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IDENTITIES FOR THE VOLUME OF THE UNIT HYPERSPHERES

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Abstract. The n-hypersphere with unit radius is defined as the set of points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying $\sum_{k=1}^{n} x_k^2 = 1$. We denote the hyper-surface area of an n-hypersphere of unit radius, and its volume, respectively by S_n and V_n. In this paper we compute the values of summations $\sum_{n=1}^{\infty} V_n^m$ and $\sum_{n=1}^{\infty} S_n^m$ for given integer $m \geqslant 1$. Meanwhile, we obtain various identities for some related summations.

1. Introduction and Summary of Results

Assume that $n \geqslant 1$ is an integer. The n-hypersphere (or simply n-sphere) with unit radius is defined as the set of points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ satisfying $\sum_{k=1}^{n} x_k^2 = 1$. We denote the hyper-surface area of an n-hypersphere of unit radius, and its volume, respectively by S_n and V_n. It is known that $S_n = nV_n$ is valid for $n \geqslant 0$, and for $n \geqslant 1$ we have

$$S_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \qquad \text{and} \qquad V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{1.1}$$

Moreover, we set $S_0 = 0$ and $V_0 = 1$ (see [8], pp 1438–1440). As usual, Γ denotes the Euler's gamma function, which is defined by $\Gamma(x) = \int_0^{\infty} \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$ for $x > 0$.

Because of connection of the volume of hyperspheres, and consequently the volume of the unit balls, to the Euler's gamma function, obtaining inequalities for the volume of the unit ball in $\mathbb{R}^n$ was the subject of some recent investigations (for example see [1], [2] and [5]). Also, studying monotonicity of functions connected with the gamma function, and consequently connected with the volume of the unit ball, was the subject of some other recent investigations (see for example [3] and [4]). In this paper, we obtain various identities for the volume of the unit hyperspheres. Indeed, we assume that $m \geqslant 1$ is integer, and we study the summation

$$S(m) := \sum_{n=1}^{\infty} V_n^m.$$

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The exact value of $S(1)$ is related by the value of $\mathrm{erf}(\sqrt{\pi})$, where the error function, $\mathrm{erf}(z)$ is the integral of the Gaussian distribution, defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x \mathrm{e}^{-t^2} \mathrm{d}t.$$

Indeed, we show the following result.

Theorem 1.1. *We have* $S(1) = \mathrm{e}^{\pi}\left(1 + \mathrm{erf}(\sqrt{\pi})\right) - 1 \approxeq 44.999$.

Next, we obtain the value of $S(2)$ in terms of the numbers $\mathbf{I}_0(2\pi)$ and $\mathbf{L}_0(2\pi)$. The function

$$\mathbf{I}_v(x) = \left(\frac{1}{2}x\right)^v \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k!\,\Gamma(v+k+1)}, \tag{1.2}$$

is a solution of the Modified Bessel's Equation $x^2y''(x) + xy'(x) - (x^2+v^2)y(x) = 0$ in $y(x)$ (see [6], pp 248–262), and we have

$$\mathbf{I}_0(x) = \frac{1}{\pi} \int\limits_0^{\pi} \cosh(x\cos t)\mathrm{d}t. \tag{1.3}$$

Also, the function

$$\mathbf{L}_v(x) = \left(\frac{1}{2}x\right)^{v+1} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{\Gamma(k+\frac{3}{2})\,\Gamma(v+k+\frac{3}{2})}, \tag{1.4}$$

is a solution of the Modified Struve's Equation

$$x^2y''(x) + xy'(x) - (x^2+v^2)y(x) = 4\frac{\left(\frac{1}{2}x\right)^{v+1}}{\sqrt{\pi}\,\Gamma(v+\frac{1}{2})},$$

in $y(x)$ (see [6], pp 287–301), and we have

$$\mathbf{L}_0(x) = \frac{2}{\pi} \int\limits_0^{\frac{\pi}{2}} \sinh(x\cos t)\mathrm{d}t = \frac{1}{\pi} \int\limits_0^{\pi} \sinh(x\sin t)\mathrm{d}t. \tag{1.5}$$

With these notations, we show the following result.

