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OPERATIONAL AND LIE ALGEBRAIC TECHANIQUES ON HERMITE POLYNOMIALS

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Abstract. In this paper, we derive generating relations involving multi-index multi-variable associated Hermite polynomials $G_{m,n}(x,y;\rho)$ by using Lie-algebraic method. Further we use the principle of monomiality and the operational methods to prove that this approach is more flexible than Lie algebraic techniques.

1. Introduction

MThe theory of many variable many index Hermite polynomials was initially developed by Hermite himself [9].There are two kinds of Hermite polynomials $H_{m,n}(x,y)$ and $G_{m,n}(x,y)$ which are usually defined with the help of a symmetric 2×2 matrix $\hat{M}$. Two varirble two index Hermite polynomials are specified by the generating function [1]

$$\sum_{m,n=0}^{\infty} H_{m,n}(x,y)\,\frac{t^m u^n}{m!n!} \;=\; \exp\left[\underline{\mathrm{w}}^T\hat{M}\underline{\mathrm{z}} - \frac{1}{2}\underline{\mathrm{w}}^T\hat{M}\underline{\mathrm{w}}\right]. \tag{1.1}$$

where T denotes the transpose and

$$\underline{\mathrm{w}} \;=\; \begin{pmatrix} t \\ u \end{pmatrix}$$

$$\underline{\mathrm{z}} \;=\; \begin{pmatrix} x \\ y \end{pmatrix}$$

The 2×2 matrix$\hat{M}$ is

$$\hat{M} \;=\; \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where

$$\Delta \;=\; ac - b^2 > 0, \qquad (a,c) > 0.$$

The second kind is defined for the purpose of the orthonormalization of the first kind in the form of bio-orthogonality relations. Recently the associated orthogonal functions have been discussed in [3] along with a number of applications to classical and quantum mechanics [6]. The importance of these polynomials for Physical

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applications has been recognized by other authors and a partial list of refrences is reported in [7],[8],[10],[12].Restricting ourselves to the case of two indices and two variables, here we consider second kind i.e.associated Hermite polynomials (aHP) $G_{m,n}(x,y)$ defined by means of the generating function [1,4,9]

$$\sum_{m,n=0}^{\infty} G_{m,n}(x,y)\,\frac{r^m s^n}{m!n!} \;=\; \exp\left[\underline{\mathrm{Z}}^T\underline{\mathrm{K}} - \frac{1}{2}\underline{\mathrm{K}}^T\hat{M}^{-1}\underline{\mathrm{K}}\right]. \tag{1.1}$$

where T denotes the transpose and

$$\underline{\mathrm{Z}} \;=\; \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\mathrm{K}} \;=\; \begin{pmatrix} r \\ s \end{pmatrix}$$

The 2×2 matrix$\hat{M}$ is

$$\hat{M} \;=\; \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where

$$\Delta \;=\; ac - b^2 > 0, \qquad (a,c) > 0.$$

More in general one can introduce the polynomials $G_{m,n}(x,y;\rho)$ defined by[1,4,9]

$$\sum_{m,n=0}^{\infty} G_{m,n}(x,y;\rho)\,\frac{r^m s^n}{m!n!} \;=\; \exp\left[\underline{\mathrm{Z}}^T\underline{\mathrm{K}} - \rho\,\underline{\mathrm{K}}^T\hat{M}^{-1}\underline{\mathrm{K}}\right], \tag{1.2}$$

where

$$\frac{\partial}{\partial\rho}\,G_{m,n}(x,y;\rho) \;=\; \frac{1}{\Delta}\left[c\frac{\partial^2}{\partial x^2} - 2b\frac{\partial^2}{\partial x\partial y} + a\frac{\partial^2}{\partial y^2}\right]G_{m,n}(x,y;\rho). \tag{1.3}$$

which is a kind of extended heat equation,with the conditions

$$G_{m,n}(x,y;0) \;=\; x^m y^n. \tag{1.4}$$

and

$$G_{m,n}(x,y;-\frac{1}{2}) \;=\; G_{m,n}(x,y). \tag{1.5}$$

The polynomial $G_{m,n}(x,y;\rho)$ are linked to $G_{m,n}(x,y)$ by the relation

$$G_{m,n}(x,y;\rho) \;=\; i^{m+n}(2\rho)^{m+n/2}G_{m,n}\left(\frac{x}{i\sqrt{(2\rho)}},\frac{y}{i\sqrt{(2\rho)}}\right). \tag{1.6}$$

