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NEW INEQUALITIES FOR A PRODUCT OF BESSEL FUNCTIONS

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Abstract. We use several methods to bound a product of modified Bessel functions. These inequalities, which do not seem to appear in the classical literature of Bessel functions, are useful in mathematical physics to estimate the number of eigenvalues of a Schrödinger operator.

1. Introduction

Let I_ν and K_ν denote the usual Bessel modified functions of first and second kind with $\nu \in \mathbb{R}$. There is an extensive literature dealing with these functions (cf. [1, 3]) and they play a very important role in many different fields in mathematical physics.

In the present paper, we focus on the zero order functions, and more specifically on the product $I_0(x)K_0(x)$ for $x > 0$. In a private communication [8], the second author proved that

$$I_0(x)K_0(x) \leq 1 - \ln x \quad (x \in (0,1]) . \tag{1.1}$$

This inequality plays a key part in a well-known mathematical problem in physics. More precisely, Françoise Truc showed [8] how (1.1) allowed her to bound the number of eigenvalues of a certain Schrödinger operator (see Section 4).

The purpose of this work is to provide an improvement on (1.1) by using two different representations of the modified Bessel functions I_0 and K_0. The first result below uses the classical series representation of these functions.

Theorem 1.1. *For any $x \in (0,1]$*

$$\ln 2 - \gamma - \ln x \leq I_0(x)K_0(x) < 0.66 - \ln x$$

where $\gamma \approx 0.577\,215\,664\ldots$ denotes the Euler constant.

Our second result uses less known Mellin-Barnes integral representation of the product $I_\nu K_\nu$ and some contour integration method. This enables us to substantially improve on (1.1) as follows.

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Theorem 1.2. *For any $x > 0$*

$$\left| I_0(x)K_0(x) + \ln \frac{x}{2} + \gamma \right| \leq \frac{x\sqrt{3}}{4}.$$

In particular, for any $x \in (0, 1]$, we have

$$I_0(x)K_0(x) < 0.55 - \ln x.$$

Translating the contour integration into the opposite direction also provides the following inequality.

Theorem 1.3. *For any $x > 0$*

$$\left| I_0(x)K_0(x) - \frac{1}{2x} \right| \leq \frac{1}{16x^2}.$$

2. Proof of Theorem 1.1

According to the series expansion [1, (9.6.10)], we have

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x^2/4)^k}{(k!)^2} \quad (x \in \mathbb{R}).$$

It follows that

$$1 \leq I_0(x) \leq e^{x^2/4} \quad (x \in \mathbb{R}). \tag{2.1}$$

Similarly, from [1, (9.6.13)], we have

$$K_0(x) = -\left(\ln \frac{x}{2} + \gamma\right) I_0(x) + \sum_{k=1}^{\infty} \frac{H_k}{(k!)^2} \left(\frac{x^2}{4}\right)^k \quad (x > 0)$$

where $H_k = \sum_{j=1}^{k} 1/j$ denotes the kth harmonic number, so that

$$\begin{aligned} \delta(x) &:= I_0(x)K_0(x) + \ln x \\ &= \left(1 - I_0(x)^2\right) \ln x + (\ln 2 - \gamma)\, I_0(x)^2 + I_0(x) \sum_{k=1}^{\infty} \frac{H_k}{(k!)^2} \left(\frac{x^2}{4}\right)^k. \end{aligned}$$

Now it follows at once from the left-hand side of (2.1) that

$$\delta(x) \geq (\ln 2 - \gamma)\, I_0(x)^2 \geq \ln 2 - \gamma.$$

On the other hand, according to (3.1) and using the easy bound $H_k \leq \ln(ek) \leq k!$ $(k \in \mathbb{Z}_{\geq 1})$, we obtain

$$\begin{aligned} \delta(x) &\leq -\left(e^{x^2/2} - 1\right) \ln x + (\ln 2 - \gamma)\, e^{x^2/2} + e^{x^2/4} \sum_{k=1}^{+\infty} \frac{(x^2/4)^k}{k!} \\ &= -\left(e^{x^2/2} - 1\right) \ln x + e^{x^2/2} \left(1 + \ln 2 - \gamma - e^{-x^2/4}\right). \end{aligned}$$

