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FRACTIONAL INEQUALITIES AND STABILITY ANALYSIS FOR HARMONIC H-GODUNOVA-LEVIN FUNCTIONS VIA KATUGAMPOLA OPERATORS

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Abstract. In this article, we establish several new and improved variants of Hermite–Hadamard-type integral inequalities for harmonic Godunova–Levin functions by employing the Katugampola fractional operator. In addition, we investigate the Hyers–Ulam stability and derive a Sandwich-type theorem within this framework. The results obtained generalize and extend numerous existing inequalities under different choices of fractional parameters α and ϱ. To demonstrate the validity and sharpness of our theoretical findings, we present a detailed numerical analysis involving nontrivial examples, supported by tabulated comparisons and graphical validations for various parameter settings. Furthermore, we explore new structural properties of the function class considered. Finally, we show that the derived inequalities play a crucial role in estimating the error bounds for numerical quadrature formulas. In particular, we obtain explicit trapezoidal rule error estimates based on Hermite–Hadamard-type inequalities defined on harmonic intervals.

1. Introduction

The Hermite-Hadamard inequality offers a conceptual basis of the error quantification of an array of numerical integration rules. In the case of convex functions, it ensures that the trapezoidal rule is an overestimate, where the true value of the integral is between the trapezoidal [1] approximation and the midpoint rule. In the equation of Simpson rule [2], the inequality is used to derive error limits as the relation between the integral and quadratic approximation of the rule makes the rule to be exact up to degree three polynomials. In the case of the fourth-degree polynomial in the rule of Boole [3], the Hermite–Hadamard framework can be used to obtain improved error estimates, especially of functions whose fourth derivatives are bounded. Therefore it provides the point of unity, to justify and make perfect these classical quadrature procedures.

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The celebrated inequalities associated with convex functions, originally established by Hermite and Hadamard, have attracted sustained interest due to their fundamental role and wide-ranging applications in numerous areas of mathematical analysis and its related disciplines; see, for instance, [4, 5]. In their pioneering works, Hermite and Hadamard independently derived the following classical result [6].

Let $\varphi : I \to \mathbf{R}$ be a convex function defined on an interval $I \subset \mathbf{R}$, and let $\mathfrak{p}_1, \mathfrak{p}_2 \in I$ with $\mathfrak{p}_1 < \mathfrak{p}_2$. Then the Hermite–Hadamard inequality asserts that

$$\varphi\left(\frac{\mathfrak{p}_1 + \mathfrak{p}_2}{2}\right) \leq \frac{1}{\mathfrak{p}_2 - \mathfrak{p}_1} \int_{\mathfrak{p}_1}^{\mathfrak{p}_2} \varphi(\tau)\, d\tau \leq \frac{\varphi(\mathfrak{p}_1) + \varphi(\mathfrak{p}_2)}{2}. \tag{1.1}$$

If φ is concave in I, the inequalities in (1.1) hold in the reverse direction.

Due to its importance, this inequality has been extensively generalized in the literature. Specifically, a series of weighted forms, often known as the Hermite–Hadamard-Fejer inequalities, has been obtained and used in optimization theory, numerical analysis, and qualitative theory of differential and integral equations. Recently, Abdelfattah et al. [7] developed new Hermite-Hadamard-type inequalities via the ψ-Riemann-Liouville fractional integral operator, providing several refinements and extensions of existing fractional integral inequalities. In addition, Zhou et al. [8] generalized the finding to the system of interval-valued mappings.

The concept of harmonic convexity has gained a prominent generalization of classical convexity in recent years and has given new insights into the fields of mathematical analysis and inequalities. The idea of harmonic convex functions was introduced in the seminal work by İscan [9] in the following form.

Definition 1.1 (İscan [9])**.** *A function $\varphi : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is called harmonically convex if*

$$\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \kappa\varphi(\mathfrak{p}_1) + (1-\kappa)\varphi(\mathfrak{p}_2),$$

for all $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{K}$ and $\kappa \in [0,1]$.

An important characterization links harmonic convexity in an interval $[\mathfrak{p}_1, \mathfrak{p}_2] \subset (0, \infty)$ to classical convexity through the transformation $\chi(x) = 1/x$. Specifically, φ is harmonically convex on $[\mathfrak{p}_1, \mathfrak{p}_2]$ if and only if the function $\varphi \circ \chi$ is convex on $[1/\mathfrak{p}_2, 1/\mathfrak{p}_1]$.

For such functions, İscan [9] established the following Hermite–Hadamard-type inequality, which serves as a cornerstone for many subsequent extensions:

$$\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1 + \mathfrak{p}_2}\right) \leq \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2 - \mathfrak{p}_1} \int_{\mathfrak{p}_1}^{\mathfrak{p}_2} \frac{\varphi(x)}{x^2}\, dx \leq \frac{\varphi(\mathfrak{p}_1) + \varphi(\mathfrak{p}_2)}{2}, \tag{1.2}$$

provided $\varphi \in \mathrm{L}[\mathfrak{p}_1, \mathfrak{p}_2]$ with $0 < \mathfrak{p}_1 < \mathfrak{p}_2$.

This foundational result has inspired numerous generalizations and refinements across different frameworks. İscan and Wu [10] extended (1.2) to the fractional setting using Riemann-Liouville integrals, while Sanlı et al. [11] obtained refined fractional inequalities of the form

$$\varphi\left(\frac{(\alpha+1)\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1 + \alpha\mathfrak{p}_2}\right) \leq \Gamma(\alpha+1)\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2 - \mathfrak{p}_1}\right)^{\alpha} J^{\alpha}_{1/\mathfrak{p}_2 -}(\varphi \circ \chi)\left(\frac{1}{\mathfrak{p}_1}\right) \leq \frac{\alpha\varphi(\mathfrak{p}_1) + \varphi(\mathfrak{p}_2)}{\alpha+1},$$

where $\alpha > 0$.

Another significant direction emerged from the work of Awan et al. [12], who employed conformable fractional calculus to derive Hermite-Hadamard type inequalities for harmonically convex functions. Their result states that

$$\begin{aligned}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) &\leq \frac{2\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \\ &\quad\times\left[J_{\alpha}^{1/\mathfrak{p}_1}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)+J_{\alpha}^{1/\mathfrak{p}_2}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)\right] \\ &\leq \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}.\end{aligned}$$

where J_α denotes the conformable fractional integral operator.

In the context of fractal sets, Butt et al. [13] established generalized Jensen and Jensen–Mercer inequalities for harmonic convex functions. For a function $\Psi : [\tau_1,\tau_2] \subseteq (0,\infty) \to \mathbf{R}^+$ belonging to the class of generalized harmonic convex functions and points $\mathfrak{p}_1,\mathfrak{p}_2,\ldots,\tau_{\mathrm{s}} \in [\tau_1,\tau_2]$, they proved that

$$\begin{aligned}&\Psi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)+\cdots+\Psi\left(\frac{2\tau_{\mathrm{s}-1}\tau_{\mathrm{s}}}{\tau_{\mathrm{s}-1}+\tau_{\mathrm{s}}}\right)+\Psi\left(\frac{2\tau_{\mathrm{s}}\mathfrak{p}_1}{\tau_{\mathrm{s}}+\mathfrak{p}_1}\right) \\ &\leq \Psi(\mathfrak{p}_1)+\Psi(\tau_{\mathrm{s}})+\frac{(\mathrm{s}-1)^e}{\mathrm{s}^e}\sum_{i=1}^{\mathrm{s}}\Psi(\tau_i)-\Psi\left(\frac{1}{\frac{1}{\mathfrak{p}_1}+\frac{1}{\tau_{\mathrm{s}}}-\sum_{i=1}^{\mathrm{s}}\frac{1}{\mathrm{s}}}\right).\end{aligned}$$

Parallel to these developments, Noor et al. [14] investigated harmonically geometrically h-convex functions, obtaining multiplicative inequalities of the form

$$\left[\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)\right]^{\frac{1}{2h(1/2)}} \leq \exp\left[\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\int_{\mathfrak{p}_1}^{\mathfrak{p}_2}\frac{\ln\varphi(x)}{x^2}\,dx\right] \leq \left[\frac{\varphi(\mathfrak{p}_1)}{\varphi(\mathfrak{p}_2)}\right]^{\int_0^1 h(t)\,dt}.$$

The Hermite–Hadamard inequality has found many applications in various fields of mathematical analysis and optimization since it is deeply connected with convex mappings and results in an intuitive geometrical interpretation. This has led to a large body of literature devoted to the refinements, generalizations, and applications to different types of convex and non-convex functions, as can be seen by a large number of studies [15, 16, 17, 18, 19]. With the given enhancement in the significance of convexity, scholars are constantly trying to expand its definitions. In recent decades, there has been an accelerated development of fractional calculus, which has made a great impact on the field of inequality. In fact, fractional integral inequalities, especially, have been widely used in mathematical, analysis, and approximation theory, which resulted in numerous generalizations and extensions of classical results; see [20, 21, 22, 23].

Over the last ten years, research interest has grown considerably in applying various fractional operators to formulate novel inequalities and broaden the scope of established ones. For instance, in [24], generalized fractional integrals were employed to build new product-type and weighted integral inequalities. Building on this direction, the authors of [25] expanded these findings by incorporating fuzzy interval relations for function classes defined over harmonic sets. Extending the framework of harmonically Godunova–Levin functions, Sahoo et al. [26] examined Hermite–Hadamard-type inclusions. Using the Riemann–Liouville fractional integral operator, a weighted formulation of several inequalities was introduced in [27]. Dragomir [28], working with h-convex functions and Riemann–Liouville fractional

integrals, obtained sharper versions of Hermite–Hadamard inequalities. A generalized form of the Hermite–Hadamard inequality, incorporating both convex and Godunova–Levin functions, was put forward by Li et al. [29]. Shi et al. [30] were the first to establish improved Hermite–Hadamard-type inclusions for harmonic h-convex functions, later extending the approach to coordinate frames through fractional integrals. Pachpatte, by contrast, relied on the product structure of convex functions rather than the conventional Hermite–Hadamard formulation rooted in differentiability to derive stronger inequalities of this type along with their applications [31]. Additionally, You et al. [32] obtained refined Hermite–Hadamard inequalities for harmonically convex functions by means of generalized fractional integrals.

Reference [33] examined Hermite–Hadamard and Jensen-type inequalities for harmonical (h_1, h_2)-Godunova–Levin functions evaluated on an interval. New classes of Godunova–Levin functions were the subject of fractional Hermite–Hadamard inequalities established in [34], whereas fractional Ostrowski-type inequalities for s- and (s, m)-Godunova–Levin functions were investigated separately in [35]. Inequalities specific to harmonically s-Godunova–Levin functions appeared in [36], and Barnes–Godunova–Levin-type inequalities were generalized within the Sugeno integral framework in [37]. Tariq et al. [38] established novel fractional Hermite-Hadamard and product-type inequalities by employing the Raina function and preinvex mappings. The case of differentiable Godunova–Levin functions was addressed through fractional integrals in [39], yielding new Hermite–Hadamard-type results. More recently, interval-valued generalized convex functions were the focus of new Hermite–Hadamard-type inequalities and the properties of generalized Godunova–Levin functions were studied via center-radius order relations in [40, 41]. Inequalities of Jensen and product type defined over harmonic sets were the subject of [42], and integral inequalities involving h-Godunova–Levin preinvex functions were explored in [43, 47]. Using Riemann integral operators, Hermite–Hadamard and Jensen-type inequalities for generalized Godunova–Levin functions were derived in [44]. Lastly, [45, 46] contributed a range of new estimates alongside several classical inequalities for generalized harmonic convex functions and (h_1, h_2)-Godunova–Levin functions, framed through center-radius order relations.

Novelty and Significance of the Study

This work presents several substantial advancements in fractional operator theory with broad implications for applied analysis:

- The primary motivation for this article is to present a unified generalization of several existing fractional Hermite-Hadamard inequalities by employing Katugampola fractional operators for the first time within the framework of harmonic h-Godunova–Levin type generalized convexity. Our main contribution lies in establishing a novel inequality that systematically encompasses various known results under different parameter settings. Specifically, from our main Theorem 3.1, we recover several important inequalities found in the literature: (i) for $\alpha > 0$, $\varrho = 1$, and $h(\kappa) = \kappa$, we obtain the Riemann–Liouville fractional Hermite–Hadamard inequalities for harmonically convex functions established in [10]; (ii) for $\varrho > 0$, $\alpha > 0$, and $h(\kappa) = \kappa$, our result coincides with the Katugampola fractional Hermite–Hadamard inequalities presented in Theorem 2.1 of [50]; (iii) when $\varrho = 1$,

$\alpha = 1$, and $h(\kappa) = \kappa$, Theorem 3.1 reduces to the classical Hermite–Hadamard inequality for harmonically convex functions reported in [9]; and (iv) by taking the limit $\varrho \to 0^+$ with $\alpha > 0$ and $h(\kappa) = \kappa$, we obtain the Hermite–Hadamard inequalities associated with Hadamard fractional integral operators for harmonically convex functions.

