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INEQUALITIES RELATED TO VARIANCE OF COMPLEX NUMBERS

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Abstract. We obtain inequalities for the variance of complex numbers and show their connection with several old and new bounds involving the eigenvalues of complex matrices.

1. Introduction

The arithmetic mean and variance of n real numbers $x_1, x_2, ..., x_n$ are respectively defined as

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.1}$$

and

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{x})^2 . \tag{1.2}$$

The Samuelson inequality says [12]

$$s^2 \geq \frac{(x_j - \overline{x})^2}{n-1}, \quad j = 1, 2, ..., n. \tag{1.3}$$

The Popoviciu [11] inequality asserts that if

$$r = M - m, \tag{1.4}$$

where $m \leq x_i \leq M \ (i = 1, 2, ..., n)$, then

$$s^2 \leq \frac{r^2}{4}. \tag{1.5}$$

A lower bound for s^2 in terms of range r is the Nagy inequality [6]

$$s^2 \geq \frac{r^2}{2n}. \tag{1.6}$$

Such simple inequalities and their applications have been studied extensively in the literature. The bounds for the extreme eigenvalues and spread of a Hermitian matrix are also being studied in accordance with these variance bounds. Likewise,

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we can locate the real roots of polynomials. For details see [1]- [4] and [13]. We discuss these inequalities in complex domain and consequently give the simple proofs of the several inequalities involving eigenvalues of complex matrices.

Let $z_1, z_2, ..., z_n$ denote n complex numbers. The arithmetic mean and variance of these complex numbers are respectively defined as

$$\widetilde{z} = \frac{1}{n}\sum_{i=1}^{n} z_i \tag{1.7}$$

and

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} \left|z_i - \widetilde{z}\right|^2. \tag{1.8}$$

Our main results provide the extensions of the above inequalities for the variance of complex numbers (Theorem 2.1-2.4, below). Refinements of the Nagy and Popoviciu inequalities are given (Theorem 2.5-2.6, below). We get alternative proofs of the corresponding bounds for the variance of real numbers. The bounds for the spread and eigenvalues of a complex matrix have been studied extensively in literature, see [2, 5] and [7]-[10]. We show that some of these bounds are essentially the inequalities involving the variance of complex numbers (Theorem 3.1-3.2, below). A lower bound for the spread of the normal matrices is proved, (Theorem 3.3, below).

2. Main Results

Theorem 2.1. Under the above notations:

$$S^2 \geq \frac{\left|z_j - \widetilde{z}\right|^2}{n-1}, \tag{2.1}$$

where $j = 1, 2, ..., n$.

Proof. We can write (1.8) in the form

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} \left|z_i\right|^2 - \left|\widetilde{z}\right|^2. \tag{2.2}$$

Since $S^2 \geq 0$, (2.2) shows that

$$\frac{1}{n}\sum_{i=1}^{n} \left|z_i\right|^2 \geq \left|\widetilde{z}\right|^2 = \left|\frac{1}{n}\sum_{i=1}^{n} z_i\right|^2. \tag{2.3}$$

We note that

$$S^2 = \frac{1}{n}\left|z_j\right|^2 + \frac{n-1}{n}\left(\frac{1}{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n} \left|z_i\right|^2\right) - \left|\widetilde{z}\right|^2 \tag{2.4}$$

and, on applying (2.3) to $(n-1)$ numbers,

$$\frac{1}{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n} \left|z_i\right|^2 \geq \left|\frac{1}{n-1}\sum_{\substack{i=1\\ i\neq j}}^{n} z_i\right|^2. \tag{2.5}$$

Combining (2.4) with (2.5) and insert

$$\sum_{\substack{i=1\\i\neq j}}^{n} z_i = n\widetilde{z} - z_j, \tag{2.6}$$

we get (2.1) on simplification.□

The inequality (2.1) is a generalization of the Samuelson inequality (1.3) for complex numbers. Geometrically, all the points z_i lie in a disc with centre $\widetilde{z}$ and radius $\sqrt{n-1}S$. We prove one more extension of the inequality (1.3).