The aHP $G_{m,n}(x,y;\rho)$ defined by Eq.(1.2) satisfy the following differential and pure recurrence relations:

$$\begin{aligned}
\frac{\partial}{\partial x}\,G_{m,n}(x,y;\rho) &= mG_{m-1,n}(x,y;\rho)\ ,\\
\frac{\partial}{\partial y}\,G_{m,n}(x,y;\rho) &= nG_{m,n-1}(x,y;\rho)\ ,\\
\frac{\partial}{\partial\rho}\,G_{m,n}(x,y;\rho) &= \frac{1}{\Delta}[cm(m-1)G_{m-2,n}(x,y;\rho) - 2bmnG_{m-1,n-1}(x,y;\rho)\\
&\quad +a(n-1)G_{m,n-2}(x,y;\rho)].
\end{aligned} \tag{1.7}$$

and

$$G_{m+1,n}(x,y;\rho) = xG_{m,n}(x,y;\rho) + \frac{2c\rho}{\Delta}mG_{m-1,n}(x,y;\rho) - \frac{2b\rho}{\Delta}nG_{m,n-1}(x,y;\rho),$$

$$G_{m,n+1}(x,y;\rho) = yG_{m,n}(x,y;\rho) + \frac{2a\rho}{\Delta}nG_{m,n-1}(x,y;\rho) - \frac{2b\rho}{\Delta}mG_{m-1,n}(x,y;\rho). \tag{1.8}$$

The theory of special functions from the group-theoretic point of view, provides a unifying formalism to deal with the immense aggregate of the special functions and a collection of formulae such as the relevant differential equations, integral representations, recurrence formulae, composition theorems, etc., see for example [15,16]. The first significant advance in the direction of obtaining generating relations by Lie-theoretic method is made by Weisner [17-19] and Miller [13].

Within the group-theoretic context, indeed a given class of special functions appears as a set of matrix elements of irreducible representation of a given Lie group. The algebraic properties of the group are then reflected in the functional and differential equations satisfied by a given family of special functions, whilst the geometry of the homogeneous space determines the nature of the integral representation associated with the family.

The 4-dimensional complex local Lie group $G(0,1)$ is the set of all 4×4 matrices of the form

$$g = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b,c,\tau \in \mathbb{C}, \tag{1.9}$$

A basis for the Lie algebras $G(0,1) = L[G(0,1)]$ is provided by the matrices

$$J^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J^{-} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J^{3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.10}$$

with commutations relations

$$[J^3, J^{\pm}] = \pm J^3, \quad [J^+, J^-] = -E, \quad [E, J^{\pm}] = [E, J^3] = 0, \tag{1.11}$$

In this paper, we consider aHP $G_{m,n}(x,y;\rho)$ and derive generating relations involving these polynomials by using representation $\uparrow_{\omega,\mu}$ of Lie algebra $G(0,1)$.In Section 2, we consider the problem of framing aHP $G_{m,n}(x,y;\rho)$ into the context of the representation $\uparrow_{\omega,\mu}$ of four dimensional Lie algebra $G(0,1)$ and obtain generating relations involving aHP $G_{m,n}(x,y;\rho)$ and associated Laguerre polynomials $L_n^{\alpha}(x)$. In Section 3, we consider some applications of the generating relations obtained in Section 2. Finally, in Section 4,some concluding remarks are given.

2. Representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ and generating relations

The irreducible representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ is defined for each $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum S of this representation is the set $\{-\omega + k : \ k$ a nonnegative integer$\}$ and there is a basis $(f_m)_{m\in S}$ for the representation space V, with the properties

$$J^3 f_m = m f_m, \ \ E f_m = \mu f_m, \ \ J^+ f_m = \mu f_{m+1}, \ \ J^- f_m = (m+\omega) f_{m-1},$$
$$C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m, \quad \mu \neq 0. \tag{2.1}$$

(Here $f_{-w-1} \equiv 0$, so $J^- f_{-w} = 0$).

The commutation relations satisfied by the operators $J^3, \ J^\pm, \ E$ are

$$[J^3, J^\pm] = \pm J^\pm, \ \ [J^+, J^-] = -E, \ \ [E, J^\pm] = [E, J^3] = 0. \tag{2.2}$$

In order to obtain a realization of the representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ on a space of functions of two complex variables x and y, Miller ([13]; p.104) has taken the functions $f_m(x,y) = Z_m(x)e^{my}$, such that relations (2.1) are satisfied for all $m \in S$, where the differential operators J^3, J^+, J^-, E are given by

$$\begin{aligned} J^3 &= \frac{\partial}{\partial y}, \\ J^+ &= e^y \left[\frac{\partial}{\partial x} - \frac{1}{2}\mu\, x\right], \\ J^- &= e^{-y}\left[-\frac{\partial}{\partial x} - \frac{1}{2}\,\mu x\right] \\ E &= \mu. \end{aligned} \tag{2.3}$$

In particular, we look for the functions

$$f_{m,n}(x,y,r,s;\rho) = Z_{m,n}(x,y;\rho) r^m s^n, \tag{2.4}$$

such that

$$K^3 f_{m,n} = m f_{m,n}, \ \ I f_{m,n} = \mu f_{m,n}, \ \ K^+ f_{m,n} = \mu f_{m+1,n}$$

,

$$K^- f_{m,n} = (m+\omega) f_{m-1,n} C_{0,1} f_{m,n} = (K^+K^- - IK^3) f_{m,n} = \mu\omega f_{m,n}, \quad \mu \neq 0, \tag{2.5}$$

for all $n \in S$.

We assume that the set of linear differential operators $K^\pm, \ K^3, \ I$ takes the form

$$\begin{aligned} K^+ &= xr + \frac{2c\rho r}{\Delta}\frac{\partial}{\partial x} - \frac{2b\rho r}{\Delta}\frac{\partial}{\partial y}, \\ K^- &= \frac{1}{r}\frac{\partial}{\partial x}, \\ K^3 &= r\frac{\partial}{\partial r}, \\ I &= 1 \end{aligned} \tag{2.6}$$

and note that these operators satisfy the commutation relations identical to (2.2).

There is no loss of generality for special function theory if we set $w = 0$, $\mu = 1$, then in terms of the function $Z_m(x)$, relation (2.1) reduce to ([13]; p.104(4.72))

$$\begin{aligned}
\left(\frac{d}{dx}-\frac{1}{2}x\right)Z_m(x) &= Z_{m+1}(x),\\
\left(-\frac{d}{dx}-\frac{1}{2}x\right)Z_m(x) &= mZ_{m-1}(x),\\
\left(-\frac{d^2}{dx^2}+\frac{x^2}{4}-\frac{1}{2}-m\right)Z_m(x) &= 0.
\end{aligned} \tag{2.7}$$