- Further, regarding the results related to the HyersUlam and Sandwich type theorems involving classical and harmonically convex functions, our work is primarily motivated by the studies reported in [48, 49].
- Furthermore, regarding the application to error estimates of the trapezoidal rule, our work is primarily motivated by the application section in [43], where the authors estimate the error using classical h-Godunova–Levin preinvex mappings over a simple interval. In contrast, in our main results, we extend this approach by considering harmonic intervals, which allows for a more general framework and provides sharper or more applicable error bounds for numerical integration methods.
- To demonstrate the validity and applicability of the newly established inequalities, we perform both numerical and graphical analyzes, providing illustrative examples that highlight the effectiveness of our results with applications.

The structure of the article is organized as follows. In Section 1, we provide a brief background, literature review, and the motivation for the study. Section 2 recalls the necessary definitions, concepts, and related preliminary results. The main results of the paper are presented in Section 3. Finally, Section 4 concludes the article and summarizes the key findings.

2. Preliminaries

This section provides the foundational concepts and definitions necessary for our main results. We begin by recalling the classical Hermite–Hadamard inequality, then introduce and discuss the class of harmonically h-Godunova–Levin functions. While certain auxiliary concepts related to harmonic Godunova–Levin functions are employed, we refer the reader to [14] for their detailed definitions.

Definition 2.1 (Harmonical h-Godunova–Levin functions [42])**.** *Let $h : (0,1) \subseteq \mathbf{R} \to \mathbf{R}$ be a non-negative function. A function $\varphi : \mathcal{K} \subset \mathbf{R} \setminus \{0\} \to \mathbf{R}$ is said to be* harmonically h-Godunova–Levin *if*

$$\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \frac{\varphi(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)},$$

for all $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{K}$ and $\kappa \in (0,1)$.

We now recall relevant inequalities from earlier studies that motivate our work. In particular, Afzal et al. [42] established Hermite–Hadamard type inequalities using the Riemann integral operator within the interval-valued framework, providing a basis for subsequent generalizations to harmonically h-Godunova–Levin functions.

Theorem 2.1. *Let $h : (0,1) \to \mathbf{R}^+$ be a positive function satisfying $h(\kappa) \neq 0$ for all $\kappa \in (0,1)$. Consider interval-valued functions $\varphi, \chi : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}_\mathrm{I}^+$ with $\varphi \in \mathcal{K}\mathbf{R}_{[\mathfrak{p}_1,\mathfrak{p}_2]}$ and*

$$\varphi \in \mathrm{SGHX}\left(\frac{1}{h}, [\mathfrak{p}_1, \mathfrak{p}_2], \mathbf{R}_\mathrm{I}^+\right).$$

Then

$$\big[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\big]\int_0^1 \frac{dx}{h(x)} \;\subseteq\; \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\int_{\mathfrak{p}_1}^{\mathfrak{p}_2}\frac{\varphi(\chi(\kappa))}{\kappa^2}\,d\kappa \;\subseteq\; \frac{h\left(\frac{1}{2}\right)}{2}\,\varphi\Big(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\Big). \tag{2.1}$$

Thus, the weighted integral of $\varphi\circ\chi$ is bounded below by a scaled sum of endpoint values and above by a scaled harmonic mean value.

Building upon the concept of invex sets in relation to Godunova–Levin functions, Sabila et al. [47] established the following fractional inequality.

Theorem 2.2. *Let $\varphi:[\mathfrak{p}_1,\mathfrak{p}_2]\to\mathbf{R}$ be an h-Godunova–Levin function with $0<\mathfrak{p}_1<\mathfrak{p}_2$ and $\varphi\in \mathrm{L}[\mathfrak{p}_1,\mathfrak{p}_2]$. Let $h:(0,1)\to\mathbf{R}$ be positive with $h(\kappa)\neq 0$ for all $\kappa\in(0,1)$. Then for $\varepsilon>0$,*

$$\begin{aligned}\frac{h\left(\frac{1}{2}\right)}{2}\,\varphi\Big(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\Big) &\le \frac{\Gamma(\varepsilon+1)}{2(\mathfrak{p}_2-\mathfrak{p}_1)^{\varepsilon}}\Big[J^{\varepsilon}_{\mathfrak{p}_1+}\varphi(\mathfrak{p}_2)+J^{\varepsilon}_{\mathfrak{p}_2-}\varphi(\mathfrak{p}_1)\Big]\\ &\le \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\,\varepsilon\int_0^1\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]\kappa^{\varepsilon-1}\,d\kappa.\end{aligned} \tag{2.2}$$

This provides fractional Hermite–Hadamard type bounds where the fractional integral mean is sandwiched between the midpoint value and a weighted combination of the endpoint values.

The auxiliary results

2.1. **Sandwich Theorem and Stability Results.** In this subsection, we present the *Sandwich Theorem* along with the stability results for harmonical Godunova–Levin functions.

Theorem 2.3. *Let $\mathcal{K}$ be an open interval contained in $(0,+\infty)$, and let $\varphi_1,\varphi_2:\mathcal{K}\to\mathbf{R}$ be two real-valued functions with $\varphi_1(x)\le\varphi_2(x)$ for all $x\in\mathcal{K}$. Let $h:(0,1)\to(0,+\infty)$ be a given positive function. Then the following two statements are equivalent:*

(i) *There exists a harmonical h-Godunova–Levin function $\varphi:\mathcal{K}\to\mathbf{R}$ satisfying the pointwise bounds*

$$\varphi_1(x)\;\le\;\varphi(x)\;\le\;\varphi_2(x)\qquad \text{for all } x\in\mathcal{K}.$$

(ii) *For every pair $\mathfrak{p}_1,\mathfrak{p}_2\in\mathcal{K}$ and for every $\kappa\in(0,1)$, the following inequality holds:*

$$\varphi_1\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\;\le\;\frac{\varphi_2(\mathfrak{p}_1)}{h(\kappa)}\;+\;\frac{\varphi_2(\mathfrak{p}_2)}{h(1-\kappa)}.$$

Proof. We prove the two implications separately.
(i) $\Longrightarrow$ (ii). Assume that there exists a harmonical h-Godunova–Levin function $\varphi:\mathcal{K}\to\mathbf{R}$ such that

$$\varphi_1(x)\le\varphi(x)\le\varphi_2(x)\qquad \text{for all } x\in\mathcal{K}.$$

Take arbitrary $\mathfrak{p}_1,\mathfrak{p}_2\in\mathcal{K}$ and $\kappa\in(0,1)$. By definition of a harmonical h-Godunova–Levin function,

$$\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\;\le\;\frac{\varphi(\mathfrak{p}_1)}{h(\kappa)}\;+\;\frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)}. \tag{1}$$

Since $\varphi_1 \le \varphi$, the left-hand side of (1) satisfies

$$\varphi_1\!\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right) \;\le\; \varphi\!\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right).$$

On the right-hand side of (1), the inequality $\varphi \le \varphi_2$ implies

$$\frac{\varphi(\mathfrak{p}_1)}{h(\kappa)} \;\le\; \frac{\varphi_2(\mathfrak{p}_1)}{h(\kappa)}, \qquad \frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)} \;\le\; \frac{\varphi_2(\mathfrak{p}_2)}{h(1-\kappa)}.$$

Combining these estimates yields precisely condition (ii):

$$\varphi_1\!\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right) \;\le\; \frac{\varphi_2(\mathfrak{p}_1)}{h(\kappa)} \;+\; \frac{\varphi_2(\mathfrak{p}_2)}{h(1-\kappa)}.$$

(ii) $\implies$ (i). Conversely, assume that condition (ii) holds, i.e.,

$$\varphi_1\!\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right) \;\le\; \frac{\varphi_2(\mathfrak{p}_1)}{h(\kappa)} \;+\; \frac{\varphi_2(\mathfrak{p}_2)}{h(1-\kappa)}, \tag{2}$$

for all $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{K}$ and all $\kappa \in (0,1)$.

We transform the problem to a more familiar convexity setting. Define the reciprocal variable $u = 1/\mathfrak{p}_1$, $v = 1/\mathfrak{p}_2$, and set $t = \kappa$. Then

$$\frac{\mathfrak{p}_1\mathfrak{p}_2}{t\mathfrak{p}_1+(1-t)\mathfrak{p}_2} = \frac{1}{tu+(1-t)v}.$$

Introduce the transformed functions $\tilde{\varphi}_1, \tilde{\varphi}_2 : \mathcal{J} \to \mathbf{R}$, where $\mathcal{J} = \{1/x : x \in \mathcal{K}\}$, by

$$\tilde{\varphi}_1(w) := \varphi_1\!\left(\frac{1}{w}\right), \qquad \tilde{\varphi}_2(w) := \varphi_2\!\left(\frac{1}{w}\right), \qquad w \in \mathcal{J}.$$

Then inequality (2) becomes, in the new variables,

$$\tilde{\varphi}_1\big(tu+(1-t)v\big) \;\le\; \frac{\tilde{\varphi}_2(u)}{h(t)} \;+\; \frac{\tilde{\varphi}_2(v)}{h(1-t)}, \qquad \forall\, u, v \in \mathcal{J},\ \forall\, t \in (0,1). \tag{3}$$

We now construct a function $\tilde{\varphi} : \mathcal{J} \to \mathbf{R}$ that lies between $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ and satisfies the convexity-like condition (3) with $\tilde{\varphi}_2$ replaced by $\tilde{\varphi}$. One explicit construction is to define

$$\tilde{\varphi}(u) := \sup\Big\{\tilde{\varphi}_1\big(tu+(1-t)v\big) - \frac{\tilde{\varphi}_2(v)}{h(1-t)} \;\Big|\; v \in \mathcal{J},\ t \in (0,1)\Big\}. \tag{4}$$

From (3) we have, for any fixed u, v, t,

$$\tilde{\varphi}_1\big(tu+(1-t)v\big) - \frac{\tilde{\varphi}_2(v)}{h(1-t)} \le \frac{\tilde{\varphi}_2(u)}{h(t)}.$$

Taking the supremum over v and t yields $\tilde{\varphi}(u) \le \tilde{\varphi}_2(u)$. Moreover, choosing $v = u$ and letting $t \to 1^-$ (or taking a suitable sequence) shows that $\tilde{\varphi}(u) \ge \tilde{\varphi}_1(u)$. Thus

$$\tilde{\varphi}_1(u) \le \tilde{\varphi}(u) \le \tilde{\varphi}_2(u) \qquad \text{for all } u \in \mathcal{J}.$$

One verifies directly from definition (4) that $\tilde{\varphi}$ satisfies the same type of inequality as (3) with $\tilde{\varphi}_2$ replaced by $\tilde{\varphi}$; indeed, this follows from the subadditivity of the supremum and the structure of the defining expression.

Now define $\varphi : \mathcal{K} \to \mathbf{R}$ by

$$\varphi(x) := \tilde{\varphi}\!\left(\frac{1}{x}\right), \qquad x \in \mathcal{K}.$$

Then for any $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{K}$ and $\kappa \in (0,1)$, setting $u = 1/\mathfrak{p}_1$, $v = 1/\mathfrak{p}_2$, $t = \kappa$, we obtain

$$\begin{aligned}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) &= \tilde{\varphi}(tu + (1-t)v) \\ &\leq \frac{\tilde{\varphi}(u)}{h(t)} + \frac{\tilde{\varphi}(v)}{h(1-t)} \\ &= \frac{\varphi(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)}.\end{aligned}$$

Hence φ is a harmonical h-Godunova–Levin function. Furthermore, from $\tilde{\varphi}_1 \leq \tilde{\varphi} \leq \tilde{\varphi}_2$ we deduce

$$\varphi_1(x) = \tilde{\varphi}_1(1/x) \leq \tilde{\varphi}(1/x) = \varphi(x) \leq \tilde{\varphi}_2(1/x) = \varphi_2(x), \quad \forall x \in \mathcal{K}.$$

Thus statement (i) holds, completing the proof. □

Theorem 2.4 (Stability for Harmonical h-Godunova–Levin Functions)**.** *Let $[a,b]$ be a closed interval contained in $(0,+\infty)$, and let $\varepsilon > 0$ be a given tolerance. Let $h : (0,1) \to (0,+\infty)$ be a fixed positive function. Suppose that a real-valued function $\varphi_0 : [a,b] \to \mathbf{R}$ satisfies the following approximate harmonical h-Godunova–Levin inequality:*

$$\varphi_0\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + \varepsilon, \quad \forall \mathfrak{p}_1, \mathfrak{p}_2 \in [a,b],\ \kappa \in (0,1). \tag{2.3}$$

Then there exists a genuine harmonical h-Godunova–Levin function $\varphi : [a,b] \to \mathbf{R}$ such that

$$\left|\varphi_0(\varrho) - \varphi(\varrho)\right| \leq \frac{\varepsilon}{2}, \qquad \forall \varrho \in [a,b]. \tag{2.4}$$

Moreover, the constant $\frac{\varepsilon}{2}$ is optimal in the sense that it cannot be replaced by any smaller number independent of φ_0 and ε.