Theorem 2.2. Let

$$w = \frac{1}{n}\sum_{i=1}^{n} |z_i| . \tag{2.7}$$

For $j = 1, 2, ..., n$, we have

$$S^2 \geq \frac{(|z_j| - w)^2}{n-1}. \tag{2.8}$$

Proof. The triangular inequality implies that $|\widetilde{z}| \leq w$. Therefore, from (2.2) we get that

$$S^2 \geq \frac{1}{n}\sum_{i=1}^{n} |z_i|^2 - w^2. \tag{2.9}$$

The right hand side expression (2.9) is the variance of the real numbers $|z_i|$. Apply the Samuelson inequality (1.3), we get (2.8) from (2.9).□

Theorem 2.3. Under the conditions of Theorem 2.1,

$$S^2 \geq \max_{l,k} \frac{|z_l - z_k|^2}{2n}, \tag{2.10}$$

where $l, k = 1, 2, ..., n$.

Proof. On applying (2.3) to $(n-2)$ numbers $z_i - \frac{z_l+z_k}{2}$, we get

$$\frac{1}{n-2}\sum_{\substack{i=1\\i\neq l,k}}^{n} \left| z_i - \frac{z_l + z_k}{2} \right|^2 \geq \left| \frac{1}{n-2}\sum_{\substack{i=1\\i\neq l,k}}^{n} \left(z_i - \frac{z_l + z_k}{2} \right) \right|^2 . \tag{2.11}$$

From (1.7) we have

$$\sum_{\substack{i=1\\i\neq l,k}}^{n} z_i = n\widetilde{z} - (z_l + z_k) . \tag{2.12}$$

Therefore,

$$\sum_{\substack{i=1\\i\neq l,k}}^{n} \left(z_i - \frac{z_l + z_k}{2} \right) = n\left(\widetilde{z} - \frac{z_l + z_k}{2} \right) \tag{2.13}$$

and

$$\left|\frac{1}{n-2}\sum_{\substack{i=1\\i\neq l,k}}^{n}\left(z_i-\frac{z_l+z_k}{2}\right)\right|^2=\left|\frac{n}{n-2}\left(\widetilde{z}-\frac{z_l+z_k}{2}\right)\right|^2. \tag{2.14}$$

Combining (2.11) and (2.14), we get

$$\frac{1}{n-2}\sum_{\substack{i=1\\i\neq l,k}}^{n}\left|z_i-\frac{z_l+z_k}{2}\right|^2\geq\left|\frac{n}{n-2}\left(\widetilde{z}-\frac{z_l+z_k}{2}\right)\right|^2. \tag{2.15}$$

From (1.8), for any complex number c

$$S^2=\frac{1}{n}\sum_{i=1}^{n}\left|z_i-c\right|^2-\left|\widetilde{z}-c\right|^2. \tag{2.16}$$

For $c=\frac{z_l+z_k}{2}$, (2.16) gives

$$S^2=\frac{1}{2n}\left|z_l-z_k\right|^2+\frac{1}{n}\sum_{\substack{i=1\\i\neq l,k}}^{n}\left|z_i-\frac{z_l+z_k}{2}\right|^2-\left|\widetilde{z}-\frac{z_l+z_k}{2}\right|^2. \tag{2.17}$$

From (2.15)

$$\frac{1}{n}\sum_{\substack{i=1\\i\neq l,k}}^{n}\left|z_i-\frac{z_l+z_k}{2}\right|^2-\left|\widetilde{z}-\frac{z_l+z_k}{2}\right|^2\geq 0. \tag{2.18}$$

Combining (2.17) and (2.18); we get

$$S^2\geq\frac{1}{2n}\left|z_l-z_k\right|^2. \tag{2.19}$$

Since (2.19) is valid for all $l,k=1,2,...,n$, (2.19) implies (2.10).□

The inequality (2.10) gives a generalization of the Nagy inequality (1.6) for the variance of complex numbers. The inequality says that the spread of n numbers z_i is at most $\sqrt{2n}S$. We now give a generalization of the Popoviciu inequality (1.5) for the variance of complex numbers.

