YOUNG’S CONVOLUTION INEQUALITIES FOR WEIGHTED MIXED (QUASI-) NORM SPACES.

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Abstract. Young’s Convolution Inequality is extended to several cases of discrete, semi-discrete and continuous convolution of sequences and functions that belong to weighted mixed quasi-norm spaces and amalgam spaces.

1. Introduction

Convolution relations play a central role in the study of the Wiener-type spaces. Under the assumption that \(1 \leq p, q, r \leq \infty\), Young’s Convolution Inequality states that for both the function and the sequence spaces,

\[
\| f \ast g \|_r \leq C(p, q) \| f \|_p \cdot \| g \|_q
\]

if and only if

\[
\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.
\]

Young’s Inequality allows for various generalizations, particularly for mixed-norm spaces. For example,

\[
\| F \ast G \|_{r_1, r_2} \leq C(p, q) \cdot \| F \|_{p_1, p_2} \cdot \| G \|_{q_1, q_2}
\]

if and only if

\[
\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}, \quad i \in \{1, 2\},
\]

where the norm \(\| F \|_{p_1, p_2}\) is defined by

\[
\| F \|_{p_1, p_2} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^{p_1} \, dx \right)^{p_2/p_1} \, d\omega \right)^{1/p_2}.
\]

When \(0 < p < 1\), the inequality (1.1) is not valid. Moreover, in this case, \(f \in L^p\) does not imply local integrability of \(f\) and, therefore, \(f \ast g\) is not even defined. However, in this case the sequence space \(l^p\) enjoys the inequality

\[
\| f \ast g \|_{l^p} \leq \| f \|_{l^p} \cdot \| g \|_{l^p},
\]

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which means that is a quasi-Banach algebra under convolution. Moreover, if we (formally) define the semi-discrete convolution of a sequence \( a \in \ell^p \) and a function \( f \in L^p \) by \((a \ast f)(x) = \sum_{k \in \mathbb{Z}^d} a(k)f(x-k)\), we have the convolution inequality

\[
\|a \ast f\|_{L^p} \leq \|a\|_{\ell^p} \cdot \|f\|_{L^p}.
\]

In this paper we investigate the question of how the convolution inequalities \((1.5)\) and \((1.6)\) can be extended to the case of weighted mixed (quasi-)norm sequence and function spaces and derive several generalized versions of the inequality \((1.3)\). Several particular cases of our results have been used in time-frequency analysis, particularly in the theory of modulation spaces (for example, in \([3, 7]\)) . The case \(0 < p \leq 1\) is of particular interest there because then the modulation space (quasi-)norm provides a flexible lower bound in a class of uncertainty principles \([3]\).

2. Definitions and Preliminary Results

When working with convolution inequalities, it is convenient to use moderate weights on \(\mathbb{R}^{2d}\) and \(\mathbb{Z}^{2d}\) defined as follows:

**Definition 1.** Let \(m\) be a weight function on \(\mathbb{R}^{2d}\) (or on \(\mathbb{Z}^{2d}\)). We call it \(\nu\)-moderate if \(m(z_1 + z_2) \leq C \nu(z_1)m(z_2)\), for all \(z_1, z_2 \in \mathbb{R}^{2d}\) (or for all \(z_1, z_2 \in \mathbb{Z}^{2d}\)).

We will measure the decay and and integrability of functions on the \(\mathbb{R}^{2d}\) plane by means of weighted mixed (quasi-)norms as follows:

**Definition 2** (weighted mixed-norm spaces). Let \(m\) be a weight function on \(\mathbb{R}^{2d}\) and let \(0 < p, q < \infty\). Then the weighted mixed (quasi-)norm space \(L_{m,q}^{p,\nu}(\mathbb{R}^{2d})\) consists of all Lebesgue measurable functions on \(\mathbb{R}^{2d}\), such that the (quasi-)norm

\[
\|F\|_{L_{m,q}^{p,\nu}} = \left( \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^{2d}} |F(x, \omega)|^p m(x, \omega)^{p/\nu} dx \right)^{q/p} d\omega \right)^{1/q}
\]

is finite. If \(p = \infty\) of \(q = \infty\), then the corresponding \(p\)-norm is replaced by the essential supremum.

\(L_{m,q}^{p,\nu}\) is a translation-invariant quasi-Banach space (Banach space if both \(p \geq 1\) and \(q \geq 1\)). The theory of mixed-norm spaces is developed in \([1]\).

We will measure the decay and summability of sequences by means of diiscrete mixed (quasi-)norms as follows:

**Definition 3** (discrete mixed (quasi-)norm spaces). The space \(\ell_{m,q}^{p,\nu}(\mathbb{Z}^{2d})\) consists of all sequences \(a = (a_{k,n})_{k,n \in \mathbb{Z}^d}\) for which the (quasi-)norm

\[
\|a\|_{\ell_{m,q}^{p,\nu}} = \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |a_{k,n}|^{p\nu} m(k, n)^p \right)^{q/p} \right)^{1/q}
\]

is finite. If \(p = \infty\) of \(q = \infty\), then the corresponding \(p\)-norm is replaced by the supremum.

