu_2}{2},1,\frac{\nu_1}{2}+1,\tilde{F}_{p,k}\right).$$

Observe that from (2.2)

$$\left(\frac{\tilde{F}_{p,k}}{\tilde{F}_{p,1}}\right)^{\frac{\nu_1}{2}}\left(\frac{1-\tilde{F}_{p,k}}{1-\tilde{F}_{p,1}}\right)^{\frac{\nu_2}{2}} = k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}}$$

and

$$\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}}k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}} = (ka)^{\frac{\nu_2}{2}}F_p^{\frac{\nu_1}{2}}(F_p+ka)^{-\frac{\nu_1+\nu_2}{2}}.$$

So

$$\begin{aligned}\frac{\partial}{\partial p}h(p|k) &= \frac{2}{\nu_1 p^2}\left(\tilde{F}_{p,1}\right)^{\frac{\nu_1}{2}}\left(1-\tilde{F}_{p,1}\right)^{\frac{\nu_2}{2}}k^{\frac{\nu_2}{2}}\left(\frac{F_p+ka}{F_p+a}\right)^{-\frac{\nu_1+\nu_2}{2}}[H_1-H_k]\\ &= \frac{2}{\nu_1 p^2}F_p^{\frac{\nu_1}{2}}(F_p+ka)^{-\frac{\nu_1+\nu_2}{2}}(ka)^{\frac{\nu_2}{2}}\times s(p|\nu_1,\nu_2,k), \qquad (3.5)\end{aligned}$$

where $s(p|\nu_1,\nu_2,k)$ was defined in Lemma 2.2.

Case I: If $k\in[0,1)$, then $\frac{2}{\nu_1}F_p^{\frac{\nu_1}{2}}(F_p+ka)^{-\frac{\nu_1+\nu_2}{2}}(ka)^{\frac{\nu_2}{2}}>0$. By Case I of Lemma 2.2 we have $s(p|\nu_1,\nu_2,k)<0$, hence $\frac{\partial}{\partial p}h(p|k)<0$ so that $h(p|k)$ is decreasing in p, which completes the proof for this case.

Case III: If $k>1$, then $\frac{2}{\nu_1}F_p^{\frac{\nu_1}{2}}(F_p+ka)^{-\frac{\nu_1+\nu_2}{2}}(ka)^{\frac{\nu_2}{2}}>0$. By Case III of Lemma 2.2 we have $s(p|\nu_1,\nu_2,k)>0$, hence $\frac{\partial}{\partial p}h(p|k)>0$ so that $h(p|k)$ is increasing in p, which completes the proof for this case. □

Figure 1 illustrates the behavior of $h(p|k)$ for different p and k values. It is clear that $h(p|k)$ decreases in p for $k\in[0,1)$ and increases in p for $k>1$, while $h(p|k)$ decreases in k for any p. Figure 2 shows $h(p,k)$ vs. p for fixed k values in a one-dimensional plot. For example, for $k\in[0,1)$ the dashed decreasing lines represent $h(p|k)$ as a function of p and they are bounded from below by the constant line $h(p|k=1)=B(\nu_1/2,\nu_2/2)$ and from above by the decreasing curve $h(p|k=0)=\frac{1}{p}B(\nu_1/2,\nu_2/2)$. On the other hand, for $k>1$ the dashed increasing lines represent $h(p|k)$ as a function of p and they are bounded by the constant lines $h(p|k=1)=B(\nu_1/2,\nu_2/2)$ and $h(p|k=\infty)=0$. If we look at $h(p|k)$ as a function of two variables p and k, it is apparent that $h(p|k)$ is a continuous function with no non-degenerate stationary points. In fact