Theorem 2.4. Let r_z be the radius of the smallest disc containing all the points z_i , $i=1,2,...,n$. Then

$$S\leq r_z. \tag{2.20}$$

Proof. Let a be the centre of the smallest disc containing all the points z_i . Since (2.16) is valid for any complex numbers c, therefore

$$S^2=\frac{1}{n}\sum_{i=1}^{n}\left|z_i-a\right|^2-\left|\widetilde{z}-a\right|^2. \tag{2.21}$$

It is then evident that

$$S^2\leq\frac{1}{n}\sum_{i=1}^{n}\left|z_i-a\right|^2. \tag{2.22}$$

Also,

$$|z_i - a| \le r_z, \tag{2.23}$$

$i = 1, 2, ..., n$. Combining (2.22) and (2.23), we immediately get (2.20).□

Theorem 2.5. For $n \ge 3$, we have

$$S^2 \ge \frac{|z_l - z_k|^2}{2n} + \frac{2}{n-2}\left|\widetilde{z} - \frac{z_l + z_k}{2}\right|^2, \tag{2.24}$$

where $l, k = 1, 2, ..., n$.
Proof. From (2.15) we find that

$$\frac{1}{n}\sum_{\substack{i=1\\ i\ne l,k}}^{n}\left|z_i - \frac{z_l + z_k}{2}\right|^2 - \left|\widetilde{z} - \frac{z_l + z_k}{2}\right|^2 \ge \frac{2}{n-2}\left|\widetilde{z} - \frac{z_l + z_k}{2}\right|^2. \tag{2.25}$$

Combining (2.17) and (2.25) we get (2.24) on simplification.□

Theorem 2.6. Under the assumptions of Theorem 2.4, we have

$$S^2 \le \left(r_z - |\widetilde{z} - a|\right)\left(r_z + |\widetilde{z} - a|\right). \tag{2.26}$$

Or equivalently

$$S^2 + |\widetilde{z} - a|^2 \le r_z^2. \tag{2.27}$$

Proof. The inequality (2.27) follows easily on combining (2.21) and (2.23), and simplifying the resulting expressions.□

The inequality (2.27) gives a refinement of the inequality (2.20). It says that if all the points z_i lie in a disc with center a and radius r_z then their arithmetic mean lies in a concentric disc of smaller radius $\sqrt{r_z^2 - S^2}$. We obtain an inequality complementary to the Popoviciu inequality in the following theorem.

Theorem 2.7 For $n \ge 2$, we have

$$S^2 \ge \frac{n-1}{2n}\min_{i\ne j}|z_i - z_j|^2. \tag{2.28}$$

Proof Use Lagrange's identity, we can write

$$S^2 = \frac{1}{n^2}\sum_{i<j}^{n}|z_i - z_j|^2. \tag{2.29}$$

The total number of terms in the right hand side expression of (2.29) is $\frac{n(n-1)}{2}$. So (2.28) follows from (2.29).□

Remark.2.1. Since $2n(n-1) \ge 1$, it follows from (2.28) that

$$S^2 \ge \frac{1}{4}\min_{i\ne j}|z_i - z_j|^2.$$

3. Related Bounds for the Spread and Eigenvalues

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. The spread of A is defined as

$$spd\,(A) = \max_{i,j} |\lambda_i - \lambda_j| \,. \tag{3.1}$$

The arithmetic mean of eigenvalues is

$$\widetilde{z}_\lambda = \frac{1}{n}\sum_{i=1}^{n} \lambda_i = \frac{trA}{n}. \tag{3.2}$$

The variance of the eigenvalues of a normal matrix A can be written as

$$\begin{aligned} S_\lambda^2 &= \frac{1}{n}\sum_{i=1}^{n} |\lambda_i|^2 - \left|\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right|^2 \\ &= \frac{trA^*A}{n} - \left|\frac{trA}{n}\right|^2. \end{aligned} \tag{3.3}$$

If A is arbitrary matrix (not necessarily normal), then

$$\sum_{i=1}^{n} |\lambda_i|^2 \le \|A\|_2^2 \,, \tag{3.4}$$

where $\|A\|_2$ is Frobenius norm,

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} . \tag{3.5}$$

From (3.3) and (3.4), we have

$$S_\lambda^2 \le \frac{\|A\|_2^2}{n} - \left|\frac{trA}{n}\right|^2 . \tag{3.6}$$

The equality holds in (3.6) if and only if A is a normal matrix and in this case (3.3) and (3.6) become identical.

