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(n,k)-FACTORIALS

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Abstract. In this paper, we define (n,k)-factorial and (n,k)-double factorial for $n \in \mathbb{N}, k > 0$. We apply these definitions to construct a k-Meclaurin series involving these newly defined factorials. Also, we express some previously proved results in the form of (n,k)-factorial and discuss some new results satisfying the classical results.

1. Introduction

The factorial notation (!) was introduced by Christian Kramp in 1808 for positive integers and is frequently used to compute the binomial coefficients. The relationship between classical gamma function and ordinary factorial is $\Gamma(n) = (n-1)!, n \in \mathbb{N}$. Also, gamma function is defined for all real numbers except $n = 0, -1, -2, ...$ After words, the German mathematician Leo Pochhammer defined shifted (rising) factorial, which was named as Pochhammer's symbol and is given by (See [1-2])

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1)\,, n \in \mathbb{N} \\ 1, \quad n = 0\,, \alpha \neq 0. \end{cases} \tag{1.1}$$

It follows that $(1)_n = n!$ and for $n, m \in \mathbb{N}$, we can derive the following expression for the rising factorial of a negative integer.

$$(-n)_m = \begin{cases} \frac{(-1)^m n!}{(n-m)!}, 1 \leq m \leq n \\ 0, \quad m \geq n+1. \end{cases} \tag{1.2}$$

In literature, definition of double factorial and its application in Meclaurin series expansion is present (See [3]). Thus, we have the definition of double factorial as

$$n!! = \begin{cases} n(n-2)(n-4)...6.4.2 \quad \text{if n is even} \\ n(n-2)(n-4)...5.3.1 \quad \text{if n is odd} \\ 1, \quad \text{if } n = 0, -1 \quad ; \quad (-2n)!! = \infty, n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Recently, Diaz and Pariguan [4] introduced the generalized k-gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \; k > 0, x \in \mathbb{C} \setminus kZ^- \tag{1.4}$$

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and also gave the properties of said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. The Γ_k is based on the repeated appearance of the expression of the following form

$$\alpha(\alpha+k)(\alpha+2k)(\alpha+3k)...(\alpha+(n-1)k). \tag{1.5}$$

The function of the variable α given by the statement (1.5), denoted by $(\alpha)_{n,k}$, is called the Pochhammer k-symbol. Thus, we have

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)(\alpha+3k)...(\alpha+(n-1)k)\,, n \in \mathbb{N},\, k>0 \\ 1, \quad n=0\,, \alpha \neq 0. \end{cases} \tag{1.6}$$

We obtain the usual Pochhammer symbol $(\alpha)_n$ by taking $k = 1$. Also, the researchers [5-9] have worked on the generalized k-gamma function and discussed the following properties:

$$\Gamma_k(\alpha k) = k^{\alpha-1}\Gamma(\alpha),\ k>0, \alpha \in \mathbb{R} \tag{1.7}$$

$$\Gamma_k(nk) = k^{n-1}(n-1)!,\ k>0,\ n \in \mathbb{N} \tag{1.8}$$

$$\Gamma_k((2n+1)\frac{k}{2}) = k^{\frac{2n-1}{2}}\frac{(2n)!\sqrt{\pi}}{2^n n!},\ k>0,\ n \in \mathbb{N} \tag{1.9}$$

$$\Gamma_k(k) = 1 \tag{1.10}$$

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k}). \tag{1.11}$$

2. Main Results

Definition (2.1): For $n \in \mathbb{N}, k>0$, we define (n,k)-factorial as

$$(n,k)! = nk(nk-k)(nk-2k)(nk-3k)...3k.2k.k \tag{2.1}$$

$$\Rightarrow (n,k)! = k^n n(n-1)(n-2)(n-3)...3.2.1 = k^n n!. \tag{2.2}$$

From the above definition of $(n,k)!$, we can easily prove that

$$(nk,k)! = k^n(nk)! \tag{2.3}$$

$$(n+a,k)! = k^n(n+a)!\,,\ a \in \mathbb{R},\ n \in \mathbb{N} \tag{2.4}$$

$$[(n+b)k,k]! = k^n[(n+b)k]!\,,\ b \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.5}$$