The most basic tool we will use when working with sequence spaces is the following variation of Young’s Inequality:

**Lemma 2.1** (Young’s Inequality). Assume that \(m\) is a \(\nu\)-moderate \(\nu\)-moderate weight on \(\mathbb{Z}^{d}\), \(0 < p \leq \infty\), and \(r = \min\{1, p\}\). Then for all \(a \in \ell_{\nu}^r\) and all \(b \in \ell_{m}^p\),

\[
\|a \ast b\|_{\ell_{r}^\nu} \leq C \|a\|_{\ell_{\nu}^r} \|b\|_{\ell_{m}^p},
\]

where \(C\) is independent of \(p\), \(a\) and \(b\). If \(m \equiv \nu \equiv 1\), then \(C = 1\).
Proof. In the case $1 \leq p \leq \infty$ the result is well known. If $0 < p < 1$, we use the inclusion $\ell^p \hookrightarrow \ell^1$ and the fact that $m(n) \leq C\nu(k)m(n-k)$ for all $k,n \in \mathbb{Z}^d$ to obtain
\[
\|a * b\|_{\ell^p_m} = \left( \sum_{n \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} a(k)b(n-k)m(n)^p \right|^p \right)^{1/p} \\
\leq C \left( \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a(k)b(n-k)|^p \nu(k)^p m(n-k)^p \right)^{1/p} \\
= C \|a\|_{\ell^p_m} \|b\|_{\ell^p_m},
\]
as desired. \hfill \Box

In order to describe the space of functions defined by local integrability and global summability, we define the so-called Wiener-type amalgam spaces as follows:

**Definition 4.** Assume that $m$ is a weight function on $\mathbb{Z}^d$ and that $0 < p, q \leq \infty$. Denote $Q = [0,1]^{2d}$ and $\chi = \chi_Q$. The Wiener-type amalgam space $W(L^p, L^q_m)$ consists of all measurable functions $f$ for which the (quasi-)norm
\[
\|f\|_{W(L^p, L^q_m)} = \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \|f \cdot T_{(k,n)}\chi\|_{L^q_m} \right)^q \right)^{1/q},
\]
where $T_{(k,n)}$ is the $2d$-dimensional translation, is finite. In particular, $W(L^\infty, L^q_m)$ consists of all measurable functions $F$ on $\mathbb{R}^{2d}$, so that the sequence of local suprema
\[
a_{kn} = \text{ess sup}_{x,\omega \in [0,1]^d} |F(x+k,\omega+n)| = \|f \cdot T_{(k,n)}\chi\|_{\infty},
\]
where $\chi = \chi_{[0,1]^{2d}}$, belongs to $\ell^q_m$. The (quasi-)norm on $W(L^p, L^q_m)$ is
\[
\|F\|_{W(L^p, L^q_m)} = \|a\|_{\ell^p_m}
\]
We also use the notation $W(L^p_m) = W(L^\infty, L^p_m)$.

**Remark 5.** We note the following inclusion relation for Wiener-type spaces: If $0 < p_1 \leq p_2 \leq \infty$, then $W(L^{p_2}, L^q_m) \subseteq W(L^{p_1}, L^q_m)$. In particular, $W(L^q_m) \subseteq W(L^1, L^q_m)$.

**Remark 6.** Given $\alpha, \beta > 0$, an equivalent quasi-norm on $W(L^p_m)$ is defined by $\|F \cdot T_{(k\alpha,n\beta)}\chi_{Q_\alpha \times Q_\beta}\|_{\ell^p_m}$, where $Q_\alpha = [0,\alpha)$, $Q_\beta = [0,\beta)$, and $\tilde{m}(k,n) = m(n\alpha, n\beta)$.

In addition to the continuous convolution of functions and the discrete convolution of sequences, we will consider the semi-discrete convolution of a sequence and a function defined as follows:

**Definition 7** (Semi-discrete convolution). Given the lattice parameters $\alpha, \beta > 0$, formally define the semi-discrete convolution of a complex-valued sequence $a = (a(k,n) : (k,n) \in \mathbb{Z}^d)$ and a continuous function $F$ on $\mathbb{R}^{2d}$, by
\[
(a * F)(x,\omega) = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} a(k,n)F(x-k\alpha,\omega-n\beta) = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} a(k,n)\cdot T_{k\alpha,n\beta}F(x,\omega),
\]
where $T_{k\alpha,n\beta}F(x,\omega)$ is the $2d$-dimensional translation of $F$. 

3. Main Results.

Our first result is a generalization the inequality (3.1) for sequence spaces for the case $0 < p_j, q_j, r_j \leq \infty$. We establish that under a certain range of parameters the generalization is exact, while for the remaining case it is almost exact (up to the interchange of the indices and the order of summation).