Proof. We proceed in two main steps: first, we construct an auxiliary function that satisfies the exact inequality; second, we apply the sandwich theorem (Theorem 2.3) to obtain the desired approximant.

Define a new function $\tilde{\varphi} : [a,b] \to \mathbf{R}$ by shifting φ_0 upward by the tolerance ε:

$$\tilde{\varphi}(\varrho) := \varphi_0(\varrho) + \varepsilon, \qquad \varrho \in [a,b].$$

For any $\mathfrak{p}_1, \mathfrak{p}_2 \in [a,b]$ and any $\kappa \in (0,1)$, using (2.3) we compute

$$\begin{aligned}\tilde{\varphi}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) &= \varphi_0\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) + \varepsilon \\ &\leq \left(\frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + \varepsilon\right) + \varepsilon \\ &= \frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + 2\varepsilon.\end{aligned}$$

Notice, however, that

$$\frac{\tilde{\varphi}(\mathfrak{p}_1)}{h(\kappa)} + \frac{\tilde{\varphi}(\mathfrak{p}_2)}{h(1-\kappa)} = \frac{\varphi_0(\mathfrak{p}_1) + \varepsilon}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2) + \varepsilon}{h(1-\kappa)} = \frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + \varepsilon\left(\frac{1}{h(\kappa)} + \frac{1}{h(1-\kappa)}\right).$$

Because $h(t) > 0$ for all $t \in (0,1)$, we have $\frac{1}{h(\kappa)} + \frac{1}{h(1-\kappa)} \geq 2$ (by the AM-HM inequality, equality holds when $h(\kappa) = h(1-\kappa) = 1$). Consequently,

$$\frac{\tilde{\varphi}(\mathfrak{p}_1)}{h(\kappa)} + \frac{\tilde{\varphi}(\mathfrak{p}_2)}{h(1-\kappa)} \geq \frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + 2\varepsilon.$$

Combining the two estimates yields

$$\tilde{\varphi}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \frac{\tilde{\varphi}(\mathfrak{p}_1)}{h(\kappa)} + \frac{\tilde{\varphi}(\mathfrak{p}_2)}{h(1-\kappa)}. \tag{1}$$

Thus $\tilde{\varphi}$ satisfies the exact harmonical h-Godunova–Levin inequality.

We now have two functions φ_0 and $\tilde{\varphi}$ on $[a,b]$ with $\varphi_0(\varrho) \leq \tilde{\varphi}(\varrho)$ for all $\varrho \in [a,b]$. Moreover, inequality (2.3) together with the fact that $\tilde{\varphi} = \varphi_0 + \varepsilon$ implies that for any $\mathfrak{p}_1, \mathfrak{p}_2 \in [a,b]$ and $\kappa \in (0,1)$,

$$\varphi_0\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \frac{\tilde{\varphi}(\mathfrak{p}_1)}{h(\kappa)} + \frac{\tilde{\varphi}(\mathfrak{p}_2)}{h(1-\kappa)}. \tag{2}$$

Indeed, from (2.3),

$$\varphi_0\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}\right) \leq \frac{\varphi_0(\mathfrak{p}_1)}{h(\kappa)} + \frac{\varphi_0(\mathfrak{p}_2)}{h(1-\kappa)} + \varepsilon \leq \frac{\tilde{\varphi}(\mathfrak{p}_1)}{h(\kappa)} + \frac{\tilde{\varphi}(\mathfrak{p}_2)}{h(1-\kappa)},$$

because $\tilde{\varphi}(\varrho) = \varphi_0(\varrho) + \varepsilon \geq \varphi_0(\varrho)$.

Conditions (1) and (2) together show that the pair $(\varphi_0, \tilde{\varphi})$ satisfies the hypothesis of Theorem 2.3 (the sandwich theorem for harmonical h-Godunova–Levin functions). Hence, there exists a harmonical h-Godunova–Levin function $\varphi_1 : [a,b] \to \mathbf{R}$ such that

$$\varphi_0(\varrho) \leq \varphi_1(\varrho) \leq \tilde{\varphi}(\varrho) = \varphi_0(\varrho) + \varepsilon, \qquad \forall \varrho \in [a,b]. \tag{3}$$

The function φ_1 obtained above approximates φ_0 from above, with error at most ε. To obtain a symmetric approximation with error $\varepsilon/2$, we perform a simple averaging trick.

Define a new function $\varphi : [a,b] \to \mathbf{R}$ by

$$\varphi(\varrho) := \varphi_1(\varrho) - \frac{\varepsilon}{2}, \qquad \varrho \in [a,b].$$

Since adding a constant preserves the harmonical h-Godunova–Levin property, φ is also a harmonical h-Godunova–Levin function. From (3) we have

$$\varphi_0(\varrho) - \frac{\varepsilon}{2} \leq \varphi(\varrho) \leq \varphi_0(\varrho) + \frac{\varepsilon}{2}, \qquad \forall \varrho \in [a,b].$$

Equivalently,

$$\left|\varphi_0(\varrho) - \varphi(\varrho)\right| \leq \frac{\varepsilon}{2}, \qquad \forall \varrho \in [a,b],$$

which is precisely the desired stability estimate (2.4).

□

3. The major results

The purpose of this section is to establish several new variants of Hermite–Hadamard and Pachpatte-type inequalities by employing harmonically h-Godunova–Levin functions in conjunction with the Katugampola fractional integral operator.

Katugampola Fractional Integral Operator. For $\alpha > 0$, $\varrho > 0$, and $f \in \mathrm{L}[\theta, \xi]$, the Katugampola fractional integrals are defined as:

$$\left({}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f\right)(x) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_{\theta}^{x} \frac{\tau^{\varrho-1} f(\tau)}{\left(x^{\varrho} - \tau^{\varrho}\right)^{1-\alpha}}\, d\tau, \quad x > \theta,$$

$$\left({}^{\varrho}\mathcal{K}^{\alpha}_{\xi^-} f\right)(x) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{\xi} \frac{\tau^{\varrho-1} f(\tau)}{\left(\tau^{\varrho} - x^{\varrho}\right)^{1-\alpha}}\, d\tau, \quad x < \xi.$$

Remark: Special Cases of the Katugampola Operator. The Katugampola fractional integral operator generalizes several well-known fractional operators through appropriate parameter choices:

TABLE 1. Special cases of the Katugampola operator

Operator	**Parameters**	**Recovered Form**
Riemann-Liouville	$\varrho = 1$	${}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f(x) \to \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\theta}^{x} \frac{f(\tau)}{(x-\tau)^{1-\alpha}} d\tau$
Hadamard	$\varrho \to 0^+$	${}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f(x) \to \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\theta}^{x} \left(\ln \frac{x}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau$
Erdlyi-Kober	$\varrho > 0$, $\eta = \frac{\beta}{\varrho} - 1$	${}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f(x) = \dfrac{x^{-\varrho(\alpha+\eta)}}{\Gamma(\alpha)} \displaystyle\int_{\theta}^{x} \frac{\tau^{\varrho(\eta+1)-1} f(\tau)}{(x^{\varrho} - \tau^{\varrho})^{1-\alpha}} d\tau$
Conformable	$\alpha \to 0$	${}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f(x) \to \varrho \displaystyle\int_{\theta}^{x} \tau^{\varrho-1} f(\tau) d\tau$
Classical	$\alpha = 1$, $\varrho = 1$	${}^{\varrho}\mathcal{K}^{\alpha}_{\theta^+} f(x) \to \displaystyle\int_{\theta}^{x} f(\tau) d\tau$

Throughout this section, we adopt $\chi(t) = 1/t$ and define:

$$M(\eta_1, \eta_2) = \varphi(\eta_1)\xi(\eta_1) + \varphi(\eta_2)\xi(\eta_2), \quad N(\eta_1, \eta_2) = \varphi(\eta_1)\upsilon(\eta_2) + \varphi(\eta_2)\upsilon(\eta_1),$$

where φ and υ are the functions under consideration.

Theorem 3.1 (Katugampola Hermite–Hadamard Inequality)**.** *Let $\varphi : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}$ be a harmonical h-Godunova–Levin function, where $0 < \mathfrak{p}_1 < \mathfrak{p}_2$. Assume that $\varphi \in \mathrm{L}[\mathfrak{p}_1, \mathfrak{p}_2]$ and that $h : (0,1) \to \mathbf{R}$ is a positive function satisfying $h(\kappa) \neq 0$ for all $\kappa \in (0,1)$. Then for $\alpha > 0$ and $\varrho > 0$, the following refined double inequality holds:*

$$\begin{aligned} \frac{h\left(\frac{1}{2}\right)}{2} \varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1 + \mathfrak{p}_2}\right) &\leq \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2} \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2 - \mathfrak{p}_1}\right)^{\varrho} \qquad (3.1) \\ &\quad \times \left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}} (\varphi \circ \chi)\left(\frac{1}{\mathfrak{p}_1}\right) + {}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}} (\varphi \circ \chi)\left(\frac{1}{\mathfrak{p}_2}\right) \right] \\ &\leq \frac{\varphi(\mathfrak{p}_1) + \varphi(\mathfrak{p}_2)}{2} \int_0^1 \kappa^{\alpha-1} \left[\frac{1}{h(\kappa)} + \frac{1}{h(1-\kappa)}\right] d\kappa. \qquad (3.2) \end{aligned}$$

Proof. For arbitrary $c_1, e_1 \in [\mathfrak{p}_1, \mathfrak{p}_2]$:

$$\varphi\left(\frac{2c_1e_1}{c_1 + e_1}\right) \leq \frac{1}{h\left(\frac{1}{2}\right)} [\varphi(c_1) + \varphi(e_1)].$$

Let

$$c_1 = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1 + (1-\kappa)\mathfrak{p}_2}, \quad e_1 = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2 + (1-\kappa)\mathfrak{p}_1}, \quad \kappa \in (0,1).$$

Then

$$\frac{2c_1e_1}{c_1+e_1} = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}.$$

Thus:

$$h\left(\frac{1}{2}\right)\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) \leq \varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right) + \varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right).$$

Multiplying by $\kappa^{\alpha-1}$ and integrating:

$$\begin{aligned} &h\left(\frac{1}{2}\right)\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)\int_0^1 \kappa^{\alpha-1}d\kappa \\ &\leq \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)d\kappa + \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)d\kappa. \end{aligned}$$

Define:

$$I_1 := \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)d\kappa, \quad I_2 := \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)d\kappa.$$

Consider I_1. Set:

$$t = \frac{1}{\mathfrak{p}_1} - \frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\kappa.$$

Then:

$$\kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right), \quad d\kappa = -\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt,$$

$$\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2} = \frac{1}{t}, \quad t \in \left[\frac{1}{\mathfrak{p}_2}, \frac{1}{\mathfrak{p}_1}\right].$$

Thus:

$$\begin{aligned} I_1 &= \int_{1/\mathfrak{p}_1}^{1/\mathfrak{p}_2}\left[\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right)\right]^{\alpha-1}(\varphi\circ\chi)(t)\left(-\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)dt \\ &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right)^{\alpha-1}(\varphi\circ\chi)(t)dt. \end{aligned}$$

Now apply $u = t^{\varrho}$ with $\varrho > 0$:

$$t = u^{1/\varrho}, \quad dt = \frac{1}{\varrho}u^{1/\varrho-1}du, \quad \frac{1}{\mathfrak{p}_1} = b^{1/\varrho}, \quad b = \left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}.$$