Several authors have given estimates for the spread and extreme eigenvalues of a complex $n \times n$ matrix A in terms of easily calculable quantities like trA and $\|A\|_2$. See [7]-[10] and [14]-[15]. We show that some of these bounds follow easily from the variance bounds of complex numbers. A lower bound for the spread is obtained, see [2].

Theorem 3.1. Let λ_i $(i = 1, 2, ..., n)$ be the eigenvalues of an $n \times n$ matrix A. Then, all the eigenvalues lie in the disc

$$\left|\lambda_i - \frac{trA}{n}\right| \le \left[\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|trA|^2}{n}\right)\right]^{\frac{1}{2}} . \tag{3.7}$$

Proof. It follows from the Samuelson inequality (2.1) for the variance of complex number that

$$|\lambda_i - \widetilde{z}_\lambda| \,| \le \sqrt{(n-1)} S_\lambda \tag{3.8}$$

Combining (3.2), (3.6) and (3.8) we immediately get (3.7).□

The inequality (3.7) in the special case when A is symmetric is obtained in [10]. See also, [7]. We now show that a main result of the Mirsky [4] follows from the Nagy inequality.

Theorem 3.2. For an $n \times n$ matrix A, we have

$$\operatorname{spd}(A) \leq \sqrt{2\left\|A\right\|_2^2 - \frac{2}{n}\left|trA\right|^2}. \tag{3.9}$$

Proof. From the Nagy inequality (2.10) we have

$$S_\lambda^2 \geq \frac{1}{2n}\max_{i,j}\left|\lambda_i - \lambda_j\right|^2. \tag{3.10}$$

From (3.6) and (3.10),

$$\max_{i,j}\left|\lambda_i - \lambda_j\right|^2 \leq 2n\left(\frac{\left\|A\right\|_2^2}{n} - \left|\frac{trA}{n}\right|^2\right). \tag{3.11}$$

Combining (3.1) and (3.11) we get (3.9).□

Theorem 3.3. Let A be an $n \times n$ normal matrix, then

$$\operatorname{spd}(A) \geq \sqrt{3}\sqrt{\frac{trA^*A}{n} - \left|\frac{trA}{n}\right|^2}. \tag{3.12}$$

Proof: Let r_λ be the radius of smallest disc containing the spectrum of A. Then, it follows from (2.20) that

$$r_\lambda \geq S_\lambda. \tag{3.13}$$

We have, see [2]

$$r_\lambda \leq \frac{\operatorname{spd}(A)}{\sqrt{3}}. \tag{3.14}$$

Combine (3.13) and (3.14) and substitute the value of S_λ from (3.3); we immediately get (3.3).□

Theorem 3.4. Let A be an $n \times n$ matrix then at least one eigenvalue lies on or inside the disk

$$\left|\lambda_k - \frac{trA}{n}\right| = \sqrt{\frac{\left\|A\right\|_2^2}{n} - \left|\frac{trA}{n}\right|^2}. \tag{3.15}$$

If A is normal then at least one eigenvalue lies on or outside the disk (3.15).
Proof. It is evident that there is one eigenvalue λ_k such that $S_\lambda \geq |\lambda_k - \widetilde{z}_\lambda|$ and an eigenvalue λ_h such that $S_\lambda \leq |\lambda_h - \widetilde{z}_\lambda|$. The assertions of the theorem are now immediate.□

Remark 3.1. If A is symmetric matrix then it follows from the above theorem that there exist eigenvalues in the interval $\left[\widetilde{z}_\lambda - \sqrt{n-1}S_\lambda, \widetilde{z}_\lambda - S_\lambda\right]$ and $\left[\widetilde{z}_\lambda + S_\lambda, \widetilde{z}_\lambda + \sqrt{n-1}S_\lambda\right]$, see also [10].

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