**Theorem 3.1.** Assume that $m$ is a $\nu$-moderate weight on $\mathbb{Z}^d$. Also assume that

$$\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i} \quad \text{for} \quad 1 \leq r_i \leq \infty$$

and

$$p_i = q_i = r_i \quad \text{for} \quad 0 < r_i < 1.$$

(a) If either $1 \leq r_1 \leq \infty$ or $0 < r_2 \leq r_1 < 1$, then

$$||a * b||_{\ell^{r_1} \cdot r_2} \leq C ||a||_{\ell^{p_1} \cdot p_2} ||b||_{\ell^{q_1} \cdot q_2}.$$  (3.2)

(b) If $0 < r_1 \leq r_2 \leq \infty$ then

$$||a * b||_{\ell^{r_1} \cdot r_2} \leq C ||a||_{\ell^{p_1} \cdot p_2} ||b||_{\ell^{q_1} \cdot q_1}$$  (3.3)

and

$$||a * b||_{\ell^{r_1} \cdot r_2} \leq C ||Ua||_{\ell^{p_2} \cdot p_1} ||Ub||_{\ell^{q_2} \cdot q_1},$$  (3.4)

where $Ua(k, n) = a(n, k), Um(k, n) = m(n, k), U\nu(k, n) = \nu(n, k)$. Therefore,

$$||Ua||_{\ell^{p_2} \cdot p_1} = \left( \sum_{k \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} |a(k, n)|^{p_2} m(k, n)^{p_2} \right)^{p_1/p_2} \right)^{1/p_1}.$$

**Proof.** (a) Since $m$ is $\nu$-moderate,

$$|(a * b) l, n\rangle \cdot m(l, n) \leq C \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |a(l, n - k) \cdot b(j, k)| \cdot \nu(l, n, k) m(j, k)$$

$$= C \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \tilde{a}_{n-k} l, j \rangle \cdot \tilde{b}_k (j)$$

$$= C \sum_{k \in \mathbb{Z}^d} \langle \tilde{a}_{n-k} \ast \tilde{b}_k \rangle (l),$$  (3.5)

where $\tilde{a}_k (j) = |a(j, k) \cdot m(j, k)|$ and $\tilde{b}_k (j) = |b(j, k) \cdot \nu(j, k)|$. We can therefore estimate the $\ell^{r_1} \cdot r_2$-norm of $a * b$ by

$$||a * b||_{\ell^{r_1} \cdot r_2} = C \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{l \in \mathbb{Z}^d} |(a * b) l, n\rangle \cdot m(l, n)|^{r_2} \right)^{r_1/r_2} \right)^{1/r_2}$$

$$\leq C \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} (\tilde{a}_{n-k} \ast \tilde{b}_k) (l) \right)^{r_2/r_2} \right)^{1/r_2}$$  (3.6)

and consider the following cases:
Case 1 ($1 \leq r_1 \leq \infty$, $0 < r_2 \leq \infty$). We use Minkowsky’s Inequality and Young’s Inequality to estimate the inner double sum in (3.6) by

$$
\left( \sum_{l \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} (\tilde{a}_{n-k} \ast \tilde{b}_k)(l) \right|^{r_1} \right)^{1/r_1} \leq C \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_{n-k} \ast \tilde{b}_k||_{r_1}
$$

where we denoted $A(k) = ||\tilde{a}_k||_{p_1}$ and $B(k) = ||\tilde{b}_k||_{q_1}$. By substituting (3.7) into (3.6) and using Young’s Inequality, we obtain

$$
||a \ast b||_{\ell_m^{r_1 \cdot r_2}} \leq C \left( \sum_{n \in \mathbb{Z}^d} |(A \ast B)(n)|^{r_2} \right)^{1/r_2}
= C||A||_{p_2} \cdot ||B||_{q_2} = C||a||_{\ell_m^{p_1 \cdot p_2}} \cdot ||b||_{\ell_m^{q_1 \cdot q_2}}.
$$

Case 2 ($0 < r_2 < r_1 < 1$). We use Young’s Inequality and interchange the order of summation to estimate the inner double sum in (3.6) by

$$
\sum_{l \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |(\tilde{a}_{n-k} \ast \tilde{b}_k)(l)| \right)^{r_1} \leq \sum_{k \in \mathbb{Z}^d} \left( \sum_{l \in \mathbb{Z}^d} |(\tilde{a}_{n-k} \ast \tilde{b}_k)(l)| \right)^{r_1}
$$

where we denoted $A(k) = ||\tilde{a}_k||_{r_1}$ and $B(n) = ||\tilde{b}_k||_{r_1}$. By substituting (3.8) into (3.6) and using Young’s Inequality, we obtain

$$
||a \ast b||_{\ell_m^{r_1 \cdot r_2}} \leq C \left( \sum_{n \in \mathbb{Z}^d} |(A \ast B)(n)|^{r_2/r_1} \right)^{1/r_2}
= C(|A||_{r_2/r_1} \cdot |B||_{r_2/r_1})^{1/r_2} = C||a||_{\ell_m^{p_1 \cdot p_2}} \cdot ||b||_{\ell_m^{q_1 \cdot q_2}}.
$$