Theorem 1.2. *We have* $S(2) = \mathbf{I}_0(2\pi) + \mathbf{L}_0(2\pi) - 1 \approxeq 173.112$.

We can rewrite results of Theorem 1.1 and Theorem 1.2 in terms of hypergeometric function, which is defined by

$${}_pF_q\left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}\,;x\right] = \sum_{n=0}^{\infty} t_n x^n, \tag{1.6}$$

in which

$$\frac{t_{n+1}}{t_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)}{(n+b_1)(n+b_2)\cdots(n+b_q)(k+1)}x.$$

For special case $a_1 = \cdots = a_p = a$, and $b_1 = b_2 = \cdots = b_q = b$, we denote the left hand side of (1.6) simply by ${}_p\mathbf{H}_q[\{a\},\{b\},x]$. Conversion of (1.2) and (1.4) in terms of hypergeometric function implies

$$\mathbf{I}_v(x) = \frac{\left(\frac{1}{2}x\right)^v}{\Gamma(v+1)}\ {}_0\mathbf{H}_1\left[\{\ \},\{v+1\},\frac{x^2}{4}\right], \tag{1.7}$$

and

$$\mathbf{L}_v(x) = \frac{(-1)^{\frac{v}{2}}}{\mathrm{e}^{\frac{\pi}{2}vi}} \frac{2}{\sqrt{\pi}} \frac{\left(\frac{1}{2}x\right)^{v+1}}{\Gamma(v+\frac{3}{2})} \, {}_1F_2 \left[\begin{matrix} 1 & & \\ \frac{3}{2} & v+\frac{3}{2} & \end{matrix} ; \frac{x^2}{4} \right]. \tag{1.8}$$

By applying (1.7) and (1.8), we obtain

$$S(2) = {}_0\mathbf{H}_1 \left[\{\ \}, \{1\}; \pi^2\right] + 4 \, {}_1\mathbf{H}_2 \left[\{1\}, \left\{\frac{3}{2}\right\}; \pi^2\right] - 1. \tag{1.9}$$

Also, considering hypergeometric representation of the error function, as below

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}} \, {}_1\mathbf{H}_1 \left[\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2}\right\}; -x^2\right],$$

we get

$$S(1) = 2\mathrm{e}^{\pi} \, {}_1\mathbf{H}_1 \left[\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2}\right\}; -\pi\right] + \mathrm{e}^{\pi} - 1. \tag{1.10}$$

More generally, the notion of hypergeometric function allows us to get an identity for $S(m)$ in general, as follows.

Theorem 1.3. *For any integer $m \geqslant 3$ we have*

$$S(m) = {}_0\mathbf{H}_{m-1} \left[\{\ \}, \{1\}; \pi^m\right] + 2^m \, {}_1\mathbf{H}_m \left[\{1\}, \left\{\frac{3}{2}\right\}; \pi^m\right] - 1.$$

We give two proofs of Theorem 1.3. The first proof is detailed and shows connection to the Euler's gamma function, and the second proof is short and direct.

2. Proof of Theorems

Our strategy for proving our results on the summation $\sum_{n=1}^{\infty} V_n^m$, is considering a reformed formula for V_n, by transferring the Γ-factor in its fraction from denominator to numerator. This allows us to write the Γ-factor as an improper integral. Below, we describe the desired formula for V_n.