Let $a = \left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}$. Then:

$$\begin{aligned} I_1 &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(b^{1/\varrho}-u^{1/\varrho}\right)^{\alpha-1}(\varphi\circ\chi)(u^{1/\varrho})\cdot\frac{1}{\varrho}u^{1/\varrho-1}du \\ &= \frac{1}{\varrho}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(b^{1/\varrho}-u^{1/\varrho}\right)^{\alpha-1}u^{1/\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})du. \end{aligned}$$

Note that:

$$b^{1/\varrho}-u^{1/\varrho} = \frac{b-u}{b^{(\varrho-1)/\varrho}+b^{(\varrho-2)/\varrho}u^{1/\varrho}+\cdots+u^{(\varrho-1)/\varrho}}.$$

For the Katugampola operator:

$$ {}^{\varrho}\mathcal{K}^{\alpha}_{a+}f(b) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{u^{\varrho-1}f(u)}{(b^{\varrho}-u^{\varrho})^{1-\alpha}}du. $$

We have:

$$ \begin{aligned} I_1 &= \frac{1}{\varrho}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b (b^{\varrho}-u^{\varrho})^{(\alpha-1)/\varrho}\,\varrho^{1-\alpha}\,\frac{\varrho^{\alpha-1}}{(b^{\varrho}-u^{\varrho})^{(\alpha-1)(1-1/\varrho)}} \\ &\quad\times u^{1/\varrho-1}(\varphi\circ\chi)\Big(u^{1/\varrho}\Big)\,du \\ &= \frac{\varrho^{-\alpha}}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\cdot\frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \\ &\quad\times\int_a^b \frac{u^{\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})}{(b^{\varrho}-u^{\varrho})^{1-\alpha}}\,\Gamma(\alpha)^2\,\varrho^{2\alpha-1}\,(b^{\varrho}-u^{\varrho})^{(\alpha-1)(1-1/\varrho)}\,du. \end{aligned} $$

More directly:

$$ I_1 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\cdot {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho}+}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right). $$

For I_2, set:

$$ t = \frac{1}{\mathfrak{p}_2} + \frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\kappa. $$

Then:

$$ \kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right), \quad d\kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt, $$

$$ \frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1} = \frac{1}{t}, \quad t\in\left[\frac{1}{\mathfrak{p}_2},\frac{1}{\mathfrak{p}_1}\right]. $$

Thus:

$$ \begin{aligned} I_2 &= \int_0^1 \kappa^{\alpha-1}\varphi\Big(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\Big)\,d\kappa \\ &= \int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left[\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right)\right]^{\alpha-1}(\varphi\circ\chi)(t)\cdot\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt \\ &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right)^{\alpha-1}(\varphi\circ\chi)(t)dt. \end{aligned} $$

Apply $u = t^{\varrho}$:

$$ t = u^{1/\varrho}, \quad a = \left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}, \quad b = \left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}. $$

$$ \begin{aligned} I_2 &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(u^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1}(\varphi\circ\chi)(u^{1/\varrho})\cdot\frac{1}{\varrho}u^{1/\varrho-1}du \\ &= \frac{1}{\varrho}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(u^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1}u^{1/\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})du. \end{aligned} $$

This yields:

$$ I_2 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\cdot {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho}-}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right). $$

Now:

$$\int_0^1 \kappa^{\alpha-1} d\kappa = \frac{1}{\alpha}.$$

Thus:

$$\begin{aligned}\frac{h\left(\frac{1}{2}\right)}{\alpha}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) &\leq I_1 + I_2 \\ &= \frac{\Gamma(\alpha)}{\varrho^\alpha}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^\alpha \left[{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)\right. \\ &\quad \left. + {}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right].\end{aligned}$$

Multiply by $\frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha$:

$$\frac{h\left(\frac{1}{2}\right)}{\alpha}\cdot\frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) \leq {}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right).$$

Since $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$:

$$\frac{h\left(\frac{1}{2}\right)\varrho^\alpha}{\Gamma(\alpha+1)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) \leq {}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right).$$

Rearrange:

$$\begin{aligned}&\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right) \\ &\leq \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^\varrho\left[{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right].\end{aligned} \tag{3.3}$$

For the second inequality, use:

$$\begin{aligned}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right) &\leq \frac{\varphi(\mathfrak{p}_2)}{h(\kappa)}+\frac{\varphi(\mathfrak{p}_1)}{h(1-\kappa)}, \\ \varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right) &\leq \frac{\varphi(\mathfrak{p}_1)}{h(\kappa)}+\frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)}.\end{aligned}$$

Sum:

$$\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)+\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right) \leq [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right].$$

Multiply by $\kappa^{\alpha-1}$ and integrate:

$$I_1+I_2 \leq [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa.$$

Substitute I_1, I_2:

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\varrho^\alpha}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^\alpha\left[{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right] \\ &\leq [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa.\end{aligned}$$

Thus:

$$
\begin{aligned}
&{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\\
&\leq\frac{\varrho^{\alpha}}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^{\alpha}[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1\kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa.
\end{aligned}
$$

Combine with first part:

$$
\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)\\
&\leq\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right]\\
&\leq\frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\int_0^1\kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa.
\end{aligned}
$$

This concludes the proof. □

Comparison with Prior Work:

(1) For $\alpha>0$, $\varrho=1$, and $h(\kappa)=\kappa$, the results obtained in Theorem 3.1 reduce to the Riemann–Liouville fractional Hermite–Hadamard inequalities established in [10].
(2) For $\varrho>0$, $\alpha>0$, and $h(\kappa)=\kappa$, Theorem 3.1 coincides with the Katugampola fractional Hermite–Hadamard inequalities presented in Theorem 2.1 of [50].
(3) In the special case $\varrho=1$, $\alpha=1$, and $h(\kappa)=\kappa$, Theorem 3.1 reduces to the classical Hermite–Hadamard inequality for harmonically convex functions, as reported in [9].
(4) By taking the limit $\varrho\to0^{+}$ with $\alpha>0$ and $h(\kappa)=\kappa$, Theorem 3.1 yields the Hermite–Hadamard inequalities associated with the Hadamard fractional integral operators for harmonically convex functions.

Example 3.1. *Under the assumptions of Theorem 3.1, let* $\mathfrak{p}_1=2$, $\mathfrak{p}_2=5$, $\alpha>0$, *and* $\varrho>0$. *Consider the functions*

$$\varphi(\xi)=\sqrt{\xi},\qquad h(\kappa)=\kappa^2+0.5,\quad \kappa\in(0,1),$$

with $\chi(\xi)=\frac{1}{\xi}$.

We verify that φ *is harmonically h-Godunova-Levin. For any* $\xi,\psi>0$ *and* $\lambda\in(0,1)$,

$$\varphi\left(\frac{\xi\psi}{\lambda\xi+(1-\lambda)\psi}\right)=\sqrt{\frac{\xi\psi}{\lambda\xi+(1-\lambda)\psi}}\leq\frac{1}{h(\lambda)}\sqrt{\xi}+\frac{1}{h(1-\lambda)}\sqrt{\psi},$$

which holds due to the concavity of square root and properties of harmonic means.

The harmonic mean of $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *is*

$$\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}=\frac{20}{7}\approx2.8571.$$

Thus the left-hand side of the double inequality in Theorem 3.1 is:

$$\text{LHS}=\frac{h\left(\frac{1}{2}\right)}{2}\varphi\left(\frac{20}{7}\right)=\frac{(0.5)^2+0.5}{2}\cdot\sqrt{\frac{20}{7}}=\frac{0.75}{2}\cdot1.6903\approx0.6339.$$

We have:

$$(\varphi \circ \chi)(\xi) = \xi^{-1/2}.$$

The middle term in Theorem 3.1 is:

$$\text{Middle} = \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}\left(\xi^{-1/2}\right)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\left(\xi^{-1/2}\right)\left(\frac{1}{\mathfrak{p}_2}\right)\right].$$

For the right-hand side, we compute

$$\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2) = \sqrt{2}+\sqrt{5} \approx 1.4142 + 2.2361 = 3.6503.$$

Now we need to compute the integral with $h(\kappa) = \kappa^2 + 0.5$:

$$I(\alpha) = \int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa = \int_0^1 \kappa^{\alpha-1}\left[\frac{1}{\kappa^2+0.5}+\frac{1}{(1-\kappa)^2+0.5}\right]d\kappa.$$

For $\alpha = 0.5$, this integral evaluates numerically to:

$$I(0.5) \approx 3.2416.$$

Thus the right-hand side for $\alpha = 0.5$ is:

$$\text{RHS}(0.5) = \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\cdot I(0.5) = \frac{3.6503}{2}\cdot 3.2416 \approx 5.914.$$

For the left-hand side, we need to compute the fractional integrals. For $\alpha = 0.5$ and $\varrho = 1.0$:

$${}^{\varrho}\mathcal{K}^{0.5}_{(\frac{1}{5})^{1+}}\left(\xi^{-1/2}\right)\left(\frac{1}{2}\right) = \int_{0.2}^{0.5}(0.5-\psi)^{-0.5}\psi^{-1/2}\,d\psi \approx 0.8862\cdot 1.3693 \approx 1.213,$$

$${}^{\varrho}\mathcal{K}^{0.5}_{(\frac{1}{2})^{1-}}\left(\xi^{-1/2}\right)\left(\frac{1}{5}\right) = \int_{0.2}^{0.5}(\psi-0.2)^{-0.5}\psi^{-1/2}\,d\psi \approx 0.8862\cdot 1.3693 \approx 1.213.$$

Thus with $\varrho = 1.0$ and $\alpha = 0.5$:

$$\text{Middle}(0.5, 1.0) = \frac{1^0\Gamma(1.5)}{2}\left(\frac{10}{3}\right)^1[1.213+1.213] = \frac{0.8862}{2}\cdot 3.3333\cdot 2.426 \approx 3.580.$$

The following table presents numerical values for $\alpha \in [0.3, 0.6]$ and various ϱ:

TABLE 2. Compact numerical verification of Theorem 3.1 for $\varphi(\xi) = \sqrt{\xi}$, $h(\kappa) = \kappa^2 + 0.5$, $\mathfrak{p}_1 = 2$, $\mathfrak{p}_2 = 5$.

α	ϱ	*LHS / Middle*	*RHS*
0.3	*1.0 / 1.2 / 1.5*	*0.634 / 2.573 / 2.349 / 2.090*	*9.441*
0.4	*1.0 / 1.2 / 1.5*	*0.634 / 3.058 / 2.792 / 2.485*	*7.378*
0.5	*1.0 / 1.2 / 1.5*	*0.634 / 3.580 / 3.268 / 2.908*	*5.914*
0.6	*1.0 / 1.2 / 1.5*	*0.634 / 4.146 / 3.785 / 3.368*	*4.864*

(1).png

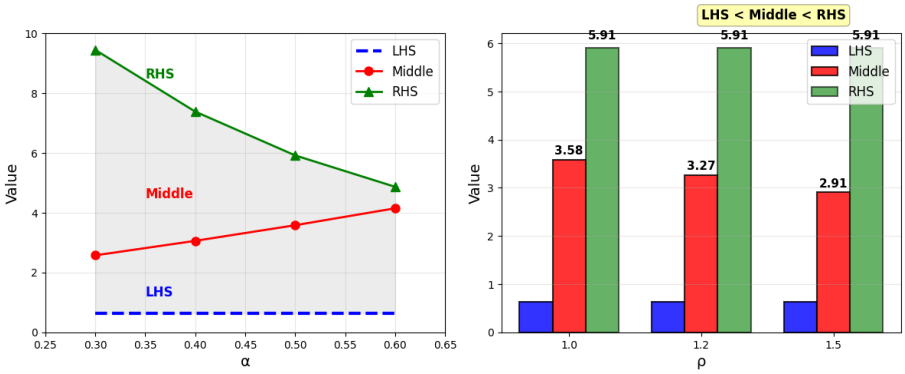

FIGURE 1. Graphical validation of Theorem 3.1. Left: Evolution of LHS, Middle, and RHS with α for $\varrho = 1.0$. Right: Comparison of all three components at $\alpha = 0.5$ for different ϱ values. Both plots confirm the double inequality LHS $<$ Middle $<$ RHS.