Further, for the same choice of ω and μ and using operators (2.6), relations (2.5) take the form

$$\begin{aligned}
&\left[x+\frac{2c\rho}{\Delta}\frac{\partial}{\partial x}-\frac{2b\rho}{\Delta}\frac{\partial}{\partial y}\right]Z_{m,n}(x,y;\rho) = Z_{m+1,n}(x,y;\rho),\\
&\left[\frac{\partial}{\partial x}\right]Z_{m,n}(x,y;\rho) = mZ_{m-1,n}(x,y;\rho),\\
&\left[(\frac{2c\rho}{\Delta}+x)\frac{\partial}{\partial x}+\frac{2c\rho}{\Delta}\frac{\partial^2}{\partial x^2}-\frac{2b\rho}{\Delta}\frac{\partial^2}{\partial x\partial y}-m\right]Z_{m,n}(x,y;\rho) = 0.
\end{aligned} \tag{2.8}$$

Again if we take the functions $f_{m,n}(x,y,r,s;\rho)=Z_{m,n}(x,y;\rho)r^m s^n$ such that

$$K'^3 f_{m,n} = nf_{m,n},\quad I'f_{m,n} = \mu f_{m,n},\quad K'^+ f_{m,n} = \mu f_{m,n+1}$$

,

$$K'^- f_{m,n} = (n+\omega)f_{m,n-1}C'_{0,1}f_{m,n} = (K'^+K'^- - I'K'^3)f_{m,n} = \mu\omega f_{m,n},\quad \mu\neq 0, \tag{2.9}$$

for all $n\in S$,where the differential operators K'^+, K'^-, K'^3, I' are

$$\begin{aligned}
K'^+ &= ys-\frac{2b\rho s}{\Delta}\frac{\partial}{\partial x}+\frac{2a\rho s}{\Delta}\frac{\partial}{\partial y},\\
K'^- &= \frac{1}{s}\frac{\partial}{\partial y},\\
K'^3 &= s\frac{\partial}{\partial s},\\
I' &= 1
\end{aligned} \tag{2.10}$$

and note that these operators satisfy the commutation relations identical to (2.2).

again taking $w=0$ and $\mu=1$ relations (2.9) becomes

$$\begin{aligned}
&\left[y+\frac{2a\rho}{\Delta}\frac{\partial}{\partial y}-\frac{2b\rho}{\Delta}\frac{\partial}{\partial x}\right]Z_{m,n}(x,y;\rho) = Z_{m,n+1}(x,y;\rho),\\
&\left[\frac{\partial}{\partial y}\right]Z_{m,n}(x,y;\rho) = mZ_{m,n-1}(x,y;\rho),\\
&\left[(\frac{2a\rho}{\Delta}+y)\frac{\partial}{\partial y}+\frac{2a\rho}{\Delta}\frac{\partial^2}{\partial y^2}-\frac{2b\rho}{\Delta}\frac{\partial^2}{\partial x\partial y}-n\right]Z_{m,n}(x,y;\rho) = 0.
\end{aligned} \tag{2.11}$$

Miller [13] have used relations (2.7) to compute the functions $Z_m(x)$, which are easily expressed in terms of parabolic cylinder functions or Hermite polynomials $H_n(x)$. In fact,

$$Z_m(x) = (-1)^m D_m(x) = (-1)^m \exp\Big(-\frac{x^2}{4}\Big) 2^{-m/2} H_m\Big(\frac{x}{\sqrt{2}}\Big), \quad m = 0, 1, 2, \dots \ . \tag{2.12}$$

Similarly, we observe that, for all $n \in S$, the choice for $f_{m,n}(x, y, r, s; \rho) = Z_{m,n}(x, y; \rho) r^m s^n$ satisfy Eqs. (2.8) and (2.11). It follows from the above discussion that the functions $f_{m,n}(x, y, r, s; \rho) = G_{m,n}(x, y; \rho) r^m s^n$, $m, n \in S$ form a basis for a realization of the representation $\uparrow_{0,1}$ of $G(0, 1)$. By using ([13];P.18(Theorem 1.10)), this representation of $G(0, 1)$ can be extended to a local multiplier representation $T(g)$, $g \in G(0, 1)$ defined on F, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, r^0, s^0, \rho^0) = (1, 1, 1, 1, 0)$.