Lemma 2.1. *For any integer $n \geqslant 0$ we have*

$$V_n = \frac{2^n \pi^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})}{n!}. \tag{2.1}$$

Proof. The identity $\Gamma(x+1) = x\Gamma(x)$ with $x = \frac{n}{2}$ gives

$$V_n = \frac{2\pi^{\frac{n}{2}}}{n\,\Gamma(\frac{n}{2})} \qquad (\text{for } n \geqslant 1).$$

Duplication formula for the Euler's Gamma function, which is a special case of Gauss's multiplication formula (see[6], page 138), asserts that

$$\Gamma(2x) = (2\pi)^{-\frac{1}{2}} 2^{2x-\frac{1}{2}} \Gamma(x) \Gamma\Big(x + \frac{1}{2}\Big).$$

By using this identity with $x = \frac{n}{2}$, and considering $\Gamma(n) = (n-1)!$ we imply

$$\frac{1}{n\,\Gamma(\frac{n}{2})} = \frac{(2\pi)^{-\frac{1}{2}} 2^{n-\frac{1}{2}} \Gamma(\frac{n+1}{2})}{n!}.$$

This gives (2.1), and completes the proof. □

Proof of Theorem 1.1. By using (2.1), we have

$$S(1)=\sum_{n=1}^{\infty}\frac{2^n\pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}{n!}=\sum_{n=1}^{\infty}\frac{2^n\pi^{\frac{n-1}{2}}}{n!}\int\limits_0^{\infty}t^{\frac{n-1}{2}}\mathrm{e}^{-t}\mathrm{d}t.$$

We change the order of summation and integration to get

$$\begin{aligned}S(1)&=\int\limits_0^{\infty}2\mathrm{e}^{-t}\sum_{n=1}^{\infty}\frac{2^{n-1}\pi^{\frac{n-1}{2}}t^{\frac{n-1}{2}}}{n!}\mathrm{d}t\\&=\int\limits_0^{\infty}2\mathrm{e}^{-t}\sum_{n=1}^{\infty}\frac{(2\pi^{\frac{1}{2}}t^{\frac{1}{2}})^{n-1}}{n!}\mathrm{d}t=\int\limits_0^{\infty}\frac{-1+\mathrm{e}^{2\pi^{\frac{1}{2}}t^{\frac{1}{2}}}}{\pi^{\frac{1}{2}}t^{\frac{1}{2}}}\mathrm{e}^{-t}\mathrm{d}t.\end{aligned}$$

Let us denote the integrand of the last improper integral by $f(t)$, which is continuous and bounded over $[0,\infty)$. Moreover, by setting

$$F(t)=-\mathrm{erf}(\sqrt{t})-\mathrm{e}^{\pi}\mathrm{erf}(\sqrt{\pi}-\sqrt{t}),$$

we observe that $\frac{\mathrm{d}}{\mathrm{d}t}F(t)=f(t)$. Also, we have

$$\lim_{t\to\infty}F(t)=\mathrm{e}^{\pi}-1,\qquad\text{and}\qquad\lim_{t\to0^+}F(t)=-\mathrm{e}^{\pi}\mathrm{erf}(\sqrt{\pi}),$$

from which we obtain $S(1)=\mathrm{e}^{\pi}\left(1+\mathrm{erf}(\sqrt{\pi})\right)-1\approxeq 44.999$. This completes the proof. □

Remark. *Similarly, for $x>0$ we obtain*

$$\sum_{n=0}^{\infty}V_nx^n=\mathrm{e}^{\pi x^2}\left(1+\mathrm{erf}(\sqrt{\pi}x)\right).$$

This is generating function for the sequence $(V_n)_{n\geqslant0}$. Note that by repeated differentiating one may compute the summation $\sum\limits_{n=1}^{\infty}P(n)V_n$ for given polynomial $P(x)\in\mathbb{Z}[x]$. For example, we have

$$\begin{aligned}\sum_{n=1}^{\infty}nV_n&=2\pi\mathrm{e}^{\pi}\left(1+\mathrm{erf}(\sqrt{\pi})\right)+2\approxeq 291.022,\\\sum_{n=1}^{\infty}n^2V_n&=4\pi\mathrm{e}^{\pi}\left(1+\pi\right)\left(1+\mathrm{erf}(\sqrt{\pi})\right)+2\approxeq 2408.592.\end{aligned}$$

Remark. *We note that*

$$\sum_{n=0}^{\infty}V_{2n}=\sum_{n=0}^{\infty}\frac{\pi^n}{n!}=\mathrm{e}^{\pi}\approxeq 23.141.$$