Using the above results, we see that

$$(0,k)! = k^0 0! = 1 \quad , \quad (-n,k)! = \infty\,,\ n \in \mathbb{N}\,, k>0. \tag{2.6}$$

By Pochhammer k-symbol and relation (2.1), we get

$$(1)_{n,k} = (\frac{1}{k}+n-1,k)!. \tag{2.7}$$

Here, we give some results already proved in literature (See [4]). These results can be written in the form of (n,k)-factorial as

$$\Gamma_k(x) = \lim_{n\to\infty}\frac{(n,k)!(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k>0, x \in C \setminus kZ^- \tag{2.8}$$

$$\Gamma_k(nk) = (n-1,k)!,\ k>0,\ n \in \mathbb{N} \tag{2.9}$$

$$\Gamma_k((2n+1)\frac{k}{2}) = \frac{(2n,k)!}{2^n(n,k)!}\sqrt{\frac{\pi}{k}},\ k>0,\ n \in \mathbb{N}. \tag{2.10}$$

In terms of classical gamma function and k-gamma function defined in [4], the (n,k)-factorial can be expressed as

$$(n,k)! = \Gamma_k(nk+k) = k^n\Gamma(n+1),\ k>0,\ n\in\mathbb{N}. \tag{2.11}$$

Remarks: If we use $k=1$, we get $(n,1)=n!$, which is the classical representation of the above results.

Proposition (2.2): For $n,m\in\mathbb{N}, 1\le m\le \frac{n}{k},\ k>0$, we have

$$(-n)_{m,k} = \frac{(-k)^m(\frac{n}{k})!}{(\frac{n}{k}-m)!}. \tag{2.12}$$

Proof: Definition of Pochhammer k-symbol gives

$$\begin{aligned}(-n)_{m,k} &= (-n)(-n+k)(-n+2k)(-n+3k)...(-n+(m-1)k)\\ &= (-1)^m k^m\Big[\frac{n}{k}(\frac{n}{k}-1)(\frac{n}{k}-2)...(\frac{n}{k}-(m-1))\Big]\end{aligned}$$

$$\Rightarrow (-n)_{m,k} = \frac{(-k)^m(\frac{n}{k})!}{(\frac{n}{k}-m)!}.$$

Remarks: In the relation (2.12) if $m\ge \frac{n}{k}+1,\ k>0$, then $(-n)_{m,k}=0$.

Definition (2.3): For $n\in\mathbb{N},\ k>0$, (n,k)-double factorial is defined as (if n is even of the form $2n$),

$$\begin{aligned}(2n,k)!! &= 2nk(2nk-2k)(2nk-4k)...6k.4k.2k\\ &= k^n 2n(2n-2)(2n-4)...6.4.2 = k^n(2n)!!\end{aligned}$$

and (if n is odd of the form $2n-1$),

$$\begin{aligned}(2n-1,k)!! &= (2nk-k)(2nk-3k)(2nk-5k)...5k.3k.k\\ &= k^n(2n-1)(2n-3)...5.3.1 = k^n(2n-1)!!.\end{aligned}$$

Here, we give some useful results which justify the above definition of $(n,k)!!$.

$$(n,k)!! = 1\,, \text{if } n=0,-1 \text{ and } \quad (-2n,k)!! = \infty\,,\ n\in\mathbb{N}.$$

Theorem (2.4): For $n,\in\mathbb{N},\ k>0$, classical gamma function $\Gamma(x)$ and k-gamma function $\Gamma_k(x)$, the following expressions involving (n,k)-double factorial hold as

$$(n,k)!!\times(n-1,k)!! = (n,k)! \tag{2.13}$$

$$(2n,k)!! = k^n 2^n n! = k^n(2n)!! \tag{2.14}$$

$$(2n-1,k)!! = k^n 2^n\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2^n\frac{\Gamma_k(nk+\frac{k}{2})}{\Gamma_k(\frac{k}{2})} \tag{2.15}$$

$$[-(2n+1),k]!! = (-1)^{-n}\frac{2}{(2n-1,k)!!}. \tag{2.16}$$

Proof: If n is even, then $(n-1)$ is odd and vice versa, so

$$(2n,k)!!\times(2n-1,k)!! = (2n,k)!. \tag{2.17}$$

Replacing $2n$ by n, we get the required result (2.13). To prove the relation (2.14), just use the definition of (n,k)-double factorial and then $n!$ for even numbers n. For the relation (2.15), consider the following

$$\begin{aligned}(2n-1,k)!! &= (2nk-k)(2nk-3k)...5k.3k.k\\ &= k^n 2^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}.\end{aligned}$$

