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INEQUALITIES FOR JACOBIAN ELLIPTIC FUNCTIONS

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Abstract. Two chains of inequalities involving elliptic functions sn, sc and am are established. These results imply, in particular, two Huygens-type and two Wilker-type inequalities for the families of higher transcendental functions discussed in this paper.

1. Introduction and Definitions

The history of Jacobian elliptic functions is long and laden with detail (see, e.g., [2]). Importance of these functions in theory of dynamical systems is well documented in [3]. For other applications of these functions in science and engineering the interested reader is referred to [12] and [14].

We begin this section recalling definitions of the Jacobian elliptic functions used in this paper. In what follows the letter k, $0 \le k \le 1$, will stand for modulus of Legendre's incomplete elliptic integral of the first kind

$$F(\varphi, k) = \int_0^{\sin\varphi} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}} \tag{1.1}$$

($0 \le \varphi \le \frac{\pi}{2}$). Legendre's complete elliptic integral of the first kind is defined as

$$K(k) \equiv K = F(\frac{\pi}{2}, k). \tag{1.2}$$

The Jacobian version of Legendre's incomplete elliptic integral of the first kind is

$$x = \int_0^{\operatorname{sn}(x,k)} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \tag{1.3}$$

where $\operatorname{sn}(x,k) \equiv \operatorname{sn}(x) \equiv \operatorname{sn}$ is the Jacobian elliptic sine function and k is the same as above, see, e.g., [2]. Other Jacobian elliptic functions used in this paper are cn and sc. They are subordinate to the Jacobian elliptic sine sn in the following sense

$$\operatorname{sn}^2(x) + \operatorname{cn}^2(x) = 1, \qquad \operatorname{sc}(x) = \frac{\operatorname{sn}(x)}{\operatorname{cn}(x)}. \tag{1.4}$$

We will also utilize Jacobian amplitude function $\operatorname{am}(x,k) \equiv \operatorname{am}(x) \equiv \operatorname{am}$. Following [12, 22.16.1]

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$$\operatorname{am}(x,k) = \sin^{-1}(\operatorname{sn}(x,k)). \tag{1.5}$$

For more details regarding Jacobian elliptic functions the interested reader is referred to [12, Ch. 22], [2] and [14].

The goal of this note is to establish new inequalities involving functions under discussion. Those results are included in Section 2. It is worth mentioning that several inequalities involving these functions have been published in the past years in mathematical literature. For more details the interested reader is referred to [4, 5, 6, 8, 9, 10, 11, 13].

2. Main Results

Let $D = (0, K)$, where K is defined in (1.2). In what follows we will always assume that the independent variable x of the four Jacobian elliptic functions $\operatorname{sn}(x,k), \operatorname{cn}(x,k), \operatorname{sc}(x,k)$ and $\operatorname{am}(x,k)$ satisfies $x \in D$. For the sake of notation the variable x and the modulus k will be suppressed when no confusion would arise.

The first chain of inequalities involving functions under discussion is obtained in the following.

Theorem 2.1. *The following inequalities*

$$\begin{aligned}(\operatorname{cn})^{1/3} &< \Big(\operatorname{cn}\frac{\operatorname{sn}}{\operatorname{am}}\Big)^{1/4} < \Big[\frac{1}{2}\Big(\operatorname{cn} + \frac{\operatorname{sn}}{\operatorname{am}}\Big)\Big]^{1/2} < \\ &< \Big(\frac{1+2\operatorname{cn}}{3}\Big)^{1/2} < \Big(\frac{1+\operatorname{cn}}{2}\Big)^{2/3} < \frac{\operatorname{sn}}{\operatorname{am}} < \frac{2+\operatorname{cn}}{3} < 1\end{aligned} \tag{2.1}$$

are valid.

Proof. We shall establish the first five inequalities in (2.1) using the following chain of inequalities for the trigonometric functions [7, Theorem 1]:

$$\begin{aligned}(\cos t)^{1/3} &< \Big(\cos t\frac{\sin t}{t}\Big)^{1/4} < \Big[\frac{1}{2}\Big(\cos t + \frac{\sin t}{t}\Big)\Big]^{1/2} < \\ &< \Big(\frac{1+2\cos t}{3}\Big)^{1/2} < \Big(\frac{1+\cos t}{2}\Big)^{2/3} < \frac{\sin t}{t}.\end{aligned} \tag{2.2}$$

For the proof we let $t = \sin^{-1}(\operatorname{sn} x)$. Using (1.5) we get $t = \operatorname{am}$ and also that $\sin t = \operatorname{sn} x$. Utilizing the first part of (1.4) we obtain $\cos t = \operatorname{cn} x$. The inequalities in question now follow. The sixth inequality in (2.1) follows from the Huygens inequality [1]:

$$\frac{\sin t}{t} < \frac{2+\cos t}{3},$$

$0 < |t| < \pi/2$. The last inequality is obvious because $\operatorname{cn} < 1$. The proof is complete. □

We shall utilize some parts of (2.1) to prove the following.