(b) Case 3 ($0 < r_1 < 1$ and $r_1 < r_2 \leq \infty$). We again substitute (3.8) into (3.6) and this time use Minkowsky’s Inequality with the exponent $r_2/r_1 \geq 1$ to obtain

$$
||a \ast b||_{\ell_m^{r_1 \cdot r_2}} \leq C \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_{n-k}||_{r_1} \cdot ||\tilde{b}_k||_{r_1} \right)^{r_2/r_1} \right)^{1/r_2}
$$

where we denoted $A(k) = ||\tilde{a}_k||_{r_1}$ and $B(k) = ||\tilde{b}_k||_{r_1}$. By substituting (3.9) into (3.6) and using Young’s Inequality, we obtain

$$
||a \ast b||_{\ell_m^{r_1 \cdot r_2}} \leq C \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_{n-k}||_{r_1} \cdot ||\tilde{b}_k||_{r_1} \right)^{r_2/r_1} \right)^{1/r_2}
= C||a||_{\ell_m^{p_1 \cdot p_2}} \cdot ||b||_{\ell_m^{q_1 \cdot q_2}}.
$$
In order to derive (3.4), we redefine \( \tilde{a}_j(k) = |a(j, k) \cdot m(j, k)| \) and \( \tilde{b}_j(k) = |b(j, k) \cdot \nu(j, k)| \) and estimate \( a \ast b \) by

\[
\|(a \ast b)(l, n) \cdot m(l, n)\| \leq C \sum_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |a(l - j, n - k) \cdot b(j, k)| \cdot \nu(l - j, n - k)m(j, k)
\]

\[
= C \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \tilde{a}_{l-j}(n-k) \cdot \tilde{b}_j(k)
\]

\[
= C \sum_{j \in \mathbb{Z}^d} (\tilde{a}_{l-j} \ast \tilde{b}_j)(n).
\]

(3.9)

Next, applying Minkowsky’s Inequality with the exponent \( r_2/r_1 \geq 1 \) and substituting (3.9) we obtain

\[
\|a \ast b\|_{\ell^{r_2}_{\| \cdot \|_q}} = C \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{l \in \mathbb{Z}^d} \|(a \ast b)(l, n) \cdot m(l, n)\|_r^{r_2/r_1} \right)^{r_1/r_2} \right)^{1/r_2}
\]

\[
\leq C \left( \sum_{l \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} \|(a \ast b)(l, n) \cdot m(l, n)\|_r^{r_2/r_1} \right)^{r_1/r_2} \right)^{1/r_1}
\]

\[
\leq C \left( \sum_{l \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q \right)^{r_1/r_2} \right)^{1/r_1}
\]

(3.10)

and consider the following two cases.

Case 1 \((0 < r_1 < r_2 < 1)\). We use Young’s Inequality and change the order of summation to estimate the inner double sum in (3.9) by

\[
\sum_{n \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q \leq \sum_{j \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q
\]

\[
\leq \sum_{j \in \mathbb{Z}^d} \left( \sum_{n \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q \right) = (A \ast B)(l),
\]

(3.11)

where we denoted \( A(j) = \|(\tilde{a}_j\|_{\ell^2_q} \) and \( B(j) = \|\tilde{b}_j\|_{\ell^2_q} \). Substituting (3.11) into (3.10) we estimate

\[
\|a \ast b\|_{\ell^{r_2}_{\| \cdot \|_q}} \leq C \left( \sum_{l \in \mathbb{Z}^d} \left( \|(A \ast B)(l)\|_{r_1/r_2} \right)^{1/r_1} \right)^{1/r_2}
\]

\[
= C \cdot \|A \ast B\|_{r_1/r_2}^{1/r_2}
\]

\[
\leq C \cdot \|A\|_{r_1/r_2}^{1/r_2} \cdot \|B\|_{r_1/r_2}^{1/r_2} = C \cdot \|Ua\|_{\ell^{r_2}_{\| \cdot \|_q}} \cdot \|Ub\|_{\ell^{r_2}_{\| \cdot \|_q}}.
\]

Case 2 \((1 \leq r_2 \leq \infty)\). We use Minkowsky’s Inequality to estimate the inner double sum in (3.10) by

\[
\left( \sum_{n \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q \right)^{1/r_2} \leq C \sum_{j \in \mathbb{Z}^d} \|\tilde{a}_{l-j} \ast \tilde{b}_j\|_q \leq C \sum_{j \in \mathbb{Z}^d} \|\tilde{a}_{l-j}\|_{\ell^2_q} \cdot \|\tilde{b}_j\|_{\ell^2_q} = C \cdot (A \ast B)(l),
\]

(3.12)
where we denoted $A(j) = \|\hat{a}_j\|_{l^2_m}$ and $B(j) = \|\hat{b}_j\|_{q_2}$. Substituting \eqref{3.10} into \eqref{3.12}, we obtain
\[
\|a * b\|_{l^{r_1,r_2}_m} \leq C \cdot \left( \sum_{l \in \mathbb{Z}^d} |(A * B)(l)|^{r_1} \right)^{1/r_1} 
\leq C\|A\|_{l^{p_1}_m} \|B\|_{l^{q_2}_m} = C\|U_a\|_{l^{p_2,q_1}_{\nu}} \|U_b\|_{l^{q_2,q_1}_{\nu}}.
\]
The theorem is proved completely. \hfill \square