$$\begin{aligned}&\frac{\partial}{\partial k}h(p,k)=0 \iff \left[\frac{\partial}{\partial x}B(x,\nu_1/2,\nu_2/2)/p\Big|_{x=\tilde{F}_{p,k}}\right]\times\left[\frac{\partial}{\partial k}\tilde{F}_{p,k}\right]=0\\ \iff\ &\frac{1}{p}\left(\tilde{F}_{p,k}\right)^{\nu_1/2-1}\left(1-\tilde{F}_{p,k}\right)^{\nu_2/2-1}\frac{-F_p}{(F_p+ka)^2}\left(\frac{\nu_2}{\nu_1}\right)=0\\ \iff\ &k=0, \qquad (3.6)\end{aligned}$$

and we have already established that $\frac{\partial}{\partial p}h(p,k)=0 \iff k=1$. That is, there is no point (p,k) at which $\frac{\partial}{\partial p}h(p,k)=\frac{\partial}{\partial k}h(p,k)=0$ which means that the surface of $h(p,k)$ is bimonotonic. Further, (3.6) implies that $h(k|p)$ is monotonic in k with a degenerate stationary point on the boundary of the domain when $k=0$. Also, one should expect $h(p|k)$ to decrease in p for fixed $k\in[0,1)$ — without examining the sign of the derivative — for the following reasons. Note that $h(p|k=0)=\frac{1}{p}B(\nu_1/2,\nu_2/2)$, $h(p|k=1)=B(\nu_1/2,\nu_2/2)$, and $h(p|k=1)<h(p|k=0)$ so that $\lim_{p\to1}h(p|k=0)=h(p|k=1)$ (i.e. $h(p,k)$ is bounded by a constant from

below and a monotonically decreasing function from above). Thus, since $h(p|k)$ is monotonic for $k \in [0,1)$, it must be decreasing in p; otherwise, by definition of infimum, it cannot be bounded by a constant from below and a monotonically decreasing function from above.

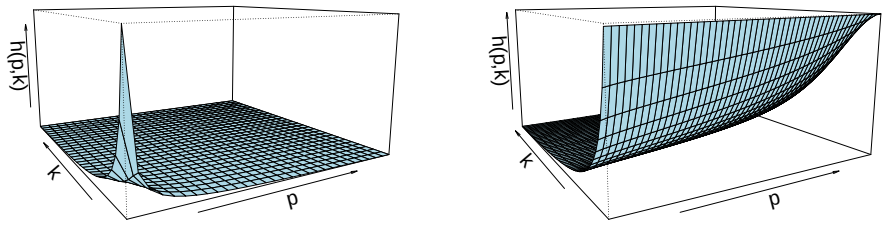

FIGURE 1. *Left*: Bimonotonic surface of the function $h(p,k)$ for $k \in [0,1)$ and $p \in (0,1)$. *Right*: Bimonotonic surface of the function $h(p,k)$ for $k > 1$ and $p \in (0,1)$.

Theorem 3.3. *Let $\alpha, p \in (0,1)$ such that $p > \alpha$ (i.e. $F_p > F_\alpha$ where $p = Pr(F < F_p)$ and $\alpha = P(F < F_\alpha)$). For $k > 1$, if F is a random variable with an $F(\nu_1, \nu_2)$ distribution, then the following inequality holds:*

$$\frac{1}{p} \times P\left(F < \frac{1}{k} F_p\right) \quad > \quad \frac{1}{\alpha} \times P\left(F < \frac{1}{k} F_\alpha\right), \tag{3.7}$$

or alternatively,

$$P\left(F < \frac{F_p}{k} \Big| F < F_p\right) \quad > \quad P\left(F < \frac{F_\alpha}{k} \Big| F < F_\alpha\right). \tag{3.8}$$

The inequality reverses for $k \in [0,1)$; it becomes equality for $k = 1$ and, trivially, for $k = \infty$.