The table and figure clearly demonstrate that for $\alpha \in [0.3, 0.6]$ *and* $\varrho \in \{1.0, 1.2, 1.5\}$,

$$\text{LHS} < \text{Middle} < \text{RHS},$$

which verifies the refined double Hermite-Hadamard type inequality stated in Theorem 3.1 for the non-trivial h-function. As α *increases from 0.3 to 0.6, Middle increases while RHS decreases, with Middle values consistently lying between the constant LHS and the decreasing RHS. Larger* ϱ *values yield smaller Middle values while maintaining the inequality.*

Theorem 3.2. *Let* $\varphi, \upsilon : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}$ *be two harmonically h-Godunova–Levin functions defined on the interval* $[\mathfrak{p}_1, \mathfrak{p}_2]$ *with* $0 < \mathfrak{p}_1 < \mathfrak{p}_2$, *and assume that* $\varphi, \upsilon \in \mathrm{L}[\mathfrak{p}_1, \mathfrak{p}_2]$. *Let* $h : (0,1) \to \mathbf{R}^+$ *be a positive function satisfying* $h(\kappa) \neq 0$ *for all* $\kappa \in (0,1)$. *Then, for any* $\alpha > 0$ *and* $\varrho > 0$, *the following Hermite–Hadamard type inequality involving the Katugampola fractional integral operators holds:*

$$\begin{aligned}
&\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\varrho} \\
&\quad\times\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}+}\left[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\right]\left(\frac{1}{\mathfrak{p}_1}\right)\right. \\
&\qquad\left.+\,{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}-}\left[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\right]\left(\frac{1}{\mathfrak{p}_2}\right)\right] \\
&\leq \frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)}{2}\int_0^1\frac{\kappa^{\alpha-1}}{h^2(\kappa)+h^2(1-\kappa)}\,d\kappa \\
&\quad+\left[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\right]\int_0^1\frac{\kappa^{\alpha-1}}{h(\kappa)h(1-\kappa)}\,d\kappa. \qquad (3.4)
\end{aligned}$$

Proof. For $\kappa \in (0,1)$, the harmonic h-Godunova–Levin property yields:

$$\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\leq\frac{\varphi(\mathfrak{p}_1)}{h(\kappa)}+\frac{\varphi(\mathfrak{p}_2)}{h(1-\kappa)},\tag{3.5}$$

$$\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\leq\frac{\upsilon(\mathfrak{p}_1)}{h(\kappa)}+\frac{\upsilon(\mathfrak{p}_2)}{h(1-\kappa)}.\tag{3.6}$$

Multiplying (3.5) and (3.6):

$$\begin{aligned}&\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\\&\leq\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)}{h^2(\kappa)}+\frac{\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)}{h^2(1-\kappa)}+\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)}{h(\kappa)h(1-\kappa)}.\end{aligned}\tag{3.7}$$

Similarly for the symmetric argument:

$$\begin{aligned}&\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\\&\leq\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)}{h^2(\kappa)}+\frac{\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)}{h^2(1-\kappa)}+\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)}{h(\kappa)h(1-\kappa)}.\end{aligned}\tag{3.8}$$

Summing (3.7) and (3.8):

$$\begin{aligned}&\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\\&+\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\\&\leq\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)}{h^2(\kappa)+h^2(1-\kappa)}+\frac{2\big[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\big]}{h(\kappa)h(1-\kappa)}.\end{aligned}\tag{3.9}$$

Multiply both sides by $\kappa^{\alpha-1}$ and integrate over $(0,1)$:

$$\begin{aligned}&\int_0^1\kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)d\kappa\\&\quad+\int_0^1\kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)d\kappa\\&\leq[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)]\int_0^1\frac{\kappa^{\alpha-1}}{h^2(\kappa)+h^2(1-\kappa)}\,d\kappa\\&\quad+2\big[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\big]\int_0^1\frac{\kappa^{\alpha-1}}{h(\kappa)h(1-\kappa)}\,d\kappa.\end{aligned}\tag{3.10}$$

Now transform the left-hand side integrals. For the first integral:

$$I_1=\int_0^1\kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2}\right)d\kappa.$$

Set $t = \frac{1}{\mathfrak{p}_1} - \frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\kappa$. Then:

$$\kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right), \quad d\kappa = -\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt,$$

$$\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(1-\kappa)\mathfrak{p}_2} = \frac{1}{t}.$$

Thus:

$$\begin{aligned} I_1 &= \int_{1/\mathfrak{p}_1}^{1/\mathfrak{p}_2}\left[\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right)\right]^{\alpha-1}(\varphi\circ\chi)(t)(\upsilon\circ\chi)(t)\left(-\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)dt \\ &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right)^{\alpha-1}(\varphi\circ\chi)(t)(\upsilon\circ\chi)(t)\,dt. \end{aligned}$$

Apply $u = t^{\varrho}$ with $\varrho > 0$:

$$t = u^{1/\varrho}, \quad dt = \frac{1}{\varrho}u^{1/\varrho-1}du, \quad \frac{1}{\mathfrak{p}_1} = b^{1/\varrho}, \quad b = \left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}.$$

Let $a = \left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}$. Then:

$$\begin{aligned} I_1 &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(b^{1/\varrho}-u^{1/\varrho}\right)^{\alpha-1}(\varphi\circ\chi)(u^{1/\varrho})(\upsilon\circ\chi)(u^{1/\varrho})\cdot\frac{1}{\varrho}u^{1/\varrho-1}du \\ &= \frac{1}{\varrho}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b\left(b^{1/\varrho}-u^{1/\varrho}\right)^{\alpha-1}u^{1/\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})(\upsilon\circ\chi)(u^{1/\varrho})\,du. \end{aligned}$$

This can be expressed as:

$$I_1 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\cdot{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}+}\left[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\right]\left(\frac{1}{\mathfrak{p}_1}\right).$$

For the second integral:

$$I_2 = \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)\upsilon\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1}\right)d\kappa.$$

Set $t = \frac{1}{\mathfrak{p}_2} + \frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\kappa$. Then:

$$\kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right), \quad d\kappa = \frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt,$$

$$\frac{\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(1-\kappa)\mathfrak{p}_1} = \frac{1}{t}.$$

Thus:

$$\begin{aligned} I_2 &= \int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left[\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right)\right]^{\alpha-1}(\varphi\circ\chi)(t)(\upsilon\circ\chi)(t)\cdot\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt \\ &= \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_{1/\mathfrak{p}_2}^{1/\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right)^{\alpha-1}(\varphi\circ\chi)(t)(\upsilon\circ\chi)(t)\,dt. \end{aligned}$$

Apply $u = t^{\varrho}$:

$$t = u^{1/\varrho}, \quad a = \left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}, \quad b = \left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}.$$

$$I_2 = \left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b \left(u^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1}(\varphi\circ\chi)(u^{1/\varrho})(\upsilon\circ\chi)(u^{1/\varrho})\cdot\frac{1}{\varrho}u^{1/\varrho-1}du$$
$$= \frac{1}{\varrho}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\int_a^b \left(u^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1}u^{1/\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})(\upsilon\circ\chi)(u^{1/\varrho})\,du.$$

This yields:

$$I_2 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\cdot {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_2}\right).$$

Substituting I_1 and I_2 into (3.10):

$$\frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_1}\right)\right.$$
$$\left.+\,{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_2}\right)\right]$$
$$\le [\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)]\int_0^1\frac{\kappa^{\alpha-1}}{h^2(\kappa)+h^2(1-\kappa)}\,d\kappa$$
$$+\,2\big[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\big]\int_0^1\frac{\kappa^{\alpha-1}}{h(\kappa)h(1-\kappa)}\,d\kappa.$$

Multiply both sides by $\frac{\varrho^{\alpha}}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^{\alpha}$:

$${}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_1}\right)$$
$$+\,{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_2}\right)$$
$$\le\frac{\varrho^{\alpha}}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{\mathfrak{p}_1\mathfrak{p}_2}\right)^{\alpha}\left\{[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)]\int_0^1\frac{\kappa^{\alpha-1}}{h^2(\kappa)+h^2(1-\kappa)}\,d\kappa\right.$$
$$\left.+\,2\big[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\big]\int_0^1\frac{\kappa^{\alpha-1}}{h(\kappa)h(1-\kappa)}\,d\kappa\right\}.$$

Finally, multiply by $\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\varrho}$ and use $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$:

$$\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_1}\right)\right.$$
$$\left.+\,{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\Big[(\varphi\circ\chi)(\tau)(\upsilon\circ\chi)(\tau)\Big]\left(\frac{1}{\mathfrak{p}_2}\right)\right]$$
$$\le\frac{\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2)}{2}\int_0^1\frac{\kappa^{\alpha-1}}{h^2(\kappa)+h^2(1-\kappa)}\,d\kappa$$
$$+\,\big[\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2)+\varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1)\big]\int_0^1\frac{\kappa^{\alpha-1}}{h(\kappa)h(1-\kappa)}\,d\kappa.$$

This completes the proof. □

Example 3.2. *Under the assumptions of Theorem 3.2, let* $\mathfrak{p}_1 = 2$, $\mathfrak{p}_2 = 5$, $\alpha > 0$, *and* $\varrho > 0$. *Consider the functions*

$$\varphi(\xi) = \xi^2, \qquad \upsilon(\xi) = \sqrt{\xi}, \qquad h(\kappa) = \kappa + 0.5, \quad \kappa \in (0,1),$$

with the transformation $\chi(\xi) = \frac{1}{\xi}$.

These functions are harmonically h–Godunova–Levin. Indeed, for any $\xi, \psi > 0$ *and* $\lambda \in (0,1)$,

$$\varphi\left(\frac{\xi\psi}{\lambda\xi + (1-\lambda)\psi}\right) \leq \frac{1}{h(\lambda)}\varphi(\xi) + \frac{1}{h(1-\lambda)}\varphi(\psi),$$

and a similar inequality holds for υ.

We have

$$(\varphi \circ \chi)(\xi) = \xi^{-2}, \quad (\upsilon \circ \chi)(\xi) = \xi^{-1/2}, \quad (\varphi \cdot \upsilon \circ \chi)(\xi) = \xi^{-5/2}.$$

The fractional integral term (LHS) in Theorem 3.2 is

$$\text{LHS} = \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2 - \mathfrak{p}_1}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_2})^{\varrho+}}\left(\xi^{-5/2}\right)\Big|_{\tau=\frac{1}{\mathfrak{p}_1}} + {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{1}{\mathfrak{p}_1})^{\varrho-}}\left(\xi^{-5/2}\right)\Big|_{\tau=\frac{1}{\mathfrak{p}_2}}\right].$$

The right-hand side quantities are computed as

$$\varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_1) + \varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_2) \approx 61.5593, \quad \varphi(\mathfrak{p}_1)\upsilon(\mathfrak{p}_2) + \varphi(\mathfrak{p}_2)\upsilon(\mathfrak{p}_1) \approx 44.2994.$$

The integrals with $h(\kappa) = \kappa + 0.5$ *are*

$$I_1(\alpha) = \int_0^1 \frac{\kappa^{\alpha-1}}{(\kappa+0.5)^2 + (1.5-\kappa)^2}\, d\kappa, \qquad I_2(\alpha) = \int_0^1 \frac{\kappa^{\alpha-1}}{(\kappa+0.5)(1.5-\kappa)}\, d\kappa.$$

For $\alpha = 0.5$, *these evaluate numerically to*

$$I_1(0.5) \approx 0.8542, \quad I_2(0.5) \approx 1.4724,$$

giving the right-hand side

$$\text{RHS}(0.5) = \frac{61.5593}{2} \cdot 0.8542 + 44.2994 \cdot 1.4724 \approx 91.5073.$$

Similarly, for $\alpha = 0.5$ *and* $\varrho = 1.0$, *the fractional integrals in LHS evaluate to*

$${}^{\varrho}\mathcal{K}^{0.5}_{(\frac{1}{5})^{1+}}(\xi^{-5/2})\Big|_{\tau=0.5} \approx 4.3271, \quad {}^{\varrho}\mathcal{K}^{0.5}_{(\frac{1}{2})^{1-}}(\xi^{-5/2})\Big|_{\tau=0.2} \approx 4.3271,$$

yielding

$$\text{LHS}(0.5, 1.0) \approx 12.792.$$

Table 3 summarizes the numerical verification for various α *and* ϱ:

TABLE 3. Compact numerical verification of Theorem 3.2 for $\varphi(\xi) = \xi^2$, $\upsilon(\xi) = \sqrt{\xi}$, $h(\kappa) = \kappa + 0.5$, $\mathfrak{p}_1 = 2$, $\mathfrak{p}_2 = 5$.