Convolution relations for general Wiener-type amalgam spaces were first proved in \cite{2}. We refer the reader to \cite{6} for an introduction to the theory of $W(L^p, L^p_{\nu})$. With the help of Theorem 3.1, we can prove the following result:

**Corollary 3.1.** Denote $X = L^{\lambda_1,\lambda_2}$, $Y = L^{\mu_1,\mu_2}$, and $Z = L^{\sigma_1,\sigma_2}$. Choose the values for $\lambda_j, \mu_j$ and $\sigma_j, j \in \{1, 2\}$ so that $X * Y \to Z$. Assume that $m$ is a $\nu$-moderate weight on $\mathbb{Z}^{2d}$, and that
\[
p_i = q_i = r_i \quad \text{for} \quad 0 < r_i < 1
\]
and
\[
\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i} \quad \text{for} \quad r_i \geq 1.
\]
(a) If either $1 \leq r_1 \leq \infty$ and $0 < r_2 \leq \infty$, or $0 < r_2 \leq r_1 < 1$, then
\[
\|F * G\|_{W(Z, L^{r_1,r_2}_m)} \leq C\|F\|_{W(X, L^{p_1,q_2}_m)} \|G\|_{W(Y, L^{r_1,q_2}_m)}, \tag{3.13}
\]
(b) If $0 < r_1 \leq r_2 \leq \infty$ then
\[
\|F * G\|_{W(Z, L^{r_1,r_2}_m)} \leq C\|F\|_{W(X, L^{p_1,q_2}_m)} \|G\|_{W(Y, L^{r_1,q_1}_m)}, \tag{3.14}
\]
and
\[
\|F * G\|_{W(Z, L^{r_1,r_2}_m)} \leq C\|U_a\|_{l^{p_2,q_1}_{\nu}} \|U_b\|_{l^{q_2,q_1}_{\nu}}, \tag{3.15}
\]
where $Ua(k,n) = a(n,k), Ub(k,n) = b(n,k), a_{kn} = \|F \cdot T(k,n)\chi_{[0,1]^{2d}}\|_{X}$, and $b_{kn} = \|G \cdot T(k,n)\chi_{[0,1]^{2d}}\|_{Y}$.

**Proof.** A particular case of Corollary 3.1 with $p_j, q_j = r_j \geq 1$ is given in \cite{6} Section 11.8. The proof of the general case follows the same steps, relies on Theorem 3.1 and does not contain any new ideas. \hfill \square

**Remark 8.** Several particular cases of Corollary 3.1 have been used extensively in time-frequency analysis, particularly in the theory of modulation spaces. We refer the reader to \cite{4} and \cite{7}. For a comprehensive introduction to time-frequency analysis we refer to \cite{5}.

When $0 < p < 1$, $f * g$ is not even defined in general because, in this case, $f \in L^p$ does not imply local integrability of $f$. However, under additional assumptions (for example, when $f$ and $g$ are assumed to be band-limited), we are able to extend Young’s inequality even in the case of $0 < p < 1$.

**Corollary 3.2.** Suppose that $0 < \lambda < 1$, $\Gamma, \Gamma'$ are compact subsets of $\mathbb{R}^{2d}$ and that supp$(F) \in \Gamma$ and supp$(G) \in \Gamma'$. Assume that $m$ is a $\nu$-moderate weight on $\mathbb{Z}^{2d}$, and that the exponents $p_j, q_j$, and $r_j$ satisfy the conditions of Theorem 3.1 (a). If either $1 \leq r_1 \leq \infty$ and $0 < r_2 \leq \infty$, or $0 < r_2 \leq r_1 < 1$, then
\[
\|F * G\|_{W(L^{r_1,r_2}_m)} \leq C\|F\|_{W(L^{p_1,q_2}_m)} \|G\|_{W(L^{r_1,q_2}_m)}, \tag{3.16}
\]
(b). If $0 < r_1 \leq r_2 \leq \infty$ then
\[ ||F \ast G||_{W(L^{r_1}, L^{r_2})} \leq C ||F||_{W(L^{r_1}, L^{r_2})} ||G||_{W(L^{r_1}, L^{r_2})} \]  
(3.17)
and
\[ ||F \ast G||_{W(L^{r_1}, L^{r_2})} \leq C ||Ua||_{L_p^{t_2-p_1}} ||Ub||_{L_p^{t_2-q_1}}, \]
(3.18)
where $Ua(k, n) = a(n, k)$, $Ub(k, n) = b(n, k)$, $\alpha_{kn} = ||F \cdot T_{(k,n)} \chi_{[0,1]}||_{L_\lambda}$, and $b_{kn} = ||G \cdot T_{(k,n)} \chi_{[0,1]}||_{L_\lambda}$.

Proof. The proof relies on Theorem [3.1] on a convolution inequality for band-limited functions which can be found in [7, Lemma 2.6], and follows the steps of [5, Theorem 11.8.3]. \qed

We next establish several versions of Young’s inequality for the semi-discrete convolution $a \ast \nu F$. In view of Remark 11.8.3 for $F \in W(L_m^{p_2}, q_2)$, replacing the function $F$ with the step function defined by the sequence of the local supremum of $F$, yields the following corollary to Theorem 3.1.