Proof. Recall that the CDF of the F-distribution with ν_1 and ν_2 degrees of freedom is

$$F(x; \nu_1, \nu_2) = I_{\frac{\nu_1 x}{\nu_1 x + d_2}}\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right),$$

where I is the regularized incomplete beta function of Definition 3.1. Thus, one can write (3.7) in terms of the incomplete beta function as

$$\frac{1}{p} \times B\left(\tilde{F}_{p,k}; \frac{\nu_1}{2}, \frac{\nu_2}{2}\right) \quad > \quad \frac{1}{\alpha} \times B\left(\tilde{F}_{\alpha,k}; \frac{\nu_1}{2}, \frac{\nu_2}{2}\right). \tag{3.9}$$

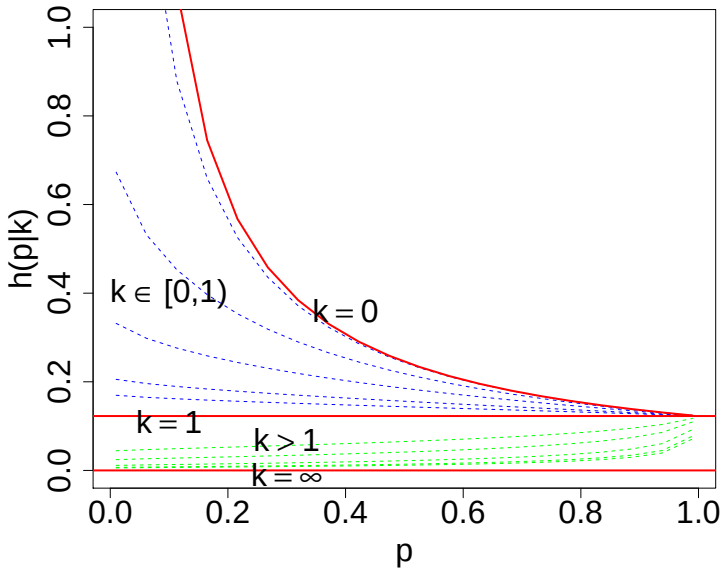

FIGURE 2. Plot of $h(p|k)$. The solid lines are the bounds at which $h(p|k)$ changes its monotonicity.

Now, since $k > 1$ and $p > \alpha$, the inequality in (3.9) follows immediately from Case III of Proposition 3.2. If $k \in [0, 1)$, then Case I of the proposition reverses the inequality; if $k = 1$, Case II changes the inequality to equality. For $k = \infty$, both sides of the inequality become zero making it equality. □

4. The G-Inequality

Similar arguments establish the G-inequality for gamma distributions.

Definition 4.1. *For $a > 0$, the lower incomplete gamma function $\gamma(a, x)$ is defined [4] as*

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$$

for $x \geq 0$ and the gamma function $\Gamma(a)$ is defined as

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

Recall that the CDF of the $Gamma(\tau, \theta)$ distribution is

$$F(x; \tau, \theta) = \frac{1}{\Gamma(\tau)} \gamma\left(\tau, \frac{x}{\theta}\right).$$

Proposition 4.2. *Let G be a random variable with the $Gamma(\tau, \theta)$ distribution. Define the GR-function as $\tilde{h}(p|k) = \frac{1}{p}\gamma\left(\tau, \frac{G_p}{k\theta}\right)$ where $p \in (0, 1)$ and G_p is the pth quantile of the gamma distribution such that $p = P(G < G_p)$. Then $\tilde{h}(p|k)$ is a monotonic function for $k \geq 0$ such that the following statements hold.*

Case I: If $0 \leq k < 1$, then $\tilde{h}(p|k)$ is strictly decreasing in p.
Case II: If $k = 1$, then $\tilde{h}(p|k) = \Gamma(\tau)$ (constant).
Case III: If $k > 1$, then $\tilde{h}(p|k)$ is strictly increasing in p.