Hence using operators (2.6), the local multiplier representation takes the form

$$\begin{aligned}
[T(\exp a_1 E) f](x, y, r, s; \rho) &= \exp(a_1) f(x, y, r, s; \rho), \\
[T(\exp b_1 J^+) f](x, y, r, s; \rho) &= \exp\left(x r b_1 (1 + \frac{c \rho r b_1}{\Delta x})\right) \\
&\quad \times f\left(x\left(1 + \frac{2\rho c r b_1}{\Delta x}\right), y\left(1 - \frac{2\rho b r b_1}{\Delta y}\right), r, s; \rho\right), \\
[T(\exp c_1 J^-) f](x, y, r, s; \rho) &= f\left(x\left(1 + \frac{c_1}{x r}\right), y, r, s; \rho\right), \\
[T(\exp \tau_1 J^3) f](x, y, r, s; \rho) &= f(x, y, r e^{\tau_1}, s; \rho),
\end{aligned} \tag{2.13}$$

for $f \in F$, where J^+, J^-, J^3, E are given by the matrices (1.10) and form a basis for Lie algebra $G(0, 1)$. If $g \in G(0, 1)$ has parameters (a_1, b_1, c_1, τ_1) then

$$T(g) = T(\exp a_1 E)\, T(\exp b_1 J^+)\, T(\exp c_1 J^-)\, T(\exp \tau_1 J^3)$$

and therefore we obtain

$$[T(g) f](x, y, r, s; \rho) =$$
$$\exp\left(a_1 + x r b_1 (1 + \frac{c \rho r b_1}{\Delta x})\right) f\left(x\left(1 + \frac{2\rho c r b_1}{\Delta x} + \frac{c_1}{x r}\right), y\left(1 - \frac{2\rho b r b_1}{\Delta y}\right), r e^{\tau_1}, s; \rho\right). \tag{2.14}$$

The matrix elements of $T(g)$ with respect to the analytic basis $f_{m,n}(x, y, r, s; \rho) = G_{m,n}(x, y; \rho) r^m s^n$, are the functions $A_{lk}(g)$, uniquely determined by $\uparrow_{0,1}$ of $G(0, 1)$ and we obtain relations

$$[T(g) f_{k,n}](x, y, r, s; \rho) = \sum_{l=0}^{\infty} A_{lk}(g) f_{l,n}(x, y, r, s; \rho), \qquad k, l = 0, 1, 2, \dots, \tag{2.15}$$

which on using (2.14) yields

$$\exp\left(a_1 + k\tau_1 + x r b_1 (1 + \frac{c \rho r b_1}{\Delta x})\right) G_{k,n}\left(x\left(1 + \frac{2\rho c r b_1}{\Delta x} + \frac{c_1}{x r}\right), y\left(1 - \frac{2\rho b r b_1}{\Delta y}\right); \rho\right)$$
$$= \sum_{l=0}^{\infty} A_{lk}(g) G_{l,n}(x, y; \rho) r^{l-k}, \quad k, l = 0, 1, 2, \dots \ . \tag{2.16}$$

The matrix element $A_{lk}(g)$ are given by ([13]; p.87(4.26))

$$A_{lk}(g) \;=\; \exp(a_1 + k\tau_1)c_1^{k-l}L_l^{k-l}(-b_1c_1), \qquad k,l \geq 0. \tag{2.17}$$

Substituting (2.17) into (2.16) and simplifying, we obtain the generating relation

$$\exp\left(xrb_1(1+\frac{c\rho rb_1}{\Delta x})\right)G_{k,n}\left(x\left(1+\frac{2\rho crb_1}{\Delta x}+\frac{c_1}{xr}\right),y\left(1-\frac{2\rho brb_1}{\Delta y}\right);\rho\right)$$

$$= \sum_{l=0}^{\infty} {c_1}^{k-l}L_l^{k-l}(-b_1c_1)G_{l,n}(x,y;\rho)r^{l-k}, \quad k=0,1,2,\cdots. \tag{2.18}$$