Corollary 3.3. Assume that $m$ is $\nu$-moderate, and that the exponents $p_j$, $q_j$, and $r_j$ satisfy the conditions of Theorem 3.1.

(a). If either $1 \leq r_1 \leq \infty$ and $0 < r_2 \leq \infty$, or $0 < r_2 \leq r_1 < 1$, then
\[ ||a \ast \nu F||_{W(L_m^{r_1}, L_m^{r_2})} \leq C ||a||_{L_p^{t_1-p_2}} ||F||_{W(L_m^{r_1}, L_m^{r_2})}. \]  
(3.19)

(b). If $0 < r_1 \leq r_2 \leq \infty$ then
\[ ||a \ast \nu F||_{W(L_m^{r_1}, L_m^{r_2})} \leq C ||a||_{L_p^{t_1-p_2}} ||F||_{W(L_m^{r_1}, L_m^{r_2})}. \]  
(3.20)
and
\[ ||a \ast \nu F||_{W(L_m^{r_1}, L_m^{r_2})} \leq C ||Ua||_{L_p^{t_2-p_1}} ||Ub||_{L_p^{t_2-q_1}}, \]
(3.21)
where $Ua(k, n) = a(n, k)$, $Ub(k, n) = b(n, k)$, and $b_{kn} = ||F \cdot T_{(k,n)} \chi_{[0,1]}||_{L_\lambda}$.

If we drop the assumption that $F \in W(L_m^{p_2}, q_2)$ and only assume that $F \in L_m^{p_2}, q_2$, we are still able to generalize Young’s Inequality for the semi-direct convolution as follows:

Theorem 3.2. Assume that $m$ is $\nu$-moderate, $0 < p, q \leq \infty$, and $\alpha, \beta > 0$. Let $r = \min\{p, q\}$, $t = \min\{q, 1\}$, and $s = \min\{p, q, 1\}$. If $F \in L_m^{p_2}$ and $a \in \ell_p^{t_2}$, where $\nu(k, n) = \nu(k\alpha, n\beta)$, then $a \ast \nu F \in L_m^{p_2}$ and
\[ ||a \ast \nu F||_{L_m^{p, q}} \leq C ||a||_{\ell_p^{t, r}} ||F||_{L_m^{p, q}}. \]  
(3.22)
Moreover, in the case $0 < p \leq q \leq \infty$, the inequality
\[ ||a \ast \nu F||_{L_m^{p, q}} \leq C ||Ua||_{L_p^{t_2-q_1}} ||UF||_{L_m^{p, q}} \]  
(3.23)
also holds.

Proof. We denote $G = a \ast \nu F$, and use a periodization idea to write
\[ ||G||_{L_m^{p, q}} = \left( \int_{0, \beta} \sum_{L \in Z^d} \left( \int_{0, \alpha} \sum_{L \in Z^d} |G(x + j\alpha, \omega + l\beta)|^p m(x + j\alpha, \omega + l\beta)^p dx \right)^{q/p} d\omega \right)^{1/q}. \]  
(3.24)
Substituting (3.28) into (3.26) and using Young’s Inequality, we obtain
\[ |G(x + j\alpha, \omega + l\beta)| \cdot m(x + j\alpha, \omega + l\beta) \leq C \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \bar{a}_n(k) \tilde{F}_{x,\omega, l-n}(j - k) \]
\[ = C \sum_{n \in \mathbb{Z}^d} (\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j), \quad (3.25) \]
where we denoted \( \bar{a}_n(k) = |a(k,n)| \cdot \tilde{v}(k,n) \) and \( \tilde{F}_{x,\omega,n}(k) = |F(x + kn, \omega + n\beta)| \cdot m(x + kn, \omega + n\beta) \). We next use (3.25) to estimate the \( L_{p,q}^\alpha \)-norm of \( G \) by
\[ \|G\|_{L_{p,q}^\alpha} \leq C \left( \int_{[0,\beta]^d} \sum_{l \in \mathbb{Z}^d} \left( \left( \int_{[0,\alpha]^d} \left| \sum_{n \in \mathbb{Z}^d} (\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j) \right|^p \right) dx \right)^{q/p} d\omega \right)^{1/q} \]
(3.26)
and consider the following two cases:
Case 1: \( p \geq 1 \). We use Minkowski’s Inequality and Young’s Inequality to estimate the inner integrand in (3.26) by
\[ \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j) \right|^p \leq \left( \sum_{n \in \mathbb{Z}^d} \|\bar{a}_n \ast \tilde{F}_{x,\omega, l-n}\|_{\ell^p} \right)^p \]
\[ \leq \left( \sum_{n \in \mathbb{Z}^d} \|\bar{a}_n\|_{\ell^r} \|\tilde{F}_{x,\omega, l-n}\|_{\ell^p} \right)^p. \quad (3.27) \]
We next use (3.27) and Minkowski’s Inequality to estimate the inner integral in (3.26) by
\[ \left( \int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j) \right|^p dx \right)^{1/p} \leq \left( \int_{[0,\alpha]^d} \left( \sum_{n \in \mathbb{Z}^d} \|\bar{a}_n\|_{\ell^r} \|\tilde{F}_{x,\omega, l-n}\|_{\ell^p} \right)^p dx \right)^{1/p} \]
\[ \leq \sum_{n \in \mathbb{Z}^d} \|\bar{a}_n\|_{\ell^r} \left( \int_{[0,\alpha]^d} \|\tilde{F}_{x,\omega, l-n}\|_{\ell^p}^p dx \right)^{1/p} \]
\[ = \sum_{n \in \mathbb{Z}^d} A(n) B_\omega(l - n) = (A \ast B_\omega)(l), \quad (3.28) \]
where we denoted \( A(n) = \|\bar{a}_n\|_{\ell^r} \) and \( B_\omega(n) = \left( \int_{[0,\alpha]^d} \|\tilde{F}_{x,\omega, n}\|_{\ell^p}^p dx \right)^{1/p} \).
Substituting (3.28) into (3.26) and using Young’s Inequality, we obtain
\[ \|G\|_{L_{p,q}^\alpha} \leq C \left( \int_{[0,\beta]^d} \|A \ast B_\omega\|_{\ell^r}^q d\omega \right)^{1/q} \]
\[ \leq C \|A\|_{\ell^r} \left( \int_{[0,\beta]^d} \|B_\omega\|_{\ell^r}^q d\omega \right)^{1/q} \]
\[ = C \|a\|_{\ell^{r,s}} \cdot \|F\|_{L_{p,q}^\alpha}. \]
Case 2: \( p < 1 \). We use the inclusion \( \ell^p \hookrightarrow \ell^1 \) and Young’s Inequality to estimate the double sum in (3.26) by
\[ \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j) \right|^p \leq \sum_{n \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |(\bar{a}_n \ast \tilde{F}_{x,\omega, l-n})(j)|^p \]
\[ \leq \sum_{n \in \mathbb{Z}^d} \|\bar{a}_n\|_{\ell^r} \|\tilde{F}_{x,\omega, l-n}\|_{\ell^p}^p. \quad (3.29) \]
Using (3.29), we estimate the inner integral in (3.26) by