Replacing $\Gamma_k(x)$ by $\Gamma(x)$, from the relation (1.11), we can get the above result as

$$(2n-1,k)!! = 2^n \frac{\Gamma_k(nk+\frac{k}{2})}{\Gamma_k(\frac{k}{2})}.$$

For the relation (2.16), we use the equation (2.17) as

$$(2n-1,k)!! = \frac{(2n,k)!}{(2n,k)!!} = \frac{k^n(2n)!}{k^n 2^n n!} = \frac{(2n)!}{2^n n!} = \frac{\Gamma(1+2n)}{2^n\Gamma(1+n)}.$$

Replacing n by $-n$, the above equation gives

$$(-2n-1,k)!! = \frac{\Gamma(1-2n)}{2^{-n}\Gamma(1-n)} \Rightarrow [-(2n+1),k]!! = \frac{2^n\Gamma[-(2n-1)]}{\Gamma[-(n-1)]}.$$

Now, using the formula $\frac{\Gamma(-n)}{\Gamma(-m)} = (-1)^{m-n}\frac{m!}{n!}$, for singular points given in [1], we get

$$[-(2n+1),k]!! = (-1)^{-n}\frac{2^n(n-1)!}{(2n-1)!} = (-1)^{-n}\frac{2^n 2n(n-1)!}{2n(2n-1)!} = (-1)^{-n}\frac{2.2^n n!}{(2n)!}.$$

Converting into (n,k)-factorial form and using (2.17), we have

$$[-(2n+1),k]!! = (-1)^{-n}\frac{2.2^n n! k^n}{(2n)! k^n} = (-1)^{-n}\frac{2.(2n,k)!!}{(2n,k)!} = (-1)^{-n}\frac{2}{(2n-1,k)!!}.$$

Definition (2.5): If a function f is such that $f, f', f'', ..., f^{(n-1)}$ are continuous on $[0,x]$ and $f^{(n)}$ exists on $(0,x)$, then the Maclaurin's series expansion of the function $f(x)$ is given by

$$f(x) = \frac{x^0}{0!}f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n}{n!}f(n)(0) + ... \quad (2.18)$$

Now, we define the Meclaurin series expansion of a function involving the (n,k)-factorial. We call this series Meclaurin k-series expansion and is defined as

$$f_k(x) = \frac{(xk)^0}{(0,k)!}f_k(0) + \frac{(xk)^1}{(1,k)!}f_k'(0) + \frac{(xk)^2}{(2,k)!}f_k''(0) + ... + \frac{(xk)^n}{(n,k)!}f_k^{(n)}(0) + ... \quad (2.19)$$

Proof: Replacing x by kx in the classical Maclaurin's series (2.18), we have

$$f(xk) = \frac{(xk)^0}{0!}f(0) + \frac{(xk)^1}{1!}f'(0) + \frac{(xk)^2}{2!}f''(0) + ... + \frac{(xk)^n}{n!}f^{(n)}(0) + ... \quad (2.20)$$

Setting $f_k(x) = f(xk)$, we see that

$$f_k'(x) = kf'(xk), \quad f_k''(x) = k^2 f''(xk), \quad f_k'''(x) = k^3 f'''(xk), \quad ....$$

$$\Rightarrow f_k(0) = f(0)\ , f_k'(0) = kf'(0)\ , \ f_k''(0) = k^2 f''(0)\ , \ f_k'''(0) = k^3 f'''(0)\ , \quad ....$$

Thus, equation (2.20) takes the following form

$$f_k(x) = \frac{(xk)^0}{0!k^0}f_k(0) + \frac{(xk)^1}{1!k}f_k'(0) + \frac{(xk)^2}{2!k^2}f_k''(0) + ... + \frac{(xk)^n}{n!k^n}f_k^{(n)}(0) + ... \quad (2.21)$$