Theorem 2.2. *We have*

$$\begin{aligned}1 &< \frac{1}{3}\Big(2\frac{\operatorname{am}}{\operatorname{sn}} + \frac{\operatorname{am}}{\operatorname{sc}}\Big) < \frac{1}{2}\Big[\Big(\frac{\operatorname{am}}{\operatorname{sn}}\Big)^2 + \frac{\operatorname{am}}{\operatorname{sc}}\Big] < \\ &< \frac{1}{3}\Big(2\frac{\operatorname{sn}}{\operatorname{am}} + \frac{\operatorname{sc}}{\operatorname{am}}\Big) < \frac{1}{2}\Big[\Big(\frac{\operatorname{sn}}{\operatorname{am}}\Big)^2 + \frac{\operatorname{sc}}{\operatorname{am}}\Big].\end{aligned} \tag{2.3}$$

Proof. In order to obtain the first inequality in (2.3) it suffices to multiply the sixth and seventh members of (2.1) by $\operatorname{am}/\operatorname{sn}$. We shall demonstrate now that the second inequality of (2.3) holds true. To this aim we rewrite the sixth inequality in (2.1) as follows

$$\operatorname{cn} > 3\frac{\operatorname{sn}}{\operatorname{am}} - 2. \tag{2.4}$$

For the brevity let

$$A := \frac{1}{2}\Big[\Big(\frac{\operatorname{am}}{\operatorname{sn}}\Big)^2 + \frac{\operatorname{am}}{\operatorname{sc}}\Big] - \frac{1}{3}\Big(2\frac{\operatorname{am}}{\operatorname{sn}} + \frac{\operatorname{am}}{\operatorname{sc}}\Big).$$

Making use of the formula $1/\operatorname{sc} = \operatorname{cn}/\operatorname{sn}$ we obtain, after a little algebra,

$$A = \frac{1}{6}\Big(\frac{\operatorname{am}}{\operatorname{sn}}\Big)^2\Big(3 + \frac{\operatorname{sn}}{\operatorname{am}}\operatorname{cn} - 4\frac{\operatorname{sn}}{\operatorname{am}}\Big).$$

Application of (2.4) to the right side of the last inequality yields

$$A > \frac{1}{6}\Big(\frac{\operatorname{am}}{\operatorname{sn}}\Big)^2\Big[3 + \frac{\operatorname{sn}}{\operatorname{am}}\Big(3\frac{\operatorname{sn}}{\operatorname{am}} - 2\Big) - 4\frac{\operatorname{sn}}{\operatorname{am}}\Big] =$$
$$= \frac{1}{6}\Big(\frac{\operatorname{am}}{\operatorname{sn}}\Big)^2\Big[3\Big(\frac{\operatorname{sn}}{\operatorname{am}}\Big)^2 - 6\frac{\operatorname{sn}}{\operatorname{am}} + 3\Big] = \frac{1}{2}\Big(\frac{\operatorname{sn}}{\operatorname{am}} - 1\Big)^2 > 0.$$

In what follows we will utilize a symbol u, where $u := \operatorname{sn}/\operatorname{am}$. This allows us to write the quotient $\operatorname{sc}/\operatorname{am}$ as

$$\frac{\operatorname{sc}}{\operatorname{am}} = \frac{\operatorname{sn}}{\operatorname{cn}\cdot\operatorname{am}} = \frac{u}{\operatorname{cn}}.$$

The difference between the the fourth and third members of (2.3) can be expressed in terms of u and cn:

$$B := \frac{1}{3}\Big(2u + \frac{u}{\operatorname{cn}}\Big) - \frac{1}{2}\Big(\frac{1}{u^2} + \frac{\operatorname{cn}}{u}\Big) = \frac{u}{\operatorname{cn}}\frac{1 + 2\operatorname{cn}}{3} - \frac{1}{2u^2}(1 + u\cdot\operatorname{cn}).$$

The third inequality of (2.1)

$$\frac{1 + 2\operatorname{cn}}{3} > \frac{1}{2}(u + \operatorname{cn})$$

is now utilized to obtain

$$B > \frac{u}{2\operatorname{cn}}(u + \operatorname{cn}) - \frac{1}{2u^2}(1 + u\cdot\operatorname{cn}) = \frac{1}{2u^2\operatorname{cn}}[u^3(u + \operatorname{cn}) - \operatorname{cn}(1 + u\cdot\operatorname{cn})].$$