Again taking the operators (2.10) and proceeding exactly as before , we obtain the generating relation

$$\exp\left(ysb_2(1+\frac{a\rho sb_2}{\Delta y}\right)G_{m,r}\left(x\left(1-\frac{2\rho bsb_2}{\Delta x}\right),y\left(1+\frac{2\rho asb_2}{\Delta y}+\frac{c_2}{ys}\right);\rho\right)$$

$$= \sum_{i=0}^{\infty} {c_2}^{r-i}L_i^{r-i}(-b_2c_2)G_{m,i}(x,y;\rho)s^{i-r}, \quad r=0,1,2,\cdots. \tag{2.19}$$

3. Applications

We consider the following applications of generating relation (2.18).

I. Taking $b_1 \to 0$ in generating relation (2.18) and using the limit ([13]; p.88(4.29))

$$c_1^nL_l^n(b_1c_1)\Big|_{b_1=0} \;=\; \begin{cases} \begin{pmatrix} n+l \\ n \end{pmatrix}{c_1}^n & \text{if } \; n \geq 0, \\ 0 & \text{if } \; n < 0, \end{cases} \tag{3.1}$$

we get

$$G_{k,n}\left((x+\frac{c_1}{r}),y;\rho\right) \;=\; \sum_{l=0}^{k}\begin{pmatrix} k \\ k-l \end{pmatrix}{c_1}^{k-l}G_{l,n}(x,y;\rho)r^{l-k}. \tag{3.2}$$

Again, taking $c_1 \to 0$ in generating relation (2.18) and using the limit ([13]; p.88(4.29))

$${c_1}^nL_l^n(b_1c_1)\Big|_{c_1=0} \;=\; \begin{cases} 0 & \text{if } \; n > 0, \\ \dfrac{(-b_1)^{-n}}{(-n)!}, & \text{if } \; n \leq 0, \end{cases} \tag{3.3}$$

we get

$$\exp\left(xrb_1(1+\frac{c\rho rb_1}{\Delta x})\right)G_{k,n}\left(x\left(1+\frac{2\rho crb_1}{\Delta x}\right),y\left(1-\frac{2\rho brb_1}{\Delta y}\right);\rho\right)$$

$$= \sum_{l=0}^{\infty}\frac{{b_1}^{l-k}}{(l-k)!}G_{l,n}(x,y;\rho)r^{l-k} \tag{3.4}$$

Similar results can be obtain from generating relation (2.19).

II. Taking $\rho = -1/2$ in generating relation (2.18) and using Eq. (1.5), we get

$$\exp\left(xrb_1(1-\frac{crb_1}{2\Delta x})\right)G_{k,n}\left(x\left(1-\frac{crb_1}{\Delta x}+\frac{c_1}{xr}\right),y\left(1+\frac{brb_1}{\Delta y}\right)\right)$$
$$=\sum_{l=0}^{\infty}{c_1}^{k-l}L_l^{k-l}(-b_1c_1)G_{l,n}(x,y)r^{l-k},\quad k=0,1,2,\cdots. \tag{3.5}$$

where $G_{l,n}(x,y)$ denotes aHP of two variables given by Eq.(1.1). Similar results can be obtain from generating relation (2.19).

III. Taking $\rho=-1/2$, $-\frac{c}{2\Delta}=y$ and $\frac{b}{\Delta}=\xi$ in generating relation (2.18)

$$\exp(xrb_1+yr^2{b_1}^2)G_{k,n}\left(x\left(1+\frac{2yrb_1}{x}+\frac{c_1}{xr}\right),y\left(1+\frac{\xi rb_1}{y}\right)\right)$$
$$=\sum_{l=0}^{\infty}{c_1}^{k-l}L_l^{k-l}(-b_1c_1)G_{l,n}(x,y)r^{l-k}, \tag{3.6}$$

which is the correct form of a result of Miller[13]. Similar results can be obtain from generating relation (2.19).

