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# A NEW OSTROWSKI-TYPE INEQUALITY FOR DOUBLE INTEGRALS

M. EMIN ÖZDEMIR, AHMET OCAK AKDEMIR AND ERHAN SET

Abstract. In this paper, we established a new Ostrowski-type inequality involving functions of two independent variables.

## 1. INTRODUCTION

In [6], Ostrowski proved the following inequality;

**Theorem 1.** *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of $I$ and $a, b \in I^o, a < b$. If $|f'| \leq M, \forall t \in [a,b]$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, M, \tag{1}$$

*for $x \in [a,b]$.*

In [7], Özdemir *et al.* proved inequalities as above for $(\alpha, m)$ −convex functions. In [1], Cheng proved the following inequality;

**Theorem 2.** *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma, x \in [a,b]$. Then we have*

$$\begin{aligned} & \left| \frac{1}{2}f(x) - \frac{(x-b)\,f(b) - (x-a)\,f(a)}{2\,(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq \quad & \frac{(x-a)^2 + (b-x)^2}{8\,(b-a)}(\Gamma - \gamma) \end{aligned}$$

*for all $x \in [a,b]$.*

Similarly, in [3], Ujevic established some double integral inequalities and in [2], Liu *et al.* proved two sharp inequalities of perturbed Ostrowski-type. Recently, in [4], Sarıkaya established following integral inequality of Ostrowski-type involving functions of two independent variables;

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**Theorem 3.** *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and supposes that there exist constants $\gamma,\Gamma\in\mathbb{R}$ with $\gamma\leq\frac{\partial^2 f(t,s)}{\partial t\partial s}\leq\Gamma$ for all $(t,s)\in[a,b]\times[c,d]$. Then, we have*

$$\begin{aligned}
&\left|\frac{1}{4}f(x,y)+\frac{1}{4}H(x,y)-\frac{1}{2(b-a)}\int_a^b f(t,y)dt-\frac{1}{2(d-c)}\int_c^d f(x,s)ds\right.\\
&-\frac{1}{2(b-a)(d-c)}\int_a^b\left[(y-c)f(t,c)+(d-y)f(t,d)\right]dt\\
&\left.-\frac{1}{2(b-a)(d-c)}\int_c^d\left[(x-a)f(a,s)+(b-x)f(b,s)\right]ds+\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\;&\frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{32(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}$$

*for all $(x,y)\in[a,b]\times[c,d]$ where*

$$\begin{aligned}
&H(x,y)\\
=\;&\frac{(x-a)\left[(y-c)f(a,c)+(d-y)f(a,d)\right]+(b-x)\left[(y-c)f(b,c)+(d-y)f(b,d)\right]}{(b-a)(d-c)}\\
&+\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}
\end{aligned}$$

In [5], Qiaoling *et al.* derived a new inequality of Ostrowski-type as following;

**Theorem 4.** *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma,\Gamma\in\mathbb{R}$ with $\gamma\leq\frac{\partial^2 f(t,s)}{\partial t\partial s}\leq\Gamma$ for all $(t,s)\in[a,b]\times[c,d]$. Then, we have*

$$\begin{aligned}
&\left|(1-\lambda)^2 f(x,y)+\frac{\lambda}{2}(1-\lambda)\left[f(a,y)+f(b,y)+f(x,c)+f(x,d)\right]\right.\\
&+\left(\frac{\lambda}{2}\right)^2\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]\\
&-\frac{1}{b-a}\left\{(1-\lambda)\int_a^b f(t,y)dt+\frac{\lambda}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt\right\}\\
&-\frac{1}{d-c}\left\{(1-\lambda)\int_c^d f(x,s)ds+\frac{\lambda}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds\right\}\\
&\left.-\frac{\Gamma+\gamma}{2}(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\;&\frac{\Gamma-\gamma}{2}\frac{1}{(b-a)(d-c)}\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\\
&\times\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(d-c)^2}{4}+\left(y-\frac{c+d}{2}\right)^2\right]
\end{aligned}$$

*for all* $(x,y) \in \left[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}\right] \times \left[c+\lambda\frac{d-c}{2}, d-\lambda\frac{d-c}{2}\right]$ *and* $\lambda \in [0,1]$.

In this paper, we proved a new Ostrowski-type inequality involving functions of two independent variables as above.