Proof. The outline of the proof is identical to that of Proposition 3.2 and therefore we begin with case II as follows.
Case II:

$$\begin{aligned} & p = P\left(G \leq G_p\right) = \frac{1}{\Gamma(\tau)}\gamma\left(\tau, \frac{G_p}{\theta}\right) \\ \iff \quad & \frac{1}{p}\gamma\left(\tau, \frac{G_p}{\theta}\right) = \Gamma(\tau). \end{aligned} \tag{4.1}$$

But the left-hand side of this equality is precisely $\tilde{h}(p|k = 1)$, which completes the proof for this case.
The result of Case II tells us that $\frac{\partial}{\partial p}\tilde{h}(p|k = 1) = 0$ for all p. So if we let f be the density function of the G random variable, we have $\frac{\partial}{\partial p}G_p = \frac{1}{f(G_p)}$. Hence the following identity holds.

$$\begin{aligned} & \frac{\partial}{\partial p}\tilde{h}(p|k = 1) = 0 \\ \iff \quad & -\frac{1}{p^2}\gamma\left(\tau, \frac{G_p}{\theta}\right) + \frac{1}{p}\left(\frac{G_p}{\theta}\right)^{\tau-1} e^{\frac{-G_p}{\theta}} \frac{1}{\theta f(G_p)} = 0 \\ \iff \quad & \left(\frac{G_p}{\theta}\right)^{\tau-1} \frac{1}{\theta f(G_p)} = \frac{1}{p}\gamma\left(\tau, \frac{G_p}{\theta}\right) e^{\frac{G_p}{\theta}}. \end{aligned} \tag{4.2}$$

One can verify (4.2) as an exercise by using (4.1) and the substitution of G_p in the density function of the gamma distribution.

Use of this identity in deriving $\frac{\partial}{\partial p}\tilde{h}(p|k)$ gives

$$\begin{aligned} \frac{\partial}{\partial p}\tilde{h}(p|k) &= -\frac{1}{p^2}\gamma\left(\tau, \frac{G_p}{k\theta}\right) + \frac{1}{p}\left(\frac{G_p}{k\theta}\right)^{\tau-1} \frac{1}{k\theta f(G_p)} e^{\frac{-G_p}{k\theta}} \\ &= -\frac{1}{p^2}\gamma\left(\tau, \frac{G_p}{k\theta}\right) + \left[\frac{1}{p}\left(\frac{G_p}{\theta}\right)^{\tau-1} \frac{1}{\theta f(G_p)}\right] \frac{1}{k^\tau} e^{\frac{-G_p}{k\theta}} \\ &= -\frac{1}{p^2}\gamma\left(\tau, \frac{G_p}{k\theta}\right) + \frac{1}{p^2}\gamma\left(\tau, \frac{G_p}{\theta}\right) e^{\frac{G_p}{\theta}} e^{\frac{-G_p}{k\theta}} k^{-\tau}. \end{aligned} \tag{4.3}$$

As shown in [6], the connection between the lower incomplete gamma function and Kummer's confluent hypergeometric function is

$$\gamma(a, x) = a^{-1}x^a e^{-x} M(1, a + 1, x).$$

Use of this representation and arguments similar to those for the F-inequality reduce (4.3) to

$$\frac{\partial}{\partial p}\tilde{h}(p|k) = \frac{1}{p^2}\tau^{-1}\left(\frac{G_p}{k\theta}\right)^\tau e^{\frac{-G_p}{k\theta}} \times S(p|\tau, \theta, k), \tag{4.4}$$

where $S(p|\tau, \theta, k)$ was defined in Lemma 2.4.
Case I: If $k \in [0, 1)$, then $\frac{1}{p^2}\tau^{-1}\left(\frac{G_p}{k\theta}\right)^\tau e^{\frac{-G_p}{k\theta}} > 0$. By Case I of Lemma 2.2 we have

$S(p|\tau,\theta,k) < 0$ and hence $\frac{\partial}{\partial p}\tilde{h}(p|k) < 0$ so that $\tilde{h}(p|k)$ is decreasing in p, which completes the proof for this case.