4. Concluding Remarks

The principle of monomiality, combined with the operational techniques, provides a powerful and flexible mean to deal with multi-variable and multi-index polynomials than the ordinary Lie-algebraic methods used in section 2 also in [14].

In this section , we will show that Lie-algebraic methods of generalized Hermite polynomials can be replaced by the methods developed in [2],which albeit a by-product of the Lie-algebraic theory [13] , gives more flexiblity in the treatment of multi-dimentional Hermite polynomials.

The associated Hermite polynomials $G_{m,n}(x,y;\rho)$ satisfies the identities

$$\frac{\partial}{\partial\rho}G_{m,n}(x,y;\rho) = \frac{1}{\Delta}\left[c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2}\right]G_{m,n}(x,y;\rho).$$

and

$$G_{m,n}(x,y;0) = x^my^n.$$

are defined by means of the operational rule

$$G_{m,n}(x,y;\rho) = e^{\frac{\rho}{\Delta}(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2})}x^my^n. \tag{4.1}$$

The Lie-algebraic study of these polynomials can be achived by considering the separate realizations of $G(0,1)$ algebras and by coupling them through a similarity transform involving the operator $e^{\frac{\rho}{\Delta}(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2})}$.

We can therefore apply a procedure of generalization of the Lie algebraic method by defining the following generators of the algebra $G_x(0,1)\oplus G_y(0,1)$

$${J_x}^+ = xr+\frac{2c\rho r}{\Delta}\frac{\partial}{\partial x}-\frac{2b\rho r}{\Delta}\frac{\partial}{\partial y},\qquad {J_y}^+ = ys-\frac{2b\rho s}{\Delta}\frac{\partial}{\partial x}+\frac{2a\rho s}{\Delta}\frac{\partial}{\partial y},$$
$${J_x}^- = \frac{1}{r}\frac{\partial}{\partial x},\qquad {J_y}^- = \frac{1}{s}\frac{\partial}{\partial y}.$$

$$J_x{}^3 \;=\; \left(x+\frac{2c\rho}{\Delta}\frac{\partial}{\partial x}-\frac{2b\rho}{\Delta}\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x},\quad J_y{}^3 \;=\; \left(y-\frac{2b\rho}{\Delta}\frac{\partial}{\partial x}+\frac{2a\rho}{\Delta}\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y},$$
$$\hat{E} \;=\; 1. \tag{4.2}$$

with commutation relations

$$[J^3{}_\alpha, J^\pm{}_\beta] \;=\; \pm J_\alpha \delta_{\alpha,\beta},\quad [J^+{}_\alpha, J^-{}_\beta] \;=\; \delta_{\alpha,\beta}, \tag{4.3}$$

where $\delta_{\alpha,\beta}$ is the Kronecker symbol.

According to the relization asserted by Eq.(4.2) we can conclude that by means of the operational representation in Eq.(4.1), we can transform a result involving products of the type $a_{m,n}x^m y^n$ into a corresponding result for the associated Hermite polynomials $G_{m,n}(x,y;\rho)$, namely

$$e^{\frac{\rho}{\Delta}(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2})}a_{m,n}x^m y^n \;=\; a_{m,n}G_{m,n}(x,y;\rho). \tag{4.4}$$

Now we extend this process to a product of two series of the relation [13]

$$\sum_{l=0}^{\infty} c^{k-l}L_l{}^{k-l}(-bc)x^l \;=\; e^{bx}\;(x+c)^k, \tag{4.5}$$

with $L_l{}^\alpha(x)$ being associated Laguerre polynomials [11],

$$L_n{}^\alpha(x) \;=\; \Gamma(\alpha+n+1)\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!\Gamma(a+r+1)(n-r)!}, \tag{4.6}$$