As $(n,k)! = k^n n!$, so above equation gives the Maclaurin k-series expansion as

$$f_k(x) = \frac{(xk)^0}{(0,k)!} f_k(0) + \frac{(xk)^1}{(1,k)!} f'_k(0) + \frac{(xk)^2}{(2,k)!} f''_k(0) + ... + \frac{(xk)^n}{(n,k)!} f_k^{(n)}(0) + ...$$

Examples (2.6): Here, we give some examples of elementary functions in the form of Maclaurin k-series involving $(n,k)!$. The results so obtained will be the classical results, if original Maclaurin series is used.

(2.6.1): Find the Meclaurin k-series expansion of the exponential function e^x. Setting $f_k(x) = f(xk) = e^{xk}$ implies

$$f_k(0) = 1 \ , f'_k(0) = k \ , \ f''_k(0) = k^2 \ , \ f'''_k(0) = k^3 \ , \quad ....$$

Using these values in equation (2.19), we have

$$e^{xk} = \frac{(xk)^0}{(0,k)!} + \frac{(xk)^1}{(1,k)!} k + \frac{(xk)^2}{(2,k)!} k^2 + \frac{(xk)^3}{(3,k)!} k^3 + ...$$

(2.6.2): Expansion by $(n,k)!$ of the logarithmic functions $\ln(1+x)$. Setting $f_k(x) = f(xk) = \ln(1+xk)$ implies

$$\ln(1+xk) = \frac{(xk)^1}{(1,k)!} k - \frac{(xk)^2}{(2,k)!} k^2 + \frac{(xk)^3}{(3,k)!} 2.k^3 - \frac{(xk)^4}{(4,k)!} 3.2.k^4 + ...$$

(2.6.3): Expansion by $(n,k)!$ of the trigonometric functions $\sin x$ and $\cos x$. Setting $f_k(x) = f(xk) = \sin xk$ and $\cos xk$ respectively implies

$$\sin xk = \frac{(xk)^1}{(1,k)!} k - \frac{(xk)^3}{(3,k)!} k^3 + \frac{(xk)^5}{(5,k)!} k^5 - \frac{(xk)^7}{(7,k)!} k^7 + ...$$

and

$$\cos xk = 1 - \frac{(xk)^2}{(2,k)!} k^2 + \frac{(xk)^4}{(4,k)!} k^4 - \frac{(xk)^6}{(6,k)!} k^6 + \frac{(xk)^8}{(8,k)!} k^8 - ....$$

(2.6.4): Expansion by $(n,k)!$ of the hyperbolic functions $\sinh x$ and $\cosh x$. Setting $f_k(x) = f(xk) = \sinh xk$ and $\cosh xk$ respectively implies

$$\sin xk = \frac{(xk)^1}{(1,k)!} k + \frac{(xk)^3}{(3,k)!} k^3 + \frac{(xk)^5}{(5,k)!} k^5 + \frac{(xk)^7}{(7,k)!} k^7 + ...$$

and

$$\cos xk = 1 + \frac{(xk)^2}{(2,k)!} k^2 + \frac{(xk)^4}{(4,k)!} k^4 + \frac{(xk)^6}{(6,k)!} k^6 + \frac{(xk)^8}{(8,k)!} k^8 + ...$$

Remarks: In the above examples, if k is replaced by 1, we have the series expansion of the classical function (as $(n,1)! = n!$).

Proposition (2.7): Using (n,k)-double factorial, the general formula for for the k-Meclaurin series expansion is given by

$$f_k(\frac{x}{2}) = \frac{(xk)^0}{(0,k)!!} f_k(0) + \frac{(xk)^1}{(2,k)!!} f'_k(0) + \frac{(xk)^2}{(4,k)!!} f''_k(0) + \frac{(xk)^3}{(6,k)!!} f'''_k(0) + ... \quad (2.22)$$