Case III: If $k > 1$, then $\frac{1}{p^2}\tau^{-1}\left(\frac{G_p}{k\theta}\right)^{\tau} e^{\frac{-G_p}{k\theta}} > 0$. By Case III of Lemma 2.2 we have $S(p|\tau,\theta,k) > 0$ and hence $\frac{\partial}{\partial p}\tilde{h}(p|k) > 0$ so that $\tilde{h}(p|k)$ is increasing in p, which completes the proof for this case. □

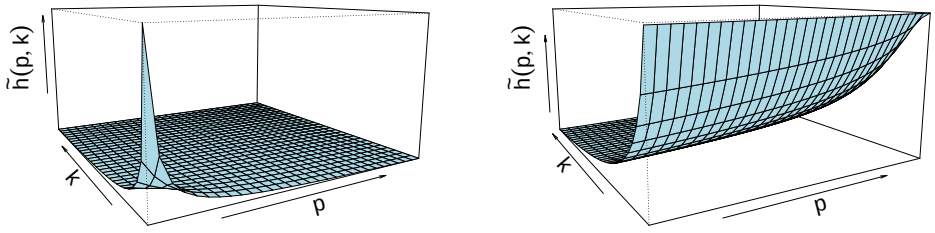

FIGURE 3. *Left*: Bimonotonic surface of the function $\tilde{h}(p,k)$ for $k \in [0,1)$ and $p \in (0,1)$. *Right*: Bimonotonic surface of the function $\tilde{h}(p,k)$ for $k > 1$ and $p \in (0,1)$.

Figure 3 illustrates the behavior of $\tilde{h}(p,k)$ for different p and k values. It is clear that $\tilde{h}(p,k)$ is decreasing in p for $k \in [0,1)$ and increasing in p for $k > 1$, while $\tilde{h}(p,k)$ is decreasing in k for any p. Figure 4 shows $\tilde{h}(p,k)$ vs. p for fixed k values in a one-dimensional plot. For example, for $k \in [0,1)$ the dashed decreasing lines represent $\tilde{h}(p|k)$ as a function of p and they are bounded from below by the constant line $\tilde{h}(p|k=1) = \Gamma(\tau)$ and from above by the decreasing line $\tilde{h}(p|k=0) = \frac{1}{p}\Gamma(\tau)$. On the other hand, for $k > 1$ the dashed increasing lines represent $h(p|k)$ as a function of p and they are bounded by the constant lines $\tilde{h}(p|k=1) = \Gamma(\tau)$ and $\tilde{h}(p|k=\infty) = 0$. If we look at $\tilde{h}(p,k)$ as a function of two variables p and k, it is easily seen that $\tilde{h}(p,k)$ is a continuous function with no non-degenerate stationary points. In fact

$$\begin{aligned}
&\frac{\partial}{\partial k}\tilde{h}(p,k) = 0 \iff \left[\frac{\partial}{\partial x}\gamma(\tau,x)/p\Big|_{x=\frac{G_p}{k\theta}}\right] \times \left[\frac{\partial}{\partial k}\frac{G_p}{k\theta}\right] = 0 \\
\iff\quad &\frac{1}{p}\left(\frac{G_p}{k\theta}\right)^{\tau-1} e^{\frac{-G_p}{k\theta}}\left(\frac{-G_p}{\theta k^2}\right) = 0 \\
\iff\quad &k = \infty, \qquad (4.5)
\end{aligned}$$

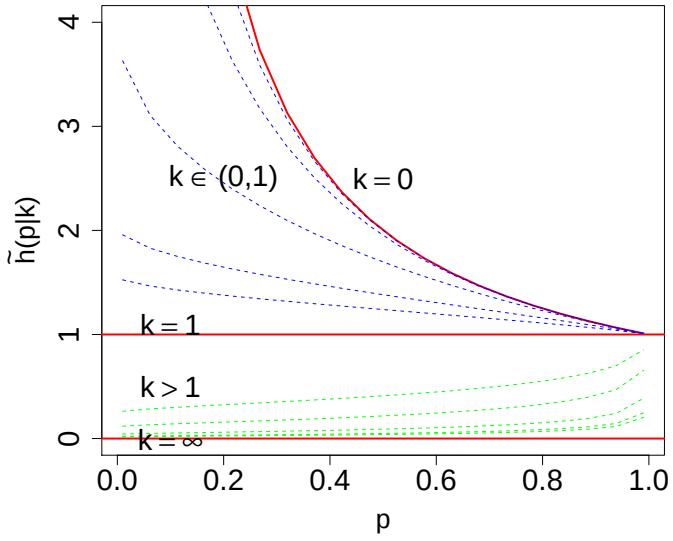

FIGURE 4. Plot of $\tilde{h}(p|k)$. The solid lines are the bounds at which $\tilde{h}(p|k)$ changes its monotonicity.