Thus, we obtain

$$\sum_{n=1}^{\infty}V_{2n-1}=\mathrm{e}^{\pi}\mathrm{erf}(\sqrt{\pi})\approxeq 22.859.$$

Proof of Theorem 1.2. We consider the notion of Euler's beta integral, which is defined by

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\qquad(\text{for }a,b>0)$$

By using (2.1), we have

$$S(2)=\sum_{n=1}^{\infty}\frac{2^{2n}\pi^{n-1}\Gamma(\frac{n+1}{2})\Gamma(\frac{n+1}{2})}{n!\,\Gamma(n+1)}$$
$$=\sum_{n=1}^{\infty}\frac{2^{2n}\pi^{n-1}B(\frac{n+1}{2},\frac{n+1}{2})}{n!}=\sum_{n=1}^{\infty}\frac{2^{2n}\pi^{n-1}}{n!}\int_0^1\big(t(1-t)\big)^{\frac{n-1}{2}}\mathrm{d}t.$$

Changing the order of summation and integration implies

$$S(2)=\int_0^1 4\sum_{n=1}^{\infty}\frac{\left(4\pi(t(1-t))^{\frac12}\right)^{n-1}}{n!}\mathrm{d}t=\int_0^1\frac{-1+\mathrm{e}^{4\pi(t(1-t))^{\frac12}}}{\pi(t(1-t))^{\frac12}}\mathrm{d}t$$
$$=\int_0^1\frac{-1}{\pi(t(1-t))^{\frac12}}\mathrm{d}t+\int_0^1\frac{\mathrm{e}^{4\pi(t(1-t))^{\frac12}}}{\pi(t(1-t))^{\frac12}}\mathrm{d}t=-1+\frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{2\pi\sin\theta}\mathrm{d}\theta,$$

where the last equality obtained by applying the change of variable $t=\sin^2\frac{\theta}{2}$. On the other hand, by using (1.3) and (1.5) we get

$$\frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{x\sin\theta}\mathrm{d}\theta=\frac{1}{\pi}\int_0^{\pi}\big(\cosh(x\sin\theta)+\sinh(x\sin\theta)\big)\mathrm{d}\theta=\mathbf{I}_0(x)+\mathbf{L}_0(x).$$

This completes the proof. □

Remark. *We follow a similar argument to get*

$$\sum_{n=0}^{\infty}V_n^2x^n=\mathbf{I}_0(2\pi x^2)+\mathbf{L}_0(2\pi x^2)=\frac{1}{\pi}\int_0^{\pi}\mathrm{e}^{2\pi x^2\sin\theta}\mathrm{d}\theta,$$

for any $x>0$. This is generating function for the sequence $(V_n^2)_{n\geqslant 0}$.

First Proof of Theorem 1.3. We have

$$\Gamma\Big(\frac{n+1}{2}\Big)=\int_0^{\infty}\mathrm{e}^{-x}x^{\frac{n-1}{2}}\mathrm{d}x.$$

Now, assume that $m\geqslant 3$ is integer. We multiply above integral representation m times to get

$$\Gamma\Big(\frac{n+1}{2}\Big)^m=\int_0^{\infty}\dots\int_0^{\infty}\mathrm{e}^{-\sum_{i=1}^{m}x_i}\prod_{i=1}^{m}x_i^{\frac{n-1}{2}}\mathrm{d}\mathbf{x},\tag{2.2}$$

where here and in what follows below, $\int\dots\int$ is m-fold integration, and $\mathrm{d}\mathbf{x}=\mathrm{d}x_1\dots\mathrm{d}x_m$. Let us set

$$\mathcal{P}:=\mathcal{P}(x_1,\dots,x_m)=\prod_{i=1}^{m}x_i^{\frac12}.$$

By using (2.1), and then (2.2) we have

$$\begin{aligned} S(m) &= \sum_{n=1}^{\infty} \frac{2^{m(n-1)}\pi^{\frac{m}{2}(n-1)}}{n!^m} \left(2\,\Gamma\Big(\frac{n+1}{2}\Big)\right)^m \\ &= \sum_{n=1}^{\infty} \frac{2^{m(n-1)}\pi^{\frac{m}{2}(n-1)}}{n!^m} \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} 2^m \mathrm{e}^{-\sum_{i=1}^m x_i} \mathcal{P}^{n-1}\mathrm{d}\mathbf{x}. \end{aligned}$$