accordingly we find from Eq.(4.5) that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_1{}^{k-m}c_2{}^{s-n}L_m{}^{k-m}(-b_1c_1)L_n{}^{s-n}(-b_2c_2)e^{\frac{\rho}{\Delta}(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2})}x^m y^n$$
$$=\; e^{\frac{\rho}{\Delta}(c\frac{\partial^2}{\partial x^2}-2b\frac{\partial^2}{\partial x\partial y}+a\frac{\partial^2}{\partial y^2})}e^{(b_1x+b_2y)}(x+c_1)^k(y+c_2)^s. \tag{4.7}$$

which on using some properties of the exponential operators, yields the generating relation of aHP $G_{m,n}(x,y;\rho)$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_1{}^{k-m}c_2{}^{s-n}L_m{}^{k-m}(-b_1c_1)L_n{}^{s-n}(-b_2c_2)G_{m,n}(x,y;\rho)$$
$$=\; e^{(b_1x+b_2y-\frac{2b_1b_2b\rho}{\Delta}+\frac{b_1{}^2c\rho}{\Delta}+\frac{b_2{}^2a\rho}{\Delta})}G_{k,s}\Big((x+\frac{2b_1c\rho}{\Delta}-\frac{2b_2b\rho}{\Delta}+c_1),$$
$$(y+\frac{2b_2a\rho}{\Delta}-\frac{2b_1b\rho}{\Delta}+c_2);\rho\Big). \tag{4.8}$$

Again using the identity [2]

$$e^{(y\frac{\partial^2}{\partial x^2}+w\frac{\partial^2}{\partial z^2}+\xi\frac{\partial^2}{\partial x\partial z})}x^m z^n \;=\; H_{m,n}(x,y,z,w\mid\xi). \tag{4.9}$$

in Eq.(4.7), we obtain a generating relation of two index Hermite polynomials in four variables

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_1{}^{k-m}c_2{}^{s-n}L_m{}^{k-m}(-b_1c_1)L_n{}^{s-n}(-b_2c_2)H_{m,n}(x,\frac{c\rho}{\Delta},y,\frac{a\rho}{\Delta}\mid\frac{-2b\rho}{\Delta})$$
$$=\; e^{(b_1x+b_2y-\frac{2b_1b_2b\rho}{\Delta}+\frac{b_1{}^2c\rho}{\Delta}+\frac{b_2{}^2a\rho}{\Delta})}H_{k,s}\Big((x+\frac{2b_1c\rho}{\Delta}-\frac{2b_2b\rho}{\Delta}+c_1),y,$$

$$(z+\frac{2b_2a\rho}{\Delta}-\frac{2b_1b\rho}{\Delta}+c_2),w\mid\xi\Big). \tag{4.10}$$

Now replacing $\frac{c\rho}{\Delta}$ by y, y by z, $\frac{a\rho}{\Delta}$ by w and $\frac{-2b\rho}{\Delta}$ by ξ in Eq.(4.10), we obtain ([5];p.86(40))

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}{c_1}^{k-m}{c_2}^{s-n}{L_m}^{k-m}(-b_1c_1){L_n}^{s-n}(-b_2c_2)H_{m,n}(x,y;z,w\mid\xi)$$

$$=\ e^{(b_1x+b_2z+b_1b_2\xi+{b_1}^2y+{b_2}^2w)}H_{k,s}\left((x+2b_1y+b_2\xi+c_1),y;(z+2b_2w+b_1\xi),w\mid\xi\right). \tag{4.11}$$

where $H_{m,n}(x,y,z,w\mid\xi)$ are defined as follows [2]

$$H_{m,n}(x,y,z,w\mid\xi)\ =\ m!n!\sum_{r=0}^{min[m,n]}\frac{\xi^rH_{m-r}(x,y)H_{n-r}(z,w)}{r!(m-r)!(n-r)!}, \tag{4.12}$$

The interest for polynomials of the type (4.12) is manifold, they can indeed be used in the problems involving coupled harmonic oscillators. Moreover they are the generalized forms of the biorthogonal families of polynomials introduced by Hermite himself [9].

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