The inequality $-\ln x \leq (aex^a)^{-1}$ [5, page 266], which holds for any $x, a > 0$, leads to

$$-\left(e^{x^2/2} - 1\right) \ln x \leq \left(e^{x^2/2} - 1\right) \min\left\{(2ex^2)^{-1}, (3ex^3)^{-1}\right\} \leq \frac{9}{8e}\left(e^{2/9} - 1\right)$$

henceforth we infer that

$$\delta(x) \leq \frac{9}{8e}\left(e^{2/9} - 1\right) + e^{1/2}\left(1 + \log 2 - \gamma - e^{-1/4}\right) < 0.66 \quad (x \in (0, 1])$$

which proves Theorem 1.1. □

3. Proofs of Theorems 1.2 and 1.3

3.1. **The Γ-function and Stirling's formula.** In this section, we set $s = \sigma + it \in \mathbb{C}$. The Euler Γ-function is defined by

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} \mathrm{d}x \quad (\sigma > 0)\,.$$

This function can be extended analytically to the whole complex plane to a meromorphic function having simple poles at $s = -k$ with $k \in \mathbb{Z}_{\geq 0}$.

For the proofs of Theorems 1.2 and 1.3, we proceed as follows: from the Mellin-Barnes integral representation of the studied function, we shift the integration contour to the left or right taking one or more poles of the Γ-functions into account. In order to apply Cauchy's theorem, the order of magnitude of the Γ-functions of the integrand has to be controlled. A key part is then played by *Stirling's formula*: *If $\sigma \in [a, b]$ is fixed, then for $|t| \geq t_0$ we get*

$$|\Gamma(\sigma + it)| = |t|^{\sigma - 1/2} e^{-\pi |t|/2} \sqrt{2\pi} \left(1 + O\left(|t|^{-1}\right)\right). \tag{3.1}$$

See [7, page 182] for a proof.

3.2. **Proof of Theorem 1.2.** Let $x > 0$. From [6, (3.4.28)], we get

$$I_0(x)\, K_0(x) = \frac{1}{2\sqrt{\pi}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)^2 \Gamma\left(\frac12 - s\right)}{\Gamma(1-s)}\, x^{-2s}\, \mathrm{d}s \quad \left(0 < c < \tfrac12\right) \tag{3.2}$$

the convergence of the integral being ensured by [6, Lemma 2.4]. We shift the integration contour to the vertical line $\operatorname{Re} s = -\frac12$, which is possible by (3.1). The integrand has a double pole at $s = 0$. With the help of the functional equation $\Gamma(s+1) = s\Gamma(s)$, we have at once

$$\begin{aligned}
\operatorname*{Res}_{s=0} \left(\frac{\Gamma(s)^2 \Gamma\left(\frac12 - s\right)}{\Gamma(1-s)}\, x^{-2s} \right) &= \left[\frac{\mathrm{d}}{\mathrm{d}s} \frac{s^2 \Gamma(s)^2 \Gamma\left(\frac12 - s\right)}{\Gamma(1-s)}\, x^{-2s} \right]_{[s=0]} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}s} \frac{\Gamma(s+1)^2 \Gamma\left(\frac12 - s\right)}{\Gamma(1-s)}\, x^{-2s} \right]_{[s=0]} \\
&= -2\sqrt{\pi} \left(\ln \frac{x}{2} + \gamma \right)
\end{aligned}$$

and then (3.2) gives

$$I_0(x)\, K_0(x) = -\ln \frac{x}{2} - \gamma + I(x)$$

with

$$\begin{aligned}
I(x) &= \frac{1}{2\sqrt{\pi}} \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\Gamma(s)^2 \Gamma\left(\frac12 - s\right)}{\Gamma(1-s)}\, x^{-2s}\, \mathrm{d}s \\
&= \frac{1}{2\pi^{3/2}} \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s)^3 \Gamma\left(\tfrac12 - s\right) \sin(\pi s)\, x^{-2s}\, \mathrm{d}s
\end{aligned}$$

where we used the reflection formula. Since

$$\left|\sin \pi \left(-\tfrac12 + it\right)\right| = \cosh \pi t \quad \text{and} \quad |\Gamma(1+it)| = \sqrt{\frac{\pi t}{\sinh \pi t}}$$

we get

$$\begin{aligned}|I(x)| &\leq \frac{x}{2\pi^{5/2}} \int_0^\infty \left|\Gamma\left(-\tfrac12+it\right)\right|^3 |\Gamma(1+it)| \cosh \pi t \,\mathrm{d}t \\ &= \frac{x}{2\pi^2} \int_0^\infty \left|\Gamma\left(-\tfrac12+it\right)\right|^3 \frac{\cosh \pi t}{\sqrt{\sinh \pi t}} \sqrt{t}\,\mathrm{d}t \\ &= \frac{x}{2\pi^2} \int_0^\infty \frac{\left|\Gamma\left(\tfrac12+it\right)\right|^3}{\left|-\tfrac12+it\right|^3} \frac{\cosh \pi t}{\sqrt{\sinh \pi t}} \sqrt{t}\,\mathrm{d}t\end{aligned}$$

