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SOLUTION OF SPACE-TIME FRACTIONAL TELEGRAPH EQUATION BY ADOMIAN DECOMPOSITION METHOD

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Abstract. In the present paper we obtain closed form solutions of space-time fractional telegraph equations using Adomian decomposition method. The space and time fractional derivatives are considered as Caputo fractional derivative and the solutions are obtained in terms of Mittag-Leffler functions.

1. Introduction

The fractional calculus is an extension of the ordinary calculus and has a history of over 300 years [24]. It represents a generalization of the ordinary differentiation and integration to arbitrary order. During the last three decades the subject has been widely used in various fields of science and engineering [18, 27, 29]. Many physicists have discovered that a number of systems-particularly those exhibiting anomalous behavior are usefully described by fractional calculus. In recent years, fractional differential equations have been investigated by many authors and some promising approximate analytical and numerical methods, such as Exp-function method [7, 12], Adomian decomposition method [1, 2, 6], variational iteration method [6, 32, 36], homotopy perturbation method [4, 22, 37], matrix method [14, 28], and homotopy analysis method [3] are proposed. In all these methods, fractional derivatives are considered in Caputo sense.

The Adomian decomposition method has been introduced and developed by Adomian [1, 2]. It is useful for obtaining closed form or numerical approximation for a wide class of stochastic and deterministic problems in science and engineering. This method has further been modified by Wazwaz [31] and more recently by Luo [17] and Zhang and Luo [38]. A considerable amount of research work has been done recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations, and integro-differential equations. For more details we refer [1, 2, 10, 11, 20, 31, 33, 34] and the references there in.

The telegraph equation is a partial differential equation with constant coefficients given by

$$u_{tt} - c^2 u_{xx} + a u_t + b u = 0, \tag{1.1}$$

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where a, b and c are constants. This equation arises in the study of propagation of electrical signals in a cable of transmission line. Both current I and voltage V satisfy an equation of the form (1.1). This equation also arises in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge [13]. Compared to the heat equation, the telegraph equation is found to be a superior model for describing certain fluid flow problems involving suspensions [14]. The classical telegraph equation and space or time fractional telegraph equations have been solved by a number of researchers namely Biazar et al. [5], Cascaval et al. [9], Kaya [16], Momani [21], Odibat and Momani [23], Orsingher and Zhao [25], Orsingher and Beghin [26], Sevimlican [30], and Yildirim [37] using various techniques such as variational iteration method, transform method, Adomian decomposition method, generalized differential transform method, and homotopy perturbation method.

In the present paper we solve homogeneous and non-homogeneous space-time fractional telegraph equation by means of Adomian decomposition method.

2. Preliminaries

Caputo fractional derivative of order α for a function $f(x)$with $x \in \mathbb{R}^{+}$is defined as [8]:

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha+1-m}} d\xi, (m-1 < \alpha \leq m), m \in \mathbb{N}, \tag{2.1}$$

$$= J_x^{m-\alpha} D^m f(x).$$

Clearly

$$D_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \mu > -1. \tag{2.2}$$

Here $D^m \equiv \frac{d^m}{dx^m}$ and J_x^{α} stands for the **Riemann-Liouville fractional integral operator** of order $\alpha > 0$ [18] defined as

$$J_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, t > 0, (m-1 < \alpha \leq m), m \in \mathbb{N}. \tag{2.3}$$

For Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$J_x^{\alpha} D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}\left(0^{+}\right) \frac{x^k}{k!}. \tag{2.4}$$

The **Mittag-Leffler function** which is a generalization of exponential function is defined as [19]:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \left(\alpha \in \mathbb{C}, Re(\alpha) > 0\right), \tag{2.5}$$

a further generalization of (2.5) is given in the form [35]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}; \left(\alpha, \beta \in C,\ Re(\alpha) > 0, Re(\beta) > 0\right). \tag{2.6}$$

Adomian decomposition method for linear differential/ integro-differential equations [33]:

We consider the linear differential equation written in an operator form as

$$Lu + Ru = g, \tag{2.7}$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is other linear operator, and g is a source term.

We, apply the inverse operator L^{-1} to both sides of equation (2.7) and use the given conditions that are assumed to be prescribed to obtain

$$u = f - L^{-1}(Ru), \tag{2.8}$$

where the function f represents the terms that arise due to application of L^{-1} to the source term g and using the given conditions. Next, we decompose the unknown function u into a sum of an infinite number of components given by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.9}$$

where the components $u_0, u_1, u_2, ...$ are usually recurrently determined. Substituting (2.9) into (2.8) leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{2.10}$$

This can be written as

$$u_0 + u_1 + u_2 + u_3 + ... = f - L^{-1}\left(R\left(u_0 + u_1 + u_2 + ...\right)\right). \tag{2.11}$$

Adomian method uses the formal recursive relations as

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= -L^{-1}\left(R\left(u_k\right)\right), k \geq 0. \end{aligned} \tag{2.12}$$