Next we apply the fifth inequality in (2.1)

$$u^3 > \Big(\frac{1 + \operatorname{cn}}{2}\Big)^2$$

to obtain

$$B > \frac{1}{2u^2\operatorname{cn}}\Big[\Big(\frac{1 + \operatorname{cn}}{2}\Big)^2(u + \operatorname{cn}) - \operatorname{cn}(1 + u\cdot\operatorname{cn})\Big] =$$
$$= \frac{1}{8u^2\operatorname{cn}}\big[(1 + \operatorname{cn})^2(u + \operatorname{cn}) - 4\operatorname{cn}(1 + u\cdot\operatorname{cn})\big] =$$
$$= \frac{1}{8u^2\operatorname{cn}}\big[u(1 + 2\operatorname{cn} - 3\operatorname{cn}^2) + \operatorname{cn}(\operatorname{cn}^2 + 2\operatorname{cn} - 3)\big] =$$
$$= \frac{1}{8u^2\operatorname{cn}}\big[u(1 + 3\operatorname{cn})(1 - \operatorname{cn}) - \operatorname{cn}(1 - \operatorname{cn})(3 + \operatorname{cn})\big] =$$
$$= \frac{1 - \operatorname{cn}}{8u^2\operatorname{cn}}[u(1 + 3\operatorname{cn}) - \operatorname{cn}(3 + \operatorname{cn})] = \frac{1 - \operatorname{cn}}{8u^2\operatorname{cn}}\varphi(\operatorname{cn}),$$

where $\varphi(\mathrm{cn}\,) = u(1 + 3\,\mathrm{cn}\,) - \mathrm{cn}\,(3 + \mathrm{cn}\,)$. Since $u > \mathrm{cn}\,^{1/3}$ (see (2.1)),

$$\begin{aligned}\varphi(\mathrm{cn}\,) > \mathrm{cn}\,^{1/3}(1 + 3\,\mathrm{cn}\,) - \mathrm{cn}\,(3 + \mathrm{cn}\,) = \\ = \mathrm{cn}\,^{1/3} + 3\,\mathrm{cn}\,^{4/3} - 3\,\mathrm{cn}\, - \mathrm{cn}\,^{2} = \\ = v + 3v^4 - 3v^3 - v^6 = v(1 - v)(1 + v - 2v^2 + v^3 + v^4),\end{aligned}$$

where $v = \mathrm{cn}\,^{1/3}$ $(0 < v < 1)$ One can easily demonstrate that $1+v-2v^2+v^3+v^4 > 0$ if $0 < v < 1$. This completes the proof of positivity of the quantity B which represents the difference between the fourth and third members of (2.3). We shall verify now validity of the last inequality in (2.3). Let

$$C := \frac{1}{2}\Big[\Big(\frac{\mathrm{sn}}{\mathrm{am}}\Big)^2 + \frac{\mathrm{sc}}{\mathrm{am}}\Big] - \frac{1}{3}\Big(2\frac{\mathrm{sn}}{\mathrm{am}} + \frac{\mathrm{sc}}{\mathrm{am}}\Big).$$

Then

$$C = \frac{u}{6\,\mathrm{cn}}(3u \cdot \mathrm{cn}\, - 4\,\mathrm{cn}\, + 1).$$

We utilize the inequality $u > \mathrm{cn}\,^{1/3}$ again to obtain

$$C > \frac{u}{6\,\mathrm{cn}}(3\,\mathrm{cn}\,^{4/3} - 4\,\mathrm{cn}\, + 1) = \frac{u}{6\,\mathrm{cn}}(3v^4 - 4v^3 + 1),$$

where v has the same meaning as in the proof of the third inequality in (2.3). An easy factorization of the above quartic polynomial completes the proof. We have

$$C > \frac{u}{6\,\mathrm{cn}}(v - 1)^2(3v^2 + 2v + 1) > 0.$$

□

Two inequalities

$$3 < 2\frac{\mathrm{sn}}{\mathrm{am}} + \frac{\mathrm{sc}}{\mathrm{am}}$$

and

$$2 < \Big(\frac{\mathrm{sn}}{\mathrm{am}}\Big)^2 + \frac{\mathrm{sc}}{\mathrm{am}}$$

follow from (2.3). The last two results can be called the first Huygens and the first Wilker inequality, respectively, for the Jacobian elliptic functions. These results have been obtained recently in [11]. Another pair of inequalities

$$3 < 2\frac{\mathrm{sn}}{\mathrm{am}} + \frac{\mathrm{sc}}{\mathrm{am}}$$

and

$$2 < \Big(\frac{\mathrm{sn}}{\mathrm{am}}\Big)^2 + \frac{\mathrm{sc}}{\mathrm{am}}$$

also follows from (2.1). We call the last two results the second Huygens and the second Wilker inequality, respectively, for the Jacobian elliptic functions.

We close this section with a remark that a chain of inequalities involving trigonometric functions follows immediately from (2.3) by taking the limiting case when $k \to 0^+$. In this case one has [12, 22.5.3]:

$$\mathrm{sn}\,(x, 0^+) = \sin x, \qquad \mathrm{sc}\,(x, 0^+) = \tan x, \qquad \text{and} \qquad \mathrm{am}\,(x, 0^+) = x.$$

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