and since $\left|\Gamma\left(\tfrac12+it\right)\right| = \sqrt{\dfrac{\pi}{\cosh \pi t}}$, we obtain

$$\begin{aligned}|I(x)| &\leq \frac{x}{2\sqrt{\pi}} \int_0^\infty \left|-\tfrac12+it\right|^{-3} \sqrt{\frac{t}{\sinh \pi t \cosh \pi t}}\,\mathrm{d}t \\ &= \frac{x}{\sqrt{2\pi}} \int_0^\infty \left|-\tfrac12+it\right|^{-3} \sqrt{\frac{t}{\sinh 2\pi t}}\,\mathrm{d}t \\ &\leq \frac{x}{\sqrt{2\pi}} \left(\int_0^\infty \left|-\tfrac12+it\right|^{-6}\mathrm{d}t\right)^{1/2} \left(\int_0^\infty \frac{t}{\sinh 2\pi t}\,\mathrm{d}t\right)^{1/2} \\ &= \frac{x\sqrt{3}}{4}\end{aligned}$$

where we used Cauchy-Schwarz's inequality. The proof is complete. □

3.3. **Proof of Theorem 1.3.** The proof is similar to that of Theorem 1.2, except that we translate the integration contour to the straight line $\operatorname{Re} s = 1$, so that the integrand has only a simple pole at $s = \frac12$. Equality (3.2) leads to

$$I_0(x)\,K_0(x) = \frac{1}{2x} + J(x)$$

with

$$|J(x)| = \left|\frac{1}{2\pi^{3/2}} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s)^3 \Gamma\left(\tfrac12 - s\right) \sin(\pi s)\, x^{-2s}\,\mathrm{d}s\right|.$$

Proceeding as in the proof of Theorem 1.2 and using

$$|\sin \pi\,(1+it)| = |\sinh \pi t|$$

we get

$$\begin{aligned}|J(x)| &\leq \frac{1}{2\pi^{5/2}x^2} \int_0^\infty |\Gamma(1+it)|^3 \left|\Gamma\left(-\tfrac12+it\right)\right| \sinh \pi t\,\mathrm{d}t \\ &= \frac{1}{x^2\sqrt{2\pi}} \int_0^\infty \left|-\tfrac12+it\right|^{-1} \sqrt{\frac{t^3}{\sinh 2\pi t}}\,\mathrm{d}t \\ &\leq \frac{1}{x^2\sqrt{2\pi}} \left(\int_0^\infty \left|-\tfrac12+it\right|^{-2}\mathrm{d}t\right)^{1/2} \left(\int_0^\infty \frac{t^3}{\sinh 2\pi t}\,\mathrm{d}t\right)^{1/2} \\ &= \frac{1}{16x^2}\end{aligned}$$

as required. □

4. Application

Let H_A^D be a Schrödinger operator and a radial potential V, under the assumptions that the resolvant of this operator is compact and that the operator $H_A^D - V$ has a discrete negative spectrum. Using the previous inequalities, we get an upper bound for the number $\mathcal{N}(H_A^D, \lambda)$ of eigenvalues of H_A^D not exceeding some fixed quantity $\lambda > 0$.

Let us go a little further. Let $B = \mathbf{b}(\mathbf{x})\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$ be a radially symmetric magnetic field over the open disc $\mathcal{D}$ with center O and radius 1. Assume that

$$K := \inf_{\mathbf{x}\in\mathcal{D}} \mathbf{b}(\mathbf{x}) > 0 \tag{4.1}$$

that $\mathbf{b}(\mathbf{x}) \longrightarrow \infty$ as $\mathbf{x}$ approaches the boundary $\delta\mathcal{D}$ of $\mathcal{D}$, and that

$$\mathbf{b}(\mathbf{x}) = O\left(D(\mathbf{x})^{-\beta}\right)$$

for some $\beta \in \left(0, \frac{3}{2}\right)$, where $D(\mathbf{x})$ is the distance of $\mathbf{x} = (x_1, x_2)$ to $\delta\mathcal{D}$. This condition implies that the magnetic field considered here is of the shape

$$\mathbf{b}(r) = \frac{b_r}{(1-r)^\beta} \tag{4.2}$$

where $r^2 = x_1^2 + x_2^2$, $\max\limits_{0\le r<1} b_r \le M$ for some $M > 0$, and $\beta \in \left(0, \frac{3}{2}\right)$. Under these assumptions, Françoise Truc [8, Theorem 2.2] established a general estimate for the number $\mathcal{N}(H_A^D, \lambda)$ introduced above. In the case $b_r = \beta = 1$, this result provides the bound

$$\mathcal{N}(H_A^D, \lambda) \le 2\lambda + \lambda^{1/2}\sqrt{2\zeta(3) - \tfrac{3}{2}} + O(1).$$

The use of the previous theorems enables us to improve on this estimate.