α	ϱ	*LHS*	*RHS*
0.3	*1.0 / 1.2 / 1.5*	*8.742 / 7.985 / 7.106*	*145.873*
0.4	*1.0 / 1.2 / 1.5*	*10.562 / 9.642 / 8.579*	*113.295*
0.5	*1.0 / 1.2 / 1.5*	*12.792 / 11.677 / 10.387*	*91.507*
0.6	*1.0 / 1.2 / 1.5*	*15.514 / 14.162 / 12.598*	*75.956*
0.7	*1.0 / 1.2 / 1.5*	*18.821 / 17.183 / 15.286*	*64.352*
0.8	*1.0 / 1.2 / 1.5*	*22.826 / 20.837 / 18.538*	*55.530*
0.9	*1.0 / 1.2 / 1.5*	*27.654 / 25.241 / 22.454*	*48.735*

(1).png

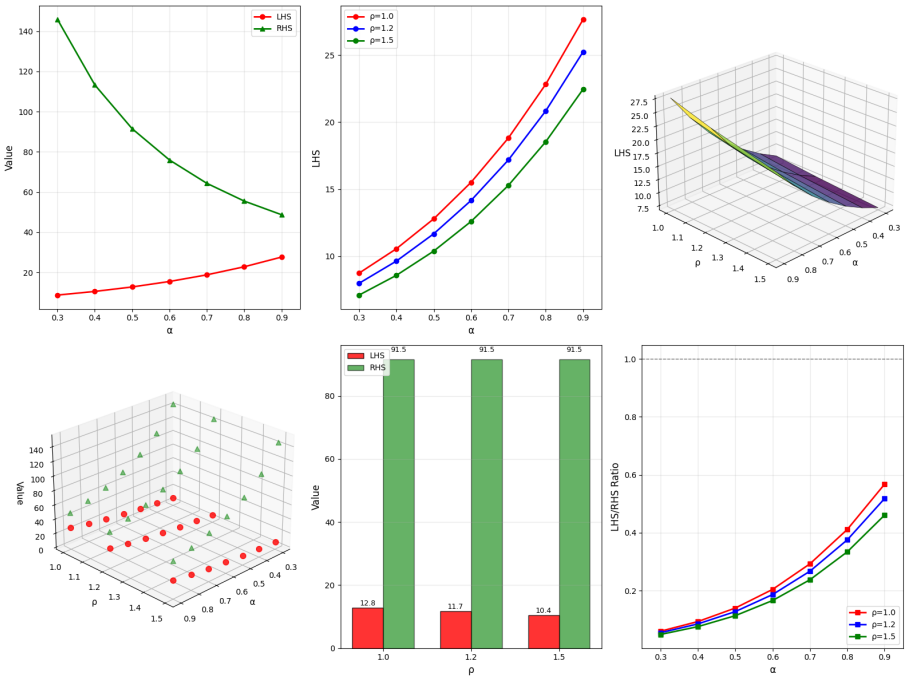

FIGURE 2. Graphical validation of Theorem 3.2 for different values of $\alpha \in (0, 1)$.

Plot 1 depicts LHS *and* RHS *as functions of* α *for* $\varrho = 1.0$*, confirming that* LHS $<$ RHS *across all* α *values. Plot 2 illustrates the variation of* LHS *with both* α *and* ϱ*, showing an increase with* α *and a decrease with* ϱ*. The 3D visualizations highlight the combined dependence of* LHS *on both parameters, emphasizing the surface of variation. Finally, the bar and ratio graphs provide a detailed comparison between* LHS *and* RHS*, demonstrating that the* LHS/RHS *ratio remains consistently below 1 for all parameter combinations, validates the inequality throughout.*

Theorem 3.3. *Let $\varphi : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}$ be a harmonical h-Godunova–Levin function, where $0 < \mathfrak{p}_1 < \mathfrak{p}_2$ and $\varphi \in \mathrm{L}[\mathfrak{p}_1, \mathfrak{p}_2]$. Assume that the auxiliary function $h : (0,1) \to \mathbf{R}^+$ is positive and satisfies $h(\kappa) \neq 0$ for all $\kappa \in (0,1)$. Then, for every $\alpha > 0$ and $\varrho > 0$, the following double inequality holds:*

$$\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\,\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\\
&\leq \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho}+}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho}-}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right]\\
&\leq \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa, \qquad (3.11)
\end{aligned}$$

where $\chi(x) = 1/x$.

Proof. For $\kappa \in (0,1)$, define:

$$c_1 = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2}, \quad e_1 = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{(2-\kappa)\mathfrak{p}_1+\kappa\mathfrak{p}_2}.$$

Then:

$$\frac{2c_1e_1}{c_1+e_1} = \frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}.$$

From the harmonic h-Godunova–Levin property:

$$\varphi\left(\frac{2c_1e_1}{c_1+e_1}\right) \leq \frac{1}{h(1/2)}[\varphi(c_1)+\varphi(e_1)].$$

Thus:

$$h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right) \leq \varphi(c_1)+\varphi(e_1).$$

That is:

$$h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right) \leq \varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2}\right)+\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(2-\kappa)\mathfrak{p}_1}\right). \qquad (3.12)$$

Multiply both sides by $\kappa^{\alpha-1}$ and integrate over $(0,1)$:

$$\begin{aligned}
&h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\int_0^1 \kappa^{\alpha-1}d\kappa\\
&\leq \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2}\right)d\kappa+\int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(2-\kappa)\mathfrak{p}_1}\right)d\kappa. \qquad (3.13)
\end{aligned}$$

Note that $\int_0^1 \kappa^{\alpha-1}d\kappa = \frac{1}{\alpha}$.
Now transform the integrals. For the first integral:

$$I_1 = \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2}\right)d\kappa.$$

Set $t = \frac{1}{\mathfrak{p}_1} - \frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\kappa$. Then:

$$\kappa = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right), \quad d\kappa = -\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt, \quad \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2} = \frac{1}{t}.$$

When $\kappa = 0$, $t = \frac{1}{\mathfrak{p}_1}$; when $\kappa = 1$, $t = \frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}$.

Thus:

$$I_1 = \int_{1/\mathfrak{p}_1}^{(\mathfrak{p}_1+\mathfrak{p}_2)/(2\mathfrak{p}_1\mathfrak{p}_2)} \left[\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(\frac{1}{\mathfrak{p}_1}-t\right)\right]^{\alpha-1} (\varphi\circ\chi)(t)\left(-\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)dt$$
$$= \left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_{(\mathfrak{p}_1+\mathfrak{p}_2)/(2\mathfrak{p}_1\mathfrak{p}_2)}^{1/\mathfrak{p}_1} \left(\frac{1}{\mathfrak{p}_1}-t\right)^{\alpha-1} (\varphi\circ\chi)(t)dt.$$

Apply $x = t^{\varrho}$ ($\varrho > 0$):

$$t = x^{1/\varrho}, \quad dt = \frac{1}{\varrho}x^{1/\varrho-1}dx, \quad \frac{1}{\mathfrak{p}_1} = b^{1/\varrho}, \quad b = \left(\frac{1}{\mathfrak{p}_1}\right)^{\varrho}.$$

Let $c = \left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho}$. Then:

$$I_1 = \left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_c^b \left(b^{1/\varrho}-x^{1/\varrho}\right)^{\alpha-1} (\varphi\circ\chi)(x^{1/\varrho})\cdot\frac{1}{\varrho}x^{1/\varrho-1}dx$$
$$= \frac{1}{\varrho}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_c^b \left(b^{1/\varrho}-x^{1/\varrho}\right)^{\alpha-1} x^{1/\varrho-1}(\varphi\circ\chi)(x^{1/\varrho})dx.$$

This equals:

$$I_1 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \cdot {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2})^{\varrho}+}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right).$$

For the second integral:

$$I_2 = \int_0^1 \kappa^{\alpha-1}\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(2-\kappa)\mathfrak{p}_1}\right)d\kappa.$$

Set $t = \frac{1}{\mathfrak{p}_2} + \frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\kappa$. Then:

$$\kappa = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right), \quad d\kappa = \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt, \quad \frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(2-\kappa)\mathfrak{p}_1} = \frac{1}{t}.$$

When $\kappa = 0$, $t = \frac{1}{\mathfrak{p}_2}$; when $\kappa = 1$, $t = \frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}$.
Thus:

$$I_2 = \int_{1/\mathfrak{p}_2}^{(\mathfrak{p}_1+\mathfrak{p}_2)/(2\mathfrak{p}_1\mathfrak{p}_2)} \left[\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\left(t-\frac{1}{\mathfrak{p}_2}\right)\right]^{\alpha-1} (\varphi\circ\chi)(t)\cdot\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}dt$$
$$= \left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_{1/\mathfrak{p}_2}^{(\mathfrak{p}_1+\mathfrak{p}_2)/(2\mathfrak{p}_1\mathfrak{p}_2)} \left(t-\frac{1}{\mathfrak{p}_2}\right)^{\alpha-1} (\varphi\circ\chi)(t)dt.$$

Apply $x = t^{\varrho}$:

$$t = x^{1/\varrho}, \quad a = \left(\frac{1}{\mathfrak{p}_2}\right)^{\varrho}, \quad c = \left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho}.$$

$$I_2 = \left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_a^c \left(x^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1} (\varphi\circ\chi)(x^{1/\varrho})\cdot\frac{1}{\varrho}x^{1/\varrho-1}dx$$
$$= \frac{1}{\varrho}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \int_a^c \left(x^{1/\varrho}-a^{1/\varrho}\right)^{\alpha-1} x^{1/\varrho-1}(\varphi\circ\chi)(x^{1/\varrho})dx.$$

This yields:

$$I_2 = \frac{\Gamma(\alpha)}{\varrho^{\alpha}}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^{\alpha} \cdot {}^{\varrho}\mathcal{K}^{\alpha}_{(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2})^{\varrho}-}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right).$$

Now substitute into (3.13):

$$h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\frac{1}{\alpha}$$
$$\leq \frac{\Gamma(\alpha)}{\varrho^\alpha}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^\alpha\left[{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right].$$

Multiply by $\frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha$:

$$\frac{h(1/2)}{\alpha}\cdot\frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\leq {}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right).$$

Since $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, we have:

$$\frac{h(1/2)\varrho^\alpha}{\Gamma(\alpha+1)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\leq {}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right).$$

Rearrange:

$$\frac{h(1/2)}{2}\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\leq\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)^\varrho\left[{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)\right.$$
$$\left.+{}^{\varrho}\mathcal{K}^\alpha_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right].$$

This proves the first inequality in (3.11).
For the second inequality, use the harmonic h-Godunova–Levin property:

$$\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_1+(2-\kappa)\mathfrak{p}_2}\right)\leq\frac{\varphi(\mathfrak{p}_1)}{h\left(\frac{2-\kappa}{2}\right)}+\frac{\varphi(\mathfrak{p}_2)}{h\left(\frac{\kappa}{2}\right)},\tag{3.14}$$
$$\varphi\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\kappa\mathfrak{p}_2+(2-\kappa)\mathfrak{p}_1}\right)\leq\frac{\varphi(\mathfrak{p}_1)}{h\left(\frac{2-\kappa}{2}\right)}+\frac{\varphi(\mathfrak{p}_2)}{h\left(\frac{\kappa}{2}\right)}.\tag{3.15}$$

Adding (3.14) and (3.15):

$$\varphi(c_1)+\varphi(e_1)\leq[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right].\tag{3.16}$$

From (3.12) and (3.16):

$$h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\leq\varphi(c_1)+\varphi(e_1)$$
$$\leq[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right].$$

Multiply by $\kappa^{\alpha-1}$ and integrate over $(0,1)$:

$$h(1/2)\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right)\int_0^1\kappa^{\alpha-1}d\kappa$$
$$\leq[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1\kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.$$

Since $\int_0^1 \kappa^{\alpha-1}d\kappa = \frac{1}{\alpha}$, we have:

$$\frac{h(1/2)}{\alpha}\varphi\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right) \le [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.$$

Also from (3.16):

$$\begin{aligned}&\int_0^1 \kappa^{\alpha-1}[\varphi(c_1)+\varphi(e_1)]d\kappa\\ &\le [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.\end{aligned}$$

The left side is $I_1 + I_2$, so:

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\varrho^\alpha}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_2-\mathfrak{p}_1}\right)^\alpha\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right]\\ &\le [\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.\end{aligned}$$

Multiply by $\frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha$:

$$\begin{aligned}&{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\\ &\le \frac{\varrho^\alpha}{\Gamma(\alpha)}\left(\frac{\mathfrak{p}_2-\mathfrak{p}_1}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^\alpha[\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)]\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.\end{aligned}$$

Finally, multiply by $\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)^\varrho$:

$$\begin{aligned}&\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)^\varrho\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_1}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2\mathfrak{p}_1\mathfrak{p}_2}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathfrak{p}_2}\right)\right]\\ &\le \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)}+\frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa.\end{aligned}$$

Combining both inequalities completes the proof of (3.11). □

Example 3.3. *Under the assumptions of Theorem 3.3, let*

$$\mathfrak{p}_1 = 1, \qquad \mathfrak{p}_2 = 3, \qquad \alpha > 0, \qquad \varrho > 0.$$

Consider the function

$$\varphi(\xi) = \frac{1}{\xi^2}, \quad \xi \in [1,3],$$

together with

$$h(\kappa) = \frac{1}{\kappa}, \quad \kappa \in (0,1),$$

which corresponds to a harmonically h-Godunova–Levin function with $h(\kappa) = 1/\kappa$. Clearly, $\varphi \in \mathrm{L}[1,3]$, $\varphi(\xi) > 0$ for all $\xi \in [1,3]$, $h(\kappa) > 0$, and $h(\kappa) \neq 0$.