## 2. MAIN RESULTS

**Theorem 5.** *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order 2 exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma$ *for all* $(t,s) \in [a,b] \times [c,d]$. *Then, we have*

$$\begin{aligned}
&\left| \frac{1}{16}K(x,y)f(x,y) + \frac{1}{16}H(x,y) \right. \\
&-\frac{1}{4(b-a)(d-c)}\int_c^d [3(x-a)f(x,s)-(b-x)f(x,s)]\,ds \\
&-\frac{1}{4(b-a)(d-c)}\int_a^b [3(y-c)f(t,y)-(d-y)f(t,y)]\,dt \\
&-\frac{1}{4(b-a)(d-c)}\int_c^d [3(b-x)f(b,s)-(x-a)f(a,s)]\,ds \qquad (2)\\
&-\frac{1}{4(b-a)(d-c)}\int_a^b [3(d-y)f(t,d)-(y-c)f(t,c)]\,dt \\
&-\frac{\left[(y-c)^2-(d-y)^2\right]\left[(x-a)^2-(b-x)^2\right]}{32(b-a)(d-c)}(\Gamma+\gamma) \\
&\left. +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt \right| \\
\leq\ &\frac{25\left[(y-c)^2+(d-y)^2\right]\left[(x-a)^2+(b-x)^2\right]}{512(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}$$

*for all* $(x,y) \in [a,b] \times [c,d]$, *where*

$$\begin{aligned}
H(x,y) = &\frac{[3(b-x)f(b,y)-(x-a)f(a,y)](3(y-c)-(d-y))}{(b-a)(d-c)} \\
&+\frac{[3(d-y)f(x,d)-(y-c)f(x,c)](3(x-a)-(b-x))}{(b-a)(d-c)} \\
&+\frac{[(y-c)f(a,c)-3(d-y)f(a,d)](x-a)}{(b-a)(d-c)} \\
&+\frac{[3(d-y)f(b,d)-(y-c)f(b,c)](b-x)}{(b-a)(d-c)}
\end{aligned}$$

*and*

$$K(x,y) = \frac{[(3(x-a)-(b-x))(3(y-c)-(d-y))]}{(b-a)(d-c)}.$$

*Proof.* We define the functions: $p:[a,b]^2 \to \mathbb{R}$ and $q:[c,d]^2 \to \mathbb{R}$ as following

$$p(x,t)=\begin{cases} t-\frac{3a+x}{4} & ,t\in[a,x] \\ t-\frac{3b+x}{4} & ,t\in(x,b] \end{cases}$$

and

$$q(y,s)=\begin{cases} s-\frac{3c+y}{4} & ,s\in[c,y] \\ s-\frac{3d+y}{4} & ,s\in(y,d] \end{cases}$$

From definitions of $p(x,t)$ and $q(y,s)$, we can write

$$\begin{aligned} &\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \qquad (3)\\ =&\int_a^x\int_c^y\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &+\int_a^x\int_y^d\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &+\int_x^b\int_c^y\left(t-\frac{3b+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\ &+\int_x^b\int_y^d\left(t-\frac{3b+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \end{aligned}$$

Computing each integral of right hand side of (3), we have

$$\begin{aligned} &\int_a^x\int_c^y\left(t-\frac{3a+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \qquad (4)\\ =&\frac{(x-a)(y-c)}{16}\left[9f(x,y)-3f(x,c)-3f(a,y)+f(a,c)\right]\\ &-\frac{(x-a)}{4}\int_c^y\left[3f(x,s)-f(a,s)\right]ds-\frac{(y-c)}{4}\int_a^x\left[3f(t,y)-f(t,c)\right]dt\\ &+\int_a^x\int_c^y f(t,s)dsdt \end{aligned}$$

$$\int_a^x\int_y^d \left(t-\frac{3a+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \tag{5}$$
$$= \frac{(x-a)(d-y)}{16}\left[9f(x,d)-3f(x,y)-3f(a,d)+f(a,y)\right]$$
$$-\frac{(x-a)}{4}\int_y^d \left[3f(x,s)-f(a,s)\right]ds-\frac{(d-y)}{4}\int_a^x \left[3f(t,d)-f(t,y)\right]dt$$
$$+\int_a^x\int_y^d f(t,s)dsdt$$

$$\int_x^b\int_c^y \left(t-\frac{3b+x}{4}\right)\left(s-\frac{3c+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \tag{6}$$
$$= \frac{(b-x)(y-c)}{16}\left[9f(b,y)-3f(b,c)-3f(x,y)+f(x,c)\right]$$
$$-\frac{(b-x)}{4}\int_c^y \left[3f(b,s)-f(x,s)\right]ds-\frac{(y-c)}{4}\int_x^b \left[3f(t,y)-f(t,c)\right]dt$$
$$+\int_x^b\int_c^y f(t,s)dsdt$$

$$\int_x^b\int_y^d \left(t-\frac{3b+x}{4}\right)\left(s-\frac{3d+y}{4}\right)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \tag{7}$$
$$= \frac{(b-x)(d-y)}{16}\left[9f(b,d)-3f(b,y)-3f(x,d)+f(x,y)\right]$$
$$-\frac{(b-x)}{4}\int_y^d \left[3f(b,s)-f(x,s)\right]ds-\frac{(d-y)}{4}\int_x^b \left[3f(t,d)-f(t,y)\right]dt$$
$$+\int_x^b\int_y^d f(t,s)dsdt$$