3. Solution of Space-Time Fractional Telegraph Equations by Adomian decomposition method

In this section we consider the space-time fractional telegraph equations in the following form

$$c^2 D_x^{\alpha} u(x,t) = D_t^{p\beta} u(x,t) + a D_t^{r\beta} u(x,t) + b u(x,t) + f(x,t), 0 < x < 1, t > 0, \tag{3.1}$$

where, $\beta = 1/q$, $p, q, r \in \mathbb{N}$, $1 < \alpha \leq 2$, $1 < p\beta \leq 2$, $0 < r\beta \leq 1$, $D_t^{p\beta} \equiv D_t^{\beta} D_t^{\beta} ... D_t^{\beta}$ (p times), $D_t^{r\beta} \equiv D_t^{\beta} D_t^{\beta} ... D_t^{\beta}$ (r times), D_x^{α}, D_t^{β} are Caputo fractional derivatives defined by equation (2.1), a, b and c are constants and $f(x,t)$ is given function.
The above space-time fractional telegraph equation is obtained by replacing the integer order space and time derivatives in telegraph equation (1.1) by Caputo fractional derivatives (2.1). Thus for $\alpha = 2, q = 1, p = 2, r = 1, f = 0$, space-time fractional telegraph equation (3.1) reduces to classical telegraph equation (1.1). We shall solve two space-time fractional telegraph equations of which first is homogeneous and second is non-homogeneous by means of Adomian decomposition method.

Example 3.1. *Consider the following homogeneous space-time fractional telegraph equation*

$$D_x^{\alpha} u(x,t) = D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t), 0 < x < 1, t > 0, \tag{3.2}$$

Where, $\beta = 1/q,\ p,q,r \in \mathbb{N}, 1 < \alpha \le 2,\ 1 < p\beta \le 2,\ 0 < r\beta \le 1, D_t^{p\beta} \equiv D_t^{\beta} D_t^{\beta} ... D_t^{\beta}$ *(p times),* $D_t^{r\beta} \equiv D_t^{\beta} D_t^{\beta} ... D_t^{\beta}$ *(r times),* $D_x^{\alpha},\ D_t^{\beta}$ *are Caputo fractional derivatives defined by equation* (2.1)*,* $p + r$ *is odd and initial conditions are given by*

$$u(0,t) = E_{\beta}\left(-t^{\beta}\right), u_x(0,t) = E_{\beta}\left(-t^{\beta}\right) \tag{3.3}$$

To apply Adomian decomposition, we take $L \equiv D_x^{\alpha}$ *and* $L^{-1} \equiv J_x^{\alpha}$ *defined by equation* (2.1) *and* (2.3) *respectively.*
Now making use of result (2.4)*, we get from equation* (2.8)

$$u(x,t) = \sum_{k=0}^{2-1} \frac{x^k}{k!} \left[D_x^k u(x,t)\right]_{x=0} + L^{-1}\left[\left(D_t^{p\beta} + D_t^{r\beta} + 1\right) u(x,t)\right]. \tag{3.4}$$

This gives the following recursive relations using equation (2.12)*,*

$$u_0(x,t) = \sum_{k=0}^{1} \frac{x^k}{k!} \left[D_x^k u(x,t)\right]_{x=0}, \tag{3.5}$$

$$u_{k+1}(x,t) = L^{-1}\left[\left(D_t^{p\beta} + D_t^{r\beta} + 1\right) u_k(x,t)\right], k = 0,1,2,... \tag{3.6}$$

Which using results (2.2)*,* (2.3) *and* (3.3) *gives*

$$u_0(x,t) = (1+x) E_{\beta}\left(-t^{\beta}\right), \tag{3.7}$$

$$u_1(x,t) = \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{x^{\alpha+1}}{\Gamma(2+\alpha)}\right) E_{\beta}\left(-t^{\beta}\right), \tag{3.8}$$

$$u_2(x,t) = \left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}\right) E_{\beta}\left(-t^{\beta}\right), \tag{3.9}$$

and so on for other components.
Substituting (3.7)*-*(3.9) *in equation* (2.9) *and making use of definitions* (2.5) *and* (2.6)*, the solution* $u(x,t)$ *of problem* (3.2) *is given by*

$$u(x,t) = \left[E_{\alpha}(x^{\alpha}) + x E_{\alpha,2}(x^{\alpha})\right] E_{\beta}\left(-t^{\beta}\right). \tag{3.10}$$

Remark 1. *Setting* $p = 2, q = r = 1$*, the space-time fractional telegraph equation* (3.2) *reduces to space fractional telegraph equation and the solution is same as obtained by Momani* [21]*, Odibat and Momani* [23] *and Yildirim* [37] *using Adomian decomposition method, generalized differential transform method and homotopy perturbation method respectively.*
2. *Setting* $\alpha = 2$*, equation* (3.2) *reduces to time fractional telegraph equation, with the meaning of various symbols and parameters as given with equation* (3.2)*, as follows*

$$D_x^2 u(x,t) = D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t), 0 < x < 1, t > 0, \tag{3.11}$$

with solution

$$u(x,t) = e^x E_{\beta}\left(-t^{\beta}\right). \tag{3.12}$$

3. *Setting* $\alpha = 2, p = 2, q = r = 1$*, equation* (3.2) *reduces to classical telegraph equation and the solution is same as obtained by Kaya* [16] *using Adomian decomposition method.*