We change the order of m-fold integration and summation, to obtain

$$\begin{aligned} S(m) &= \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} 2^m \mathrm{e}^{-\sum_{i=1}^m x_i} \sum_{n=1}^{\infty} \frac{\left(2^m\pi^{\frac{m}{2}}\mathcal{P}\right)^{n-1}}{n!^m}\,\mathrm{d}\mathbf{x} \\ &= \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \frac{\mathrm{e}^{-\sum_{i=1}^m x_i}}{\pi^{\frac{m}{2}}\mathcal{P}} \left(\sum_{n=0}^{\infty}\mathcal{J}_n(m;x_1,\dots,x_m)-1\right)\mathrm{d}\mathbf{x} \end{aligned}$$

where

$$\mathcal{J}_n := \mathcal{J}_n(m;x_1,\dots,x_m) = \frac{\left(2^m\pi^{\frac{m}{2}}\mathcal{P}\right)^n}{n!^m}.$$

We observe that

$$\frac{\mathcal{J}_{n+1}}{\mathcal{J}_n} = \frac{2^m\pi^{\frac{m}{2}}\mathcal{P}}{(n+1)(n+1)^{m-1}}.$$

Thus, the summation $\sum \mathcal{J}_n$ is indeed a hypergeometric function. More precisely, by taking $x = 2^m\pi^{\frac{m}{2}}\mathcal{P}$ and $t_n = n!^{-m}$ in (1.6) we obtain

$$S(m) = \int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \frac{\mathrm{e}^{-\sum_{i=1}^m x_i}}{\pi^{\frac{m}{2}}\mathcal{P}} \left({}_0\mathbf{H}_{m-1}\left[\{\ \},\{1\};2^m\pi^{\frac{m}{2}}\mathcal{P}\right]-1\right)\mathrm{d}\mathbf{x}.$$

We apply the change of variables $x_i = u_i^2$ to deduce that

$$\int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \frac{\mathrm{e}^{-\sum_{i=1}^m x_i}}{\mathcal{P}}\mathrm{d}\mathbf{x} = \left(2\int_0^{\infty}\mathrm{e}^{-t^2}\mathrm{d}t\right)^m = \pi^{\frac{m}{2}}.$$

Moreover, we get

$$S(m) = \frac{2^m}{\pi^{\frac{m}{2}}}\int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \mathrm{e}^{-\sum_{i=1}^m u_i^2}\ {}_0\mathbf{H}_{m-1}\left[\{\ \},\{1\};2^m\pi^{\frac{m}{2}}\prod_{i=1}^m u_i\right]\mathrm{d}\mathbf{u}-1,$$

where $\mathrm{d}\mathbf{u} = \mathrm{d}u_1\dots\mathrm{d}u_m$. By induction on $m \geq 3$ one may show that

$$\begin{aligned} &\frac{2^m}{\pi^{\frac{m}{2}}}\int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \mathrm{e}^{-\sum_{i=1}^m u_i^2}\ {}_0\mathbf{H}_{m-1}\left[\{\ \},\{1\};2^m\pi^{\frac{m}{2}}\prod_{i=1}^m u_i\right]\mathrm{d}\mathbf{u} \\ &\qquad = {}_0\mathbf{H}_{m-1}\left[\{\ \},\{1\};\pi^m\right] + 2^m\ {}_1\mathbf{H}_m\left[\{1\},\left\{\frac{3}{2}\right\};\pi^m\right]. \end{aligned}$$