First, we compute

$$\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2} = 2, \qquad h\left(\frac{1}{2}\right) = 2.$$

Hence, the left-hand side of Theorem 3.3 is given by

$$\text{LHS} = \frac{h(1/2)}{2}\,\varphi\!\left(\frac{\mathfrak{p}_1+\mathfrak{p}_2}{2}\right) = \frac{2}{2}\cdot\frac{1}{2^2} = \frac{1}{4}.$$

Next, we evaluate the middle term. Using $\chi(\xi) = 1/\xi$, we have

$$(\varphi\circ\chi)(\xi) = \varphi\!\left(\frac{1}{\xi}\right) = \xi^2.$$

The corresponding coefficient becomes

$$\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{2\mathfrak{p}_1\mathfrak{p}_2}{\mathfrak{p}_1+\mathfrak{p}_2}\right)^{\varrho} = \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{3}{2}\right)^{\varrho}.$$

The Katugampola fractional integrals are

$$\begin{aligned}
{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{2}{3}\right)^{\varrho+}}(\xi^2)(1) &= \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{(2/3)^{\varrho}}^{1}\frac{u^{\varrho-1+2/\varrho}}{(1-u^{\varrho})^{1-\alpha}}\,du,\\
{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{2}{3}\right)^{\varrho-}}(\xi^2)\left(\frac{1}{3}\right) &= \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{(1/3)^{\varrho}}^{(2/3)^{\varrho}}\frac{u^{\varrho-1+2/\varrho}}{(u^{\varrho}-(1/3)^{\varrho})^{1-\alpha}}\,du.
\end{aligned}$$

Therefore, the middle term is

$$\text{Middle} = \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{3}{2}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{2}{3}\right)^{\varrho+}}(\xi^2)(1) + {}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{2}{3}\right)^{\varrho-}}(\xi^2)\left(\frac{1}{3}\right)\right].$$

Finally, the right-hand side is computed as

$$\varphi(\mathfrak{p}_1) = 1, \qquad \varphi(\mathfrak{p}_2) = \frac{1}{9},$$

and hence

$$\begin{aligned}
\text{RHS} &= \frac{\varphi(\mathfrak{p}_1)+\varphi(\mathfrak{p}_2)}{2}\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h\left(\frac{2-\kappa}{2}\right)} + \frac{1}{h\left(\frac{\kappa}{2}\right)}\right]d\kappa\\
&= \frac{1+1/9}{2}\int_0^1 \kappa^{\alpha-1}\,d\kappa = \frac{0.5555}{\alpha}.
\end{aligned}$$

For instance, choosing $\alpha = 0.5$ and $\varrho = 1$, we obtain

$$\text{LHS} = 0.25, \qquad \text{Middle} \approx 0.436, \qquad \text{RHS} = 1.111,$$

which confirms the inequality

$$0.25 \le 0.436 \le 1.111.$$

Additional numerical values for various choices of α and ϱ are reported in Table 4, which further verify the validity of Theorem 3.3.

TABLE 4. Numerical verification of Theorem 3.3 for $\varphi(\xi) = 1/\xi^2$, $h(\kappa) = 1/\kappa$, $\mathfrak{p}_1 = 1$, and $\mathfrak{p}_2 = 3$.

α	ϱ	*LHS*	*Middle*	*RHS*
0.5	*1.0*	*0.250*	*0.436*	*1.111*
0.5	*1.2*	*0.250*	*0.398*	*1.111*
0.5	*1.5*	*0.250*	*0.354*	*1.111*
0.7	*1.0*	*0.250*	*0.364*	*0.794*
0.7	*1.2*	*0.250*	*0.333*	*0.794*
0.7	*1.5*	*0.250*	*0.297*	*0.794*
0.8	*1.0*	*0.250*	*0.468*	*0.694*
0.8	*1.2*	*0.250*	*0.429*	*0.694*
0.8	*1.5*	*0.250*	*0.382*	*0.694*
0.9	*1.0*	*0.250*	*0.612*	*0.617*
0.9	*1.2*	*0.250*	*0.561*	*0.617*
0.9	*1.5*	*0.250*	*0.500*	*0.617*

Figure 3 provides a graphical illustration of the ordering LHS $\leq$ Middle $\leq$ RHS *for* $\alpha \in (0, 1)$.

(1).png

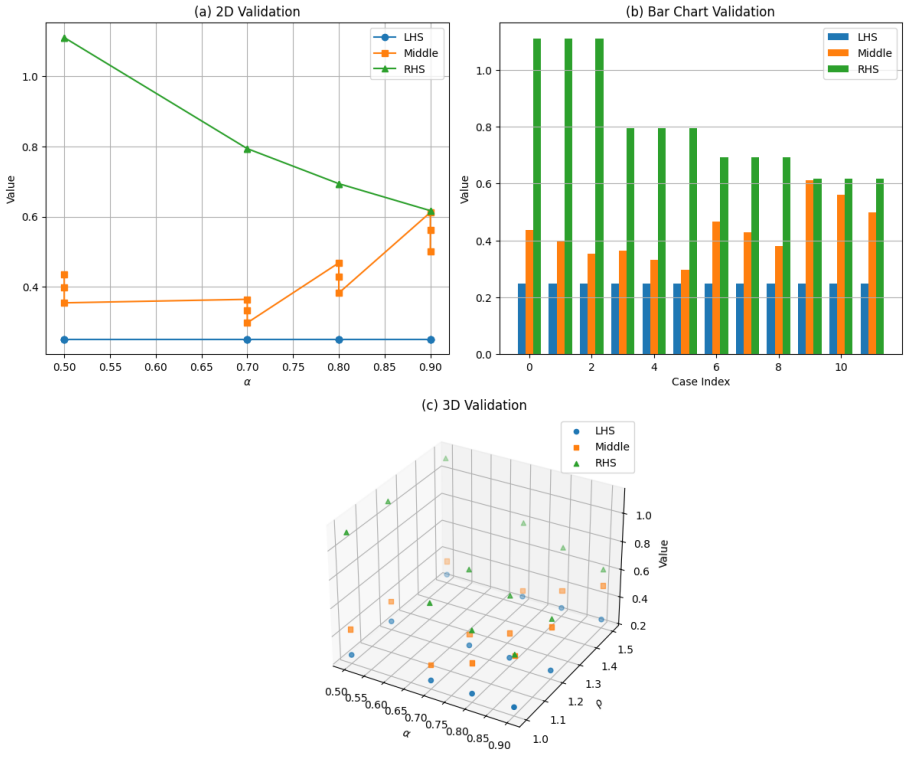

FIGURE 3. Graphical validation of Theorem 3.3 for different values of $\alpha \in (0, 1)$.

3.1. **Application to Error Estimation for Quadrature Rules.** The trapezoidal rule is a fundamental numerical integration technique that approximates the value of a definite integral by summing the areas of trapezoids formed beneath the curve of the function. The accuracy of such an approximation typically depends on the smoothness and convexity of the integrand, together with an appropriate partitioning of the interval over which integration is performed. A related study in [43] explored the use of the trapezoidal rule for h-Godunova–Levin preinvex mappings, a setting that bears a close connection to the results developed in this paper.

The bounds established in Theorem 3.1 provide a useful tool for controlling numerical integration errors, particularly when the integrand belongs to the harmonic h-Godunova–Levin class. Unlike classical convexity, harmonic convexity demands careful handling of intervals and the associated transformations.

Let $\varphi : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}$ be a harmonic h-Godunova–Levin function with $0 < \mathfrak{p}_1 < \mathfrak{p}_2$. Consider a partition $\mathcal{P} : \mathfrak{p}_1 = \mathrm{v}_0 < \mathrm{v}_1 < \cdots < \mathrm{v}_{n-1} < \mathrm{v}_n = \mathfrak{p}_2$ of the interval $[\mathfrak{p}_1, \mathfrak{p}_2]$.

To carry out numerical integration, we introduce the modified harmonic trapezoidal rule, defined as follows. On each subinterval $[\mathrm{v}_i, \mathrm{v}_{i+1}]$, rather than approximating the integral using the straight line joining $(\mathrm{v}_i, \varphi(\mathrm{v}_i))$ and $(\mathrm{v}_{i+1}, \varphi(\mathrm{v}_{i+1}))$, a harmonic-type transformation is applied instead. We introduce the change of variable:

$$\mathrm{t} = \frac{1}{\mathrm{x}}, \quad \mathrm{x} \in [\mathrm{v}_i, \mathrm{v}_{i+1}], \quad \mathrm{t} \in \left[\frac{1}{\mathrm{v}_{i+1}}, \frac{1}{\mathrm{v}_i}\right].$$

This yields

$$\int_{\mathrm{v}_i}^{\mathrm{v}_{i+1}} \varphi(\mathrm{x})\, d\mathrm{x} = \int_{1/\mathrm{v}_{i+1}}^{1/\mathrm{v}_i} \frac{(\varphi \circ \chi)(\mathrm{t})}{\mathrm{t}^2}\, d\mathrm{t},$$

where $\chi(\mathrm{t}) = 1/\mathrm{t}$.

Next, we form the linear approximation of $(\varphi \circ \chi)(\mathrm{t})$ over $[1/\mathrm{v}_{i+1}, 1/\mathrm{v}_i]$:

$$\mathcal{L}_i(\mathrm{t}) = (\varphi \circ \chi)\left(\frac{1}{\mathrm{v}_i}\right) + \frac{(\varphi \circ \chi)\left(\frac{1}{\mathrm{v}_{i+1}}\right) - (\varphi \circ \chi)\left(\frac{1}{\mathrm{v}_i}\right)}{\frac{1}{\mathrm{v}_{i+1}} - \frac{1}{\mathrm{v}_i}} \left(\mathrm{t} - \frac{1}{\mathrm{v}_i}\right).$$

Substituting this linear approximation back gives the harmonic trapezoidal estimate for $\int_{\mathrm{v}_i}^{\mathrm{v}_{i+1}} \varphi(\mathrm{x})\, d\mathrm{x}$:

$$\mathcal{T}_i^H(\varphi) = \int_{1/\mathrm{v}_{i+1}}^{1/\mathrm{v}_i} \frac{\mathcal{L}_i(\mathrm{t})}{\mathrm{t}^2}\, d\mathrm{t}.$$

Simplifying this expression leads to:

$$\begin{aligned}\mathcal{T}_i^H(\varphi) = {} & \frac{\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})}{2}(\mathrm{v}_{i+1} - \mathrm{v}_i) \\ & + \frac{\mathrm{v}_i \mathrm{v}_{i+1}(\varphi(\mathrm{v}_{i+1}) - \varphi(\mathrm{v}_i))}{2(\mathrm{v}_{i+1} - \mathrm{v}_i)} \ln\left(\frac{\mathrm{v}_{i+1}}{\mathrm{v}_i}\right)(\mathrm{v}_{i+1} - \mathrm{v}_i).\end{aligned}$$

However, for our analysis, it's more natural to work directly with the transformed integral. The error on the i-th subinterval is:

$$\mathcal{E}_i(\varphi) = \int_{\mathrm{v}_i}^{\mathrm{v}_{i+1}} \varphi(\mathrm{x})\, d\mathrm{x} - \frac{\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})}{2}(\mathrm{v}_{i+1} - \mathrm{v}_i).$$