and we have already established that $\frac{\partial}{\partial p}\tilde{h}(p,k) = 0 \iff k = 1$. That is, there is no point (p,k) at which $\frac{\partial}{\partial p}\tilde{h}(p,k) = \frac{\partial}{\partial k}\tilde{h}(p,k) = 0$ which means that the surface of $\tilde{h}(p,k)$ is bimonotonic. Also, one should expect $\tilde{h}(p|k)$ to be decreasing in p for fixed $k \in [0,1)$, without examining the sign of the derivative. Indeed, from Cases I and II we know that $\tilde{h}(p|k=0) = \frac{1}{p}\Gamma(\tau)$, $\tilde{h}(p|k=1) = \Gamma(\tau)$, and $\tilde{h}(p|k=1) < \tilde{h}(p|k=0)$ so that $\lim_{p\to 1}\tilde{h}(p|k=0) = \tilde{h}(p|k=1)$ (i.e. $\tilde{h}(p,k)$ is bounded by a constant from below and a monotonically decreasing function from above). Since $\tilde{h}(p|k)$ is monotonic for $k \in [0,1)$, it must be be decreasing in p; otherwise, by definition of infimum, it cannot be bounded by a constant from below and a monotonically decreasing function from above.

Theorem 4.3. *Let $\alpha, p \in (0,1)$ such that $p > \alpha$ (i.e. $G_p > G_\alpha$ where $p = P(G < G_p)$ and $\alpha = P(G < G_\alpha)$). For $k > 1$, if G is a gamma random variable with parameters τ and θ, then the following inequality holds:*

$$\frac{1}{p} \times P\left(G < \frac{G_p}{k}\right) \quad > \quad \frac{1}{\alpha} \times P\left(G < \frac{G_\alpha}{k}\right), \tag{4.6}$$

or alternatively,

$$P\left(G < \frac{G_p}{k} \middle| G < G_p\right) \quad > \quad P\left(G < \frac{G_\alpha}{k} \middle| G < G_\alpha\right). \tag{4.7}$$

The inequality reverses for $k \in [0,1)$; it becomes equality for $k = 1$ and, trivially, for $k = \infty$.

Proof. Recall that the CDF of the $Gamma(\tau, \theta)$ distribution is

$$F(x; \tau, \theta) = \frac{1}{\Gamma(\tau)} \gamma\left(\tau, \frac{x}{\theta}\right),$$

where γ is the lower incomplete gamma function and Γ is the gamma function of Definition 4.1. Thus, one can write (4.6) in terms of the lower incomplete gamma function as

$$\frac{1}{p}\gamma\left(\tau, \frac{G_p}{k\theta}\right) \quad > \quad \frac{1}{\alpha}\gamma\left(\tau, \frac{G_\alpha}{k\theta}\right). \tag{4.8}$$

Now, since $k > 1$ and $p > \alpha$, the inequality in (4.8) follows immediately from Case III of Proposition 4.2. If $k \in [0, 1)$ then Case I of the same proposition reverses the inequality while, for $k = 1$, Case II changes the inequality to equality. For $k = \infty$, both sides of the inequality become zero making it equality. □

Remark. *We call random variables that inherit such inequality to be quantile-stochastic. This new theory of quantile-stochastic distributions will make it possible for the first time to compare distributions in terms of the ratio of their cumulative distribution function and quantiles and it opens new windows to distribution theory.*

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