By using these inequalities in (3), we get

$$\begin{aligned}
&\int_a^b\int_c^d p(x,t)\,q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
=\;& \frac{1}{16}\{[(3(x-a)-(b-x))(3(y-c)-(d-y))]\,f(x,y) \\
&+[3(b-x)f(b,y)-(x-a)f(a,y)]\,(3(y-c)-(d-y)) \\
&+[3(d-y)f(x,d)-(y-c)f(x,c)]\,(3(x-a)-(b-x)) \\
&+[(y-c)f(a,c)-3(d-y)f(a,d)]\,(x-a) \\
&+[3(d-y)f(b,d)-(y-c)f(b,c)]\,(b-x)\} \\
&-\frac{1}{4}\int_c^d[3(x-a)f(x,s)-(b-x)f(x,s)]\,ds \\
&-\frac{1}{4}\int_a^b[3(y-c)f(t,y)-(d-y)f(t,y)]\,dt \\
&-\frac{1}{4}\int_c^d[3(b-x)f(b,s)-(x-a)f(a,s)]\,ds \\
&-\frac{1}{4}\int_a^b[3(d-y)f(t,d)-(y-c)f(t,c)]\,dt \\
&+\int_a^b\int_c^d f(t,s)dsdt
\end{aligned} \tag{8}$$

We also have

$$\int_a^b\int_c^d p(x,t)\,q(y,s)\,dsdt=\frac{\left[(y-c)^2-(d-y)^2\right]\left[(x-a)^2-(b-x)^2\right]}{16} \tag{9}$$

Let $M=\frac{\Gamma+\gamma}{2}$. From (8) and (9), we can write

$$\begin{aligned}
&\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]dsdt \qquad (10)\\
=\;&\int_a^b\int_c^d p(x,t)\,q(y,s)\,\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
&-\frac{\Gamma+\gamma}{2}\,\frac{\left[(y-c)^2-(d-y)^2\right]\left[(x-a)^2-(b-x)^2\right]}{16}
\end{aligned}$$

On the other hand, we have

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]dsdt\right| \tag{11}$$

$$\leq \max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right|\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt$$

Since

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right|\leq\frac{\Gamma-\gamma}{2} \tag{12}$$

and

$$\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt=\frac{25\left[(y-c)^2+(d-y)^2\right]\left[(x-a)^2+(b-x)^2\right]}{256} \tag{13}$$

By using (12) and (13) in (11), we get

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^2 f(t,s)}{\partial t\partial s}-M\right]dsdt\right| \tag{14}$$

$$\leq \frac{25\left[(y-c)^2+(d-y)^2\right]\left[(x-a)^2+(b-x)^2\right]}{512}(\Gamma-\gamma)$$

From (10) and (14), we get the required result. □

**Corollary 6.** *Under the assumptions of Theorem 5, if we choose $x=a$ and $y=c$, we obtain the following inequality;*

$$\left|\frac{3\left[f(b,d)-f(b,c)-f(a,d)\right]+f(a,c)}{16}+\frac{1}{4(d-c)}\int_c^d f(a,s)ds\right.$$

$$+\frac{1}{4(b-a)}\int_a^b f(t,c)dt-\frac{3}{4(d-c)}\int_c^d f(b,s)ds-\frac{3}{4(b-a)}\int_a^b f(t,d)dt$$

$$\left.-\frac{(b-a)(d-c)}{32}(\Gamma+\gamma)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \frac{25(b-a)(d-c)}{512}(\Gamma-\gamma).$$

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M. Emin Özdemir, Ataturk University, K. K. Education Faculty, Department of Mathematics, 25640, Kampus, Erzurum, Turkey

*E-mail address*: emos@atauni.edu.tr

Ahmet Ocak Akdemir, Ağri İbrahim Çeçen University, Faculty of Science and Letters, Department of Mathematics, 04100, Ağri, Turkey

*E-mail address*: ahmetakdemir@agri.edu.tr

Erhan Set, Düzce University, Faculty of Science and Arts, Department of Mathematics, Konuralp Campus, Düzce-Turkey

*E-mail address*: erhanset@yahoo.com