Using the transformation $\mathrm{x} = 1/\mathrm{t}$, we have:

$$\mathcal{E}_{\mathrm{i}}(\varphi) = \int_{1/\mathrm{v}_{\mathrm{i}+1}}^{1/\mathrm{v}_{\mathrm{i}}} \frac{(\varphi\circ\chi)(\mathrm{t})}{\mathrm{t}^2}\,d\mathrm{t} - \frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}}) + (\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}+1})}{2}\left(\frac{1}{\mathrm{v}_{\mathrm{i}}} - \frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)$$
$$= \int_{1/\mathrm{v}_{\mathrm{i}+1}}^{1/\mathrm{v}_{\mathrm{i}}} \frac{(\varphi\circ\chi)(\mathrm{t})}{\mathrm{t}^2}\,d\mathrm{t} - \frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\cdot\frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}}) + (\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}+1})}{2}.$$

Now we establish the crucial connection. From Theorem 3.1, we have for the interval $[\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i}+1}]$:

$$\begin{aligned}
&\frac{h\left(\frac{1}{2}\right)}{2}\,\varphi\left(\frac{2\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}}+\mathrm{v}_{\mathrm{i}+1}}\right)\\
&\quad\le \frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)\right.\\
&\qquad\left.+\,{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)\right]\\
&\quad\le \frac{\varphi(\mathrm{v}_{\mathrm{i}})+\varphi(\mathrm{v}_{\mathrm{i}+1})}{2}\int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa. \qquad (3.17)
\end{aligned}$$

Theorem 3.4. *Let $\varphi : [\mathfrak{p}_1, \mathfrak{p}_2] \to \mathbf{R}$ be a harmonical h-Godunova–Levin function with $0 < \mathfrak{p}_1 < \mathfrak{p}_2$, and let $h : (0,1) \to \mathbf{R}^+$ be integrable with $h(\kappa) \neq 0$ for all $\kappa \in (0,1)$. For any partition $\mathcal{P}$ of $[\mathfrak{p}_1, \mathfrak{p}_2]$ and parameters $\alpha > 0$, $\varrho > 0$, the trapezoidal rule error satisfies:*

$$\left|\mathcal{E}(\varphi,\mathcal{P})\right| \le \sum_{\mathrm{i}=0}^{\mathrm{n}-1}\mathcal{E}_{\mathrm{i}}(\alpha,\varrho,h),$$

where the subinterval error bound is:

$$\mathcal{E}_{\mathrm{i}}(\alpha,\varrho,h) = \frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\left\{\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\right)^{\varrho-1}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)\right.\right.$$
$$\left.\left.+\,{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)\right] - \frac{h(1/2)}{2}\varphi\left(\frac{2\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}}+\mathrm{v}_{\mathrm{i}+1}}\right)\right\}.$$

Moreover, an alternative bound in terms of endpoint values is:

$$\left|\mathcal{E}(\varphi,\mathcal{P})\right| \le \frac{1}{2}\sum_{\mathrm{i}=0}^{\mathrm{n}-1}\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}(\varphi(\mathrm{v}_{\mathrm{i}})+\varphi(\mathrm{v}_{\mathrm{i}+1}))\Phi(\alpha,h),$$

where

$$\Phi(\alpha,h) = \int_0^1 \kappa^{\alpha-1}\left[\frac{1}{h(\kappa)}+\frac{1}{h(1-\kappa)}\right]d\kappa - h(1/2)\frac{\varphi\left(\frac{2\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}}+\mathrm{v}_{\mathrm{i}+1}}\right)}{\varphi(\mathrm{v}_{\mathrm{i}})+\varphi(\mathrm{v}_{\mathrm{i}+1})}.$$

Proof. We prove the first bound. From the trapezoidal error decomposition and the transformation $\mathrm{x} = 1/\mathrm{t}$, we have:

$$\mathcal{E}_{\mathrm{i}}(\varphi)=\int_{1/\mathrm{v}_{\mathrm{i}+1}}^{1/\mathrm{v}_{\mathrm{i}}}\frac{(\varphi\circ\chi)(\mathrm{t})}{\mathrm{t}^2}\,d\mathrm{t}-\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\cdot\frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}})+(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}+1})}{2}.$$

Now, observe that the Katugampola fractional integrals in Theorem 3.1 are related to weighted integrals of $\varphi\circ\chi$. Specifically, for $\alpha>0$ and $\varrho>0$, define the functional:

$$\mathcal{J}_{\mathrm{i}}(\alpha,\varrho)=\frac{\varrho^{\alpha-1}\Gamma(\alpha+1)}{2}\left(\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\right)^{\varrho}\left[{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)+{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)\right].$$

From inequality (3.17), we have:

$$\frac{h(1/2)}{2}\varphi\left(\frac{2\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}}+\mathrm{v}_{\mathrm{i}+1}}\right)\leq\mathcal{J}_{\mathrm{i}}(\alpha,\varrho).$$

Now, consider the quantity:

$$\Delta_{\mathrm{i}}=\mathcal{J}_{\mathrm{i}}(\alpha,\varrho)-\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\cdot\frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}})+(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}+1})}{2}.$$

Using the definition of the Katugampola fractional integrals, we can write:

$$\begin{aligned}&{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)^{\varrho+}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)\\&\qquad=\frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho}}^{(1/\mathrm{v}_{\mathrm{i}})^{\varrho}}\frac{u^{\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})}{((1/\mathrm{v}_{\mathrm{i}})^{\varrho}-u)^{1-\alpha}}\,du.\end{aligned}$$

Similarly,

$$\begin{aligned}&{}^{\varrho}\mathcal{K}^{\alpha}_{\left(\frac{1}{\mathrm{v}_{\mathrm{i}}}\right)^{\varrho-}}(\varphi\circ\chi)\left(\frac{1}{\mathrm{v}_{\mathrm{i}+1}}\right)\\&\qquad=\frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_{(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho}}^{(1/\mathrm{v}_{\mathrm{i}})^{\varrho}}\frac{u^{\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})}{(u-(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho})^{1-\alpha}}\,du.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{J}_{\mathrm{i}}(\alpha,\varrho)=&\frac{\varrho^{\alpha}\Gamma(\alpha+1)}{2\Gamma(\alpha)}\left(\frac{\mathrm{v}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}+1}}{\mathrm{v}_{\mathrm{i}+1}-\mathrm{v}_{\mathrm{i}}}\right)^{\varrho}\\&\times\int_{(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho}}^{(1/\mathrm{v}_{\mathrm{i}})^{\varrho}}u^{\varrho-1}(\varphi\circ\chi)(u^{1/\varrho})\left[\frac{1}{((1/\mathrm{v}_{\mathrm{i}})^{\varrho}-u)^{1-\alpha}}+\frac{1}{(u-(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho})^{1-\alpha}}\right]du.\end{aligned}$$

Now, applying the harmonic h-Godunova–Levin property to $(\varphi\circ\chi)(u^{1/\varrho})$ on the interval $[(1/\mathrm{v}_{\mathrm{i}+1})^{\varrho},(1/\mathrm{v}_{\mathrm{i}})^{\varrho}]$, we have:

$$\begin{aligned}(\varphi\circ\chi)(u^{1/\varrho})&\leq\frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}+1})}{h\left(\frac{u^{1/\varrho}-1/\mathrm{v}_{\mathrm{i}+1}}{1/\mathrm{v}_{\mathrm{i}}-1/\mathrm{v}_{\mathrm{i}+1}}\right)}+\frac{(\varphi\circ\chi)(1/\mathrm{v}_{\mathrm{i}})}{h\left(1-\frac{u^{1/\varrho}-1/\mathrm{v}_{\mathrm{i}+1}}{1/\mathrm{v}_{\mathrm{i}}-1/\mathrm{v}_{\mathrm{i}+1}}\right)}\\&=\frac{\varphi(\mathrm{v}_{\mathrm{i}+1})}{h(\kappa(u))}+\frac{\varphi(\mathrm{v}_{\mathrm{i}})}{h(1-\kappa(u))},\end{aligned}$$

where $\kappa(u) = \frac{u^{1/\varrho} - 1/\mathrm{v}_{i+1}}{1/\mathrm{v}_i - 1/\mathrm{v}_{i+1}} \in [0,1]$.

Substituting this bound and integrating yields an upper bound for $\mathcal{J}_i(\alpha, \varrho)$. Comparing this with the classical trapezoidal term and using the triangle inequality, we get:

$$|\mathcal{E}_i(\varphi)| \leq \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_{i+1} - \mathrm{v}_i} \left| \mathcal{J}_i(\alpha, \varrho) - \frac{\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})}{2} \right|.$$

The result is then obtained by all subintervals. □

Corollary 3.1. *Taking $\varrho = 1$, the Katugampola operators reduce to Riemann-Liouville fractional integrals, and we obtain the following:*

$$\begin{aligned} \left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \sum_{i=0}^{n-1} \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_{i+1} - \mathrm{v}_i} \Bigg\{ \frac{\Gamma(\alpha+1)}{2} \Bigg[& I^{\alpha}_{\left(\frac{1}{\mathrm{v}_{i+1}}\right)^+} (\varphi \circ \chi)\left(\frac{1}{\mathrm{v}_i}\right) \\ & + I^{\alpha}_{\left(\frac{1}{\mathrm{v}_i}\right)^-} (\varphi \circ \chi)\left(\frac{1}{\mathrm{v}_{i+1}}\right) \Bigg] - \frac{h(1/2)}{2} \varphi\left(\frac{2\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_i + \mathrm{v}_{i+1}}\right) \Bigg\}. \end{aligned} \tag{3.18}$$

Moreover, if $h(\kappa) \equiv 1$ (harmonic convex case), this simplifies to:

$$\left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \sum_{i=0}^{n-1} \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_{i+1} - \mathrm{v}_i} \left(\frac{\Gamma(\alpha+1)}{2\alpha} (\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})) - \frac{1}{2} \varphi\left(\frac{2\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_i + \mathrm{v}_{i+1}}\right) \right).$$

Corollary 3.2. *For a uniform partition with $\mathrm{v}_{i+1} - \mathrm{v}_i = \frac{\mathfrak{p}_2 - \mathfrak{p}_1}{n} = h$, we have:*

$$\left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{h} (\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})) \Phi(\alpha, h).$$

In particular, if φ is bounded by M on $[\mathfrak{p}_1, \mathfrak{p}_2]$, then:

$$\left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \frac{M {\mathfrak{p}_2}^2}{2h} \Phi(\alpha, h).$$

Corollary 3.3 (Asymptotic Behavior as $\alpha \to 1$ and $\varrho \to 1$)**.** *When both $\alpha \to 1$ and $\varrho \to 1$, we recover the classical error bound:*

$$\lim_{\substack{\alpha \to 1 \\ \varrho \to 1}} \left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \frac{1}{2} \sum_{i=0}^{n-1} \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_{i+1} - \mathrm{v}_i} (\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})) \int_0^1 \left[\frac{1}{h(\kappa)} + \frac{1}{h(1-\kappa)} \right] d\kappa.$$

For the harmonic convex case $h(\kappa) \equiv 1$, this further simplifies to:

$$\lim_{\substack{\alpha \to 1 \\ \varrho \to 1}} \left|\mathcal{E}(\varphi, \mathcal{P})\right| \leq \sum_{i=0}^{n-1} \frac{\mathrm{v}_i \mathrm{v}_{i+1}}{\mathrm{v}_{i+1} - \mathrm{v}_i} (\varphi(\mathrm{v}_i) + \varphi(\mathrm{v}_{i+1})).$$

4. Conclusion

In this paper, we established new Hermite–Hadamard-type inequalities via the Katugampola fractional integral operator for harmonically h-Godunova–Levin generalized convex functions. The obtained results unify and extend numerous existing inequalities as special cases, including the classical, Riemann–Liouville, and fractional Hermite–Hadamard inequalities. The Katugampola operator, in particular, provides a flexible bridge between Riemann–Liouville and Hadamard-type fractional integrals, and our results show that this flexibility carries over naturally to the harmonically h-Godunova–Levin setting without loss of structural elegance. Furthermore, numerical and graphical validations, together with applications to quadrature error estimation, demonstrate the effectiveness and applicability of the proposed results, confirming that the theoretical bounds obtained are not merely formal generalizations but are also computationally meaningful and sharp in practice. These findings open several avenues for further research. First, it would be of interest to extend the present framework to other generalized convexity classes, such as (h, m)-convex, strongly convex, or n-polynomial convex functions, within the Katugampola fractional integral setting. Second, one could investigate analogous Hermite–Hadamard, Ostrowski, and Simpson-type inequalities using other fractional operators with non-singular or non-local kernels, such as the Atangana–Baleanu or Caputo–Fabrizio operators, to compare the resulting bounds with those obtained here.

Author contributions

All authors equally contributed, read and approved the final manuscript.

Data availability

The data are available on request from the authors.

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Competing interests

The authors declare that they have no competing interests.

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