$$\int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\tilde{a}_n * \tilde{F}_{x,\omega,l-n})(j) \right|^p dx \leq \int_{[0,\alpha]^d} \sum_{n \in \mathbb{Z}^d} |\tilde{a}_n|^p \left| \tilde{F}_{x,\omega,l-n} \right|^p_{\ell^p} dx$$

$$= \sum_{n \in \mathbb{Z}^d} |\tilde{a}_n|^p \int_{[0,\alpha]^d} \left| \tilde{F}_{x,\omega,l-n} \right|^p_{\ell^p} dx$$

$$= \sum_{n \in \mathbb{Z}^d} A(n) B_\omega(l-n) = (A * B_\omega)(l),$$

(3.30)

where we denoted $A(n) = |\tilde{a}_n|^p$ and $B_\omega(n) = \int_{[0,\alpha]^d} |\tilde{F}_{x,\omega,n}|^p_{\ell^p} dx$.

Substituting (3.30) into (3.26), we obtain

$$\int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\tilde{a}_n * \tilde{F}_{x,\omega,l-n})(j) \right|^q dx \leq \left( \int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d} (\tilde{a}_n * \tilde{F}_{x,\omega,l-n})(j) \right|^p dx \right)^{q/p}$$

$$= \left( \int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left( \int_{[0,\alpha]^d} \sum_{l \in \mathbb{Z}^d} |G(x,j\alpha,\omega + l\beta)|^q m(x,j\alpha,\omega + l\beta) d\omega \right)^{p/q} dx \right)^{1/p}.$$

(3.31)

In order to derive (3.23), we use Minkowsky’s Inequality with the exponent $q/p \geq 1$ and the periodization idea to estimate $\|G\|_{L^{p,q}_{m,\nu}}$ by

$$\|G\|_{L^{p,q}_{m,\nu}} \leq \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |G(x,\omega)|^q m(x,\omega) d\omega \right)^{p/q} dx \right)^{1/p}$$

$$= \left( \int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left( \int_{[0,\alpha]^d} \sum_{l \in \mathbb{Z}^d} |G(x,j\alpha,\omega + l\beta)|^q m(x,j\alpha,\omega + l\beta) d\omega \right)^{p/q} dx \right)^{1/p}.$$

(3.32)

Since $m$ is $\nu$-moderate, we can estimate the inner summand in (3.31) by

$$|G(x,j\alpha,\omega + l\beta)| \cdot m(x,j\alpha,\omega + l\beta) \leq C \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} a_k(n) \hat{F}_{x,\omega,j-k}(l-n)$$

$$= C \sum_{k \in \mathbb{Z}^d} (\hat{a}_k * \hat{F}_{x,\omega,j-k})(l),$$

(3.33)

where we denoted $a_k(n) = |a(k,n)| \cdot \nu(k,n)$ and $F_{x,\omega,k}(n) = (F(x + k\alpha, \omega + n\beta) \cdot m(x + k\alpha, \omega + n\beta)$. We next use (3.32) to estimate the $L^{p,q}_{m}\nu$-norm of $G$ by

$$\|G\|_{L^{p,q}_{m,\nu}} \leq C \left( \int_{[0,\alpha]^d} \sum_{j \in \mathbb{Z}^d} \left( \int_{[0,\alpha]^d} \sum_{l \in \mathbb{Z}^d} |(\hat{a}_k * \hat{F}_{x,\omega,j-k})(l)|^q d\omega \right)^{p/q} dx \right)^{1/p}.$$