Corollary 4.1. *Let B be a radially symmetric magnetic field as above satisfying the assumptions* (4.1) *and* (4.2) *with $b_r = \beta = 1$. Then*

$$\mathcal{N}(H_A^D, \lambda) < \lambda\left(\ln 2 - \gamma + 1 + \tfrac{1}{2\sqrt{3}}\right) + \lambda^{1/2}\sqrt{2\zeta(3) - \tfrac{3}{2}} + \tfrac{1}{2}.$$

Observe that $\ln 2 - \gamma + 1 + \frac{1}{2\sqrt{3}} \approx 1.404\,6\ldots$

Proof. From [8, Theorem 2.2] we have

$$\mathcal{N}(H_A^D, \lambda) \le \frac{2\lambda}{K}\int_0^{\sqrt{K}} r\, I_0(r) K_0(r)\,\mathrm{d}r + \frac{\lambda}{2\sqrt{1-\alpha}} + \frac{\sqrt{1-\alpha}}{\alpha}\int_0^1 \frac{A(r)^2}{r}\,\mathrm{d}r \tag{4.3}$$

where $\alpha \in (0,1)$ is a parameter at our disposal and

$$A(r) := \int_0^r u\,\mathbf{b}(u)\,\mathrm{d}u \quad (r \in [0,1))\,.$$

To estimate the integral of the main term of (4.3), we proceed as follows.

▷ If $0 < K \le 1$, we use Theorem 1.2 implying

$$\frac{2\lambda}{K}\int_0^{\sqrt{K}} r\, I_0(r) K_0(r)\,\mathrm{d}r \le \lambda\left\{\tfrac{1}{2\sqrt{3}}\sqrt{K} - \tfrac{1}{2}\ln\tfrac{K}{e} + \ln 2 - \gamma\right\}. \tag{4.4}$$

▷ If $K > 1$, we split the integral into two integrals of the shape

$$\int_0^{\sqrt{K}} = \int_0^1 + \int_1^{\sqrt{K}}$$

and apply Theorem 1.2, *resp.* Theorem 1.3, for the first, *resp.* second, integral, giving

$$\frac{2\lambda}{K}\int_0^{\sqrt{K}} r\, I_0(r)K_0(r)\,\mathrm{d}r \leq \frac{\lambda}{K}\left\{\sqrt{K}+\tfrac{1}{16}\ln K+\ln 2-\gamma-\tfrac{1}{2}\left(1-\tfrac{1}{\sqrt{3}}\right)\right\}.$$

Since $b_r=\beta=1$, we have $K=1$ and

$$A(r) = -\ln(1-r)-r \quad (r\in[0,1))$$

so that

$$\int_0^1 \frac{A(r)^2}{r}\,\mathrm{d}r = 2\zeta(3)-\tfrac{3}{2}.$$

Inserting in (4.3) we get

$$\begin{aligned}\mathcal{N}(H_A^D,\lambda) &\leq \lambda\left(\ln 2-\gamma+\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\right)+\frac{\lambda}{2\sqrt{1-\alpha}}+\frac{\sqrt{1-\alpha}}{\alpha}\left(2\zeta(3)-\tfrac{3}{2}\right)\\ &\leq \lambda\left(\ln 2-\gamma+\tfrac{1}{2}\left(1+\tfrac{1}{\sqrt{3}}\right)\right)+\frac{\lambda(2-\alpha)}{4(1-\alpha)}+\frac{1-\alpha/2}{\alpha}\left(2\zeta(3)-\tfrac{3}{2}\right)\end{aligned}$$

where we used the inequalities

$$\frac{2-2\alpha}{2-\alpha}\leq\sqrt{1-\alpha}\leq 1-\frac{\alpha}{2}\quad(\alpha\in[0,1]).$$

Setting $\kappa := 2\zeta(3)-\frac{3}{2}$ and choosing

$$\alpha=\frac{2\sqrt{\kappa}}{2\sqrt{\kappa}+\sqrt{\lambda}}$$

give the asserted result. □

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