This completes the proof of Theorem 1.3. □

Second Proof of Theorem 1.3. As we see in Remark 2, computing the sum of terms with even subscript in $S(1)$ is straightforward. Same situation is valid when we compute $S(m)$. Indeed, based on this fact, one may give a direct proof of Theorem 1.3 as follows. From the functional equation of the gamma function, we have

$$\frac{V_{2n+2}}{V_{2n}} = \frac{\pi}{n+1}, \qquad \text{and} \qquad \frac{V_{2n+3}}{V_{2n+1}} = \frac{\pi}{n+\frac{3}{2}}.$$

Together with $V_0 = 1$ and $V_1 = 2$, we imply that

$$\sum_{n=0}^{\infty} V_{2n}^m = {}_0\mathbf{H}_{m-1}\left[\{\ \}, \{1\}; \pi^m\right],$$

and

$$\sum_{n=1}^{\infty} V_{2n-1}^m = 2^m\ {}_1\mathbf{H}_m\left[\{1\}, \left\{\frac{3}{2}\right\}; \pi^m\right].$$

Combining these we get the identity for $S(m)$. □

3. Further remarks

Remark. *One may do similar analysis on* $\sum_{n=1}^{\infty} S_n^m$ *for given positive integer* m. *Following similar arguments as in the proof of the above theorems, we imply*

$$\sum_{n=1}^{\infty} S_n = 2\left(1 + \pi e^{\pi}\left(1 + \operatorname{erf}(\sqrt{\pi})\right)\right) \approxeq 291.022,$$

$$\sum_{n=1}^{\infty} S_n^2 = 4\left(1 + \pi^2\left(\mathbf{I}_0(2\pi) + \mathbf{L}_0(2\pi)\right)\right) \approxeq 6877.681.$$

More generally, for $m \geqslant 2$ *we have*

$$\sum_{n=1}^{\infty} S_n^m = (2\pi)^m\ {}_0\mathbf{H}_{m-1}\left[\{\ \}, \{1\}; \pi^m\right] + (4\pi)^m\ {}_1\mathbf{H}_m\left[\{1\}, \left\{\frac{3}{2}\right\}; \pi^m\right] + 2^m.$$

Remark. *A combinatorial recurrence argument (see* [7], *pp 135–136), implies* $V_n = 2I_nV_{n-1}$ *for* $n \geqslant 1$, *with*

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t\ \mathrm{d}t.$$

This gives $V_n = 2^n \prod_{k=1}^n I_k$ *for* $n \geqslant 1$. *Thus, we have*

$$\prod_{k=1}^{n} I_k = \frac{V_n}{2^n} = \frac{\left(\frac{\pi}{4}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Again, following similar arguments as in the proof of the above theorems, imply

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} I_k\right) = e^{\frac{\pi}{4}}\left(1 + \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\right)\right) - 1 \approxeq 2.925,$$

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} I_k\right)^2 = \mathbf{I}_0\left(\frac{\pi}{2}\right) + \mathbf{L}_0\left(\frac{\pi}{2}\right) - 1 \approxeq 2.021.$$

More generally, for $m \geqslant 3$ we have

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} I_k\right)^m = \left(\frac{\pi}{4}\right)^m {}_1\mathbf{H}_m\left[\{1\},\{2\};\left(\frac{\pi}{4}\right)^m\right] + \left(\frac{\pi}{6}\right)^m {}_1\mathbf{H}_m\left[\{1\},\left\{\frac{5}{2}\right\};\left(\frac{\pi}{4}\right)^m\right] + 1.$$

Meanwhile, by using Stirling's approximation for the Gamma function, we observe that

$$\lim_{n\to\infty} n\left(\prod_{k=1}^{n} I_k\right)^{\frac{1}{n}} = \frac{\mathrm{e}\pi}{4}.$$

Finally, we note that since the equality $\int_0^{\frac{\pi}{2}} \sin^r t \,\mathrm{d}t = \int_0^{\frac{\pi}{2}} \cos^r t \,\mathrm{d}t$ is valid for any real number r, thus all identities of present remark are valid for $I_n = \int_0^{\frac{\pi}{2}} \cos^n t \,\mathrm{d}t$, too.

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