Example 3.2. *Consider the following non-homogeneous space-time fractional telegraph equation*

$$D_x^{\alpha}u(x,t) = D_t^{p\beta}u(x,t)+D_t^{r\beta}u(x,t)+u(x,t)-2E_{\alpha}(x^{\alpha})E_{\beta}\left(-t^{\beta}\right), 0<x<1, t>0, \tag{3.13}$$

where $\beta = 1/q$, $p,q,r \in \mathbb{N}$,$1<\alpha\le 2$, $1<p\beta\le 2$,$0<r\beta\le 1$,$D_t^{p\beta} \equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ *(p times)*, $D_t^{r\beta} \equiv D_t^{\beta}D_t^{\beta}...D_t^{\beta}$ *(r times)*, D_x^{α},D_t^{β} *are Caputo fractional derivatives defined by equation* (2.1), p *and* r *are even and initial conditions are given by*

$$u(0,t) = E_{\beta}\left(-t^{\beta}\right), u_x(0,t) = 0. \tag{3.14}$$

To apply Adomian decomposition, we take $L \equiv D_x^{\alpha}$ *and* $L^{-1} \equiv J_x^{\alpha}$ *defined by equation* (2.1) *and* (2.3) *respectively.*
Now making use of result (2.4), *we get from equation* (2.8)

$$\begin{aligned} u(x,t) = {} & \textstyle\sum_{k=0}^{2-1} \frac{x^k}{k!}\left[D_x^k u(x,t)\right]_{x=0} + L^{-1}\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u(x,t)\right] \\ & -2L^{-1}\left[E_{\alpha}(x^{\alpha})E_{\beta}\left(-t^{\beta}\right)\right], \end{aligned} \tag{3.15}$$

This gives the following recursive relations using equation (2.12),

$$u_0(x,t) = \sum_{k=0}^{1}\frac{x^k}{k!}\left[D_x^k u(x,t)\right]_{x=0} - 2L^{-1}\left[E_{\alpha}(x^{\alpha})E_{\beta}\left(-t^{\beta}\right)\right], \tag{3.16}$$

$$u_{k+1}(x,t) = L^{-1}\left[\left(D_t^{p\beta}+D_t^{r\beta}+1\right)u_k(x,t)\right], k=0,1,2,... \tag{3.17}$$

Which using results (2.2), (2.3) *and* (3.14) *gives*

$$u_0(x,t) = E_{\alpha}(x^{\alpha})E_{\beta}\left(-t^{\beta}\right) - 3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)}, \tag{3.18}$$

$$u_1(x,t) = 3\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)} - 3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+2)}}{\Gamma(\alpha(k+2)+1)}\right], \tag{3.19}$$

$$u_2(x,t) = 3^2\left[E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\left(\frac{x^{\alpha(k+2)}}{\Gamma(\alpha(k+2)+1)}\right) - 3E_{\beta}\left(-t^{\beta}\right)\sum_{k=0}^{\infty}\frac{x^{\alpha(k+3)}}{\Gamma(\alpha(k+3)+1)}\right], \tag{3.20}$$

and so on for other components.
Substituting (3.18)-(3.20) *in equation* (2.9), *the solution* $u(x,t)$ *of problem* (3.13) *is given by*

$$u(x,t) = E_{\alpha}(x^{\alpha})E_{\beta}\left(-t^{\beta}\right). \tag{3.21}$$

Remark 1. *Setting* $q=2, p=4, r=2$, *equation* (3.13) *reduces to non-homogeneous space fractional telegraph equation, with the meaning of various symbols and parameters as given with equation* (3.13), *as follows*

$$D_x^{\alpha}u(x,t) = D_t^2 u(x,t)+D_t u(x,t)+u(x,t)-2E_{\alpha}(x^{\alpha})e^{-t}, 0<x<1, t>0, \tag{3.22}$$

with solution

$$u(x,t) = E_{\alpha}(x^{\alpha})e^{-t}. \tag{3.23}$$

2. *Setting* $\alpha = 2$, *equation* (3.13) *reduces to non-homogeneous time fractional telegraph equation, with the meaning of various symbols and parameters as given with equation* (3.13), *as follows*

$$D_x^2 u(x,t) = D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t) - 2e^x E_\beta\left(-t^\beta\right), 0 < x < 1, t > 0, \quad (3.24)$$

with solution

$$u(x,t) = e^x E_\beta\left(-t^\beta\right). \quad (3.25)$$

3. *Setting* $\alpha = 2, q = 2, p = 4, r = 2$, *equation* (3.13) *reduces to non-homogeneous telegraph equation, with the meaning of various symbols and parameters as given with equation* (3.13), *as follows*

$$D_x^2 u(x,t) = D_t^2 u(x,t) + D_t u(x,t) + u(x,t) - 2e^x E_{1/2}\left(-t^{1/2}\right), 0 < x < 1, t > 0, \quad (3.26)$$

with solution

$$u(x,t) = e^x E_{1/2}\left(-t^{1/2}\right). \quad (3.27)$$

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