(3.34)

and consider the following two cases:

Case 1 ($0 < p \leq q < 1$): We use the inclusion $\ell^q \hookrightarrow \ell^1$ and Young’s Inequality to estimate the double sum in (3.33) by

$$\sum_{l \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} (\hat{a}_k * \hat{F}_{x,\omega,j-k})(l) \right|^q \leq \sum_{k \in \mathbb{Z}^d} \|\hat{a}_k * \hat{F}_{x,\omega,j-k}\|_{\ell^q}^q$$

$$\leq \sum_{k \in \mathbb{Z}^d} \|\tilde{a}_k\|_{\ell^q}^q \cdot \|\tilde{F}_{x,\omega,j-k}\|_{\ell^q}^q.$$

(3.34)
Next, we use (3.34) to estimate the inner integral in (3.33) by
\[
\int_{[0, \beta]^d} \sum_{l \in \mathbb{Z}^d} | \sum_{k \in \mathbb{Z}^d} (\tilde{a}_k \ast \tilde{F}_{x, \omega, j - k})(l) |^q d\omega \leq \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_k||_{L^q_{\alpha}}^q \int_{[0, \beta]^d} ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}}^q d\omega
\]
where \( A(k) = ||\tilde{a}_k||_{L^q_{\alpha}}^q \) and \( B_x(k) = \int_{[0, \beta]^d} ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}}^q d\omega \). Therefore, by Young's Inequality with \( p/q < 1 \),
\[
||G||_{L_p^{\alpha, q}} \leq C \left( \int_{[0, \beta]^d} \sum_{j \in \mathbb{Z}^d} (A \ast B_x)(j) |^p/| dx \right)^{1/p}
\]
\[
\leq C \left( \int_{[0, \beta]^d} ||A||_{L^{p/q}_{\alpha}}^p \cdot ||B_x||_{L^{p/q}_{\alpha}}^p dx \right)^{1/p}
\]
\[
= C ||A||_{L^{p/q}_{\alpha}}^p \left( \int_{[0, \beta]^d} ||B_x||_{L^{p/q}_{\alpha}}^p dx \right)^{1/p} = C ||Ua||_{L^{p/q}_{\alpha}}^p \cdot ||UF||_{L^{p/q}_{\alpha}}^p
\]
Case 2 (1 \( \leq q < \infty \)): We use Minkowsky's Inequality with the exponent \( q \geq 1 \) to estimate the double sum in (3.33) by
\[
\sum_{l \in \mathbb{Z}^d} | \sum_{k \in \mathbb{Z}^d} (\tilde{a}_k \ast \tilde{F}_{x, \omega, j - k})(l) |^q \leq \left( \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_k \ast \tilde{F}_{x, \omega, j - k}||_{L^q} \right)^q
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_k||_{L^q_{\alpha}} \cdot ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}} \right)^q \quad (3.35)
\]
and next use (3.35) and Minkowsky's Inequality to estimate the inner integral in (3.33) by
\[
\left( \int_{[0, \beta]^d} \sum_{l \in \mathbb{Z}^d} | \sum_{k \in \mathbb{Z}^d} (\tilde{a}_k \ast \tilde{F}_{x, \omega, j - k})(l) |^q d\omega \right)^{1/q} \leq \left( \int_{[0, \beta]^d} \left( \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_k||_{L^q_{\alpha}} \cdot ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}} \right)^q d\omega \right)^{1/q}
\]
\[
\leq \sum_{k \in \mathbb{Z}^d} ||\tilde{a}_k||_{L^q_{\alpha}} \left( \int_{[0, \beta]^d} ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}}^q d\omega \right)^{1/q} = (A \ast B_x)(j),
\]
where \( A(k) = ||\tilde{a}_k||_{L^q_{\alpha}} \) and \( B_x(k) = \int_{[0, \beta]^d} ||\tilde{F}_{x, \omega, j - k}||_{L^q_{\alpha}}^q d\omega \). Therefore, by Young's Inequality
\[
||G||_{L_p^{\alpha, q}} \leq C \left( \int_{[0, \beta]^d} \sum_{j \in \mathbb{Z}^d} (A \ast B_x)(j) |^p/| dx \right)^{1/p}
\]
\[
\leq C ||A||_{L^{p/q}_{\alpha}} \left( \int_{[0, \beta]^d} ||B_x||_{L^{p/q}_{\alpha}}^p dx \right)^{1/p} = C ||Ua||_{L^{p/q}_{\alpha}}^p \cdot ||UF||_{L^{p/q}_{\alpha}}^p. \quad (3.36)
\]
The theorem is proved completely. □

References


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