Proof: To prove the series (2.22), we use the Maclaurin k-series defined in equation (2.21) as

$$f_k(x) = \frac{(xk)^0}{0!k^0} f_k(0) + \frac{(xk)^1}{1!k} f'_k(0) + \frac{(xk)^2}{2!k^2} f''_k(0) + ... + \frac{(xk)^n}{n!k^n} f_k^{(n)}(0) + ...$$

$$\Rightarrow f_k(x) = \frac{(xk)^0 2^0}{0!k^0 2^0} f_k(0) + \frac{(xk)^1 2^1}{1!k2^1} f'_k(0) + \frac{(xk)^2 2^2}{2!k^2 2^2} f''_k(0) + \frac{(xk)^3 2^3}{3!k^3 2^3} f^3_k(0) + ...$$

As $(n,k)!! = k^n(2n)!! = k^n 2^n n!$, so above equation becomes

$$f_k(x) = \frac{(2xk)^0}{(0,k)!!} f_k(0) + \frac{(2xk)^1}{(2,k)!!} f'_k(0) + \frac{(2xk)^2}{(4,k)!!} f''_k(0) + \frac{(2xk)^3}{(6,k)!!} f'''_k(0) + ...$$

Replacing x by $\frac{x}{2}$, we get the required equation (2.22).

Examples (2.8): Now we express some elementary functions in terms of $(n,k)!!$ by using the Meclaurin k-series expansion formula (2.22) in the form of the following examples.

(2.8.1): Expansion by $(n,k)!!$ of the exponential function is given by

$$e^{\frac{xk}{2}} = \frac{(xk)^0}{(0,k)!!} + \frac{(xk)^1}{(2,k)!!}k + \frac{(xk)^2}{(4,k)!!}k^2 + \frac{(xk)^3}{(6,k)!!}k^3 + \frac{(xk)^4}{(8,k)!!}k^4 + ...$$

(2.8.2): Expansion by $(n,k)!!$ of the logarithmic functions $\ln(1+x)$. Setting $f_k(\frac{x}{2}) = f(\frac{xk}{2}) = \ln(1+\frac{xk}{2})$ implies

$$\ln(1+\frac{xk}{2}) = \frac{(xk)^1}{(2,k)!!}k - \frac{(xk)^2}{(4,k)!!}k^2 + \frac{(xk)^3}{(6,k)!!}2.k^3 - \frac{(xk)^4}{(8,k)!!}3.2.k^4 + \frac{(xk)^5}{(10,k)!!}4.3.2.k^5 - ....$$

(2.8.3): Expansion by $(n,k)!!$ of the trigonometric functions $\sin x$ and $\cos x$ is given by

$$\sin(\frac{xk}{2}) = \frac{(xk)^1}{(2,k)!!}k - \frac{(xk)^3}{(6,k)!!}k^3 + \frac{(xk)^5}{(10,k)!!}k^5 - \frac{(xk)^7}{(14,k)!!}k^7 + ...$$

and

$$\cos(\frac{xk}{2}) = \frac{(xk)^0}{(0,k)!!}k^0 - \frac{(xk)^2}{(4,k)!!}k^2 + \frac{(xk)^4}{(8,k)!!}k^4 - \frac{(xk)^6}{(12,k)!!}k^6 + \frac{(xk)^8}{(16,k)!!}k^8 - ....$$

(2.8.4): Expansion by $(n,k)!!$ of the hyperbolic functions $\sinh x$ and $\cosh x$ is given by

$$\sinh(\frac{xk}{2}) = \frac{(xk)^1}{(2,k)!!}k + \frac{(xk)^3}{(6,k)!!}k^3 + \frac{(xk)^5}{(10,k)!!}k^5 + \frac{(xk)^7}{(14,k)!!}k^7 + ...$$

and

$$\cosh(\frac{xk}{2}) = \frac{(xk)^0}{(0,k)!!}k^0 + \frac{(xk)^2}{(4,k)!!}k^2 + \frac{(xk)^4}{(8,k)!!}k^4 + \frac{(xk)^6}{(12,k)!!}k^6 + \frac{(xk)^8}{(16,k)!!}k^8 + ...$$

Remarks: In the above examples, if k is replaced by 1, we have the series expansion of the classical functions (as $(n,1)! = n!$).

Note: The newly defined $(n,k)!$ can be extended up to a finite number of higher factorials as $(n,k)!!!$, $(n,k)!_4$, ... $(n,k)!_r$.

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