Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 15 Issue 2(2024), Pages 1-6
https://doi.org/10.54379/jiasf-2024-2-1

LOWER AND UPPER BOUNDS FOR BESSEL FUNCTIONS OF THE FIRST KIND

ULRICH REIF

Abstract. The partial sums of the series expansion of the Bessel function J_α of the first kind provide lower and upper bounds on the positive real axis for order $\alpha \geq -1/2$.

1. Introduction

Let $J_\alpha(x)$ denote the Bessel function of the first kind of order α at x. In general, α and x can attain complex values, but here we are only interested in the real case $\alpha \in \mathbb{R}, x \in \mathbb{R}_{>0}$. Apparently, only a few explicit bounds on the absolute value are present in the vast literature on this important special function. In that respect, as of April 2024, the Digital Library of Mathematical Functions [DLMF, §10.14] quotes the following:

$$|J_\alpha(x)| \leq 1, \qquad 0 \leq \alpha,\ 0 \leq x \tag{1.1}$$

$$|J_\alpha(x)| \leq J_\alpha^0(x) := \frac{x^\alpha}{2^\alpha\,\Gamma(\alpha+1)}, \qquad -1/2 \leq \alpha,\ 0 < x \tag{1.2}$$

$$|J_\alpha(x)| \leq \left(\frac{(x/\alpha)e^{\sqrt{1-(x/\alpha)^2}}}{(1+\sqrt{1-(x/\alpha)^2})}\right)^\alpha, \qquad 0 \leq \alpha,\ 0 \leq x \leq \alpha \tag{1.3}$$

The latter formula is sometimes referred to as *Kapteyn's inequality*. The only inclusion of the function values themselves we are aware of is due to Neuman [Neu04]. Assuming $-1/2 \leq \alpha$ and $0 < x \leq \pi/2$, he shows that

$$J_\alpha(x) \geq J_\alpha^0(x)\cos\left(\frac{x}{\sqrt{2\alpha+2}}\right) \tag{1.4}$$

$$J_\alpha(x) \leq \frac{J_\alpha^0(x)}{3\alpha+3}\left(2\alpha+1+(\alpha+2)\cos\left(\sqrt{\frac{3}{2\alpha+4}}\,x\right)\right). \tag{1.5}$$

In this short note, we present a new family of lower and upper bounds for order $\alpha \geq -1/2$, which is based on the partial sums of the well-known series expansion of J_α. A variant thereof is monotonic and thus yields a convergent sequence of nested

1991 *Mathematics Subject Classification.* 33C10, 26D07.
Key words and phrases. Bessel function, lower bound, upper bound.

Submitted April 16, 2024. Accepted May 13, 2024. Published June 5, 2024.
Communicated by N. Braha.

intervals for all $x > 0$. After presenting our main result in the next section, we compare our findings with the bounds of Neuman and Kapteyn in Section 3.

2. Result

The partial sums of the series expansion of the Bessel function J_α of the first kind are given by

$$J_\alpha^n(x) := \sum_{m=0}^{n} (-1)^m \frac{(x/2)^{2m+\alpha}}{m!\,\Gamma(\alpha+m+1)}, \quad n \in \mathbb{N}_0.$$

They converge to $J_\alpha(x)$ as $n \to \infty$ for any $x \neq 0$, but it seems to be unknown that they yield the following bounds:

Theorem 2.1. *Let $\alpha \geq -1/2$ and $x > 0$. Then the inclusions*

$$J_\alpha^{2n-1}(x) \leq J_\alpha(x) \leq J_\alpha^{2n}(x) \tag{2.1}$$

and

$$\max\{J_\alpha^{2n-1}(x), -J_\alpha^0(x)\} \leq J_\alpha(x) \leq \min\{J_\alpha^{2n}(x), J_\alpha^0(x)\} \tag{2.2}$$

hold true for all $n \in \mathbb{N}$, where $J_\alpha^0(x) = (x/2)^\alpha/\Gamma(\alpha+1)$. In the latter inequality, the lower bounds are monotone increasing and the upper bounds are monotone decreasing with n.

For $x < 2\sqrt{\alpha+1}$, the inclusion (2.1) follows immediately from the alternating series estimation theorem and (1.2). The modest contribution of this note concerns extension of the range of validity to all $x > 0$. The proof provided below is completely elementary:

Proof. Define the coefficients

$$a_m := \frac{\Gamma(\alpha+1)}{4^m m!\,\Gamma(\alpha+m+1)}$$

and the associated alternating power series

$$P_\alpha(x) := \sum_{m=0}^{\infty} (-1)^m a_m x^{2m}, \quad P_\alpha^n(x) := \sum_{m=0}^{n} (-1)^m a_m x^{2m}.$$

Dividing (2.1) and (2.2) by $J_\alpha^0(x)$, we obtain the equivalent inequalities

$$P_\alpha^{2n-1}(x) \leq P_\alpha(x) \leq P_\alpha^{2n}(x) \tag{2.3}$$

and

$$\max\bigl\{P_\alpha^{2n-1}(x), -1\bigr\} \leq P_\alpha(x) \leq \min\bigl\{P_\alpha^{2n}(x), 1\bigr\}, \tag{2.4}$$

respectively, which are now verified.

With

$$x_m := \sqrt{\frac{a_{m+1}}{a_{m+2}}} = 2\sqrt{(m+2)(\alpha+m+2)}, \quad m \in \mathbb{N}_0,$$

we obtain

$$P_\alpha^{2n+2}(x) = P_\alpha^{2n}(x) + a_{2n+2}x^{4n+2}(x^2 - x_{2n}^2) \tag{2.5}$$

and

$$P_\alpha^{2n+1}(x) = P_\alpha^{2n-1}(x) - a_{2n+1}x^{4n}(x^2 - x_{2n-1}^2). \tag{2.6}$$

To establish the upper bound in (2.3), we distinguish two cases: First, consider $x < x_{2n}$. We have

$$P_\alpha^{2n}(x) - P_\alpha(x) = \sum_{m=2n}^{\infty} (-1)^m a_{m+1} x^{2m+2}. \tag{2.7}$$

The series on the right hand side is alternating, and the ratios of absolute values of consecutive summands are bounded by

$$\frac{a_{m+2} x^{2m+4}}{a_{m+1} x^{2m+2}} = \frac{x^2}{x_m^2} \leq \frac{x^2}{x_{2n}^2} < 1.$$

Hence, the series converges and its value is positive since so is the first summand. This implies $P_\alpha(x) \leq P_\alpha^{2n}(x)$ for $x < x_{2n}$. Second, consider $x \geq x_{2n}$. We claim that

$$P_\alpha^{2n}(x) \geq 1, \quad x \geq x_{2n}. \tag{2.8}$$

The proof is by induction on n, starting from $P_\alpha^0(x) = 1$. Now, let us assume that (2.8) is valid for some $n \in \mathbb{N}_0$. For $x \geq x_{2n+2}$, (2.5) yields

$$\begin{aligned} P_\alpha^{2n+2}(x) &= P_\alpha^{2n}(x) + a_{2n+2} x^{4n+2}(x^2 - x_{2n}^2) \\ &\geq 1 + a_{2n+2} x^{4n+2}(x_{2n+2}^2 - x_{2n}^2) \geq 1, \end{aligned}$$

where we used that $P_\alpha^{2n}(x) \geq 1$ by the induction hypothesis. Inequality (1.2) implies $P_\alpha(x) \leq 1$ and thus $P_\alpha(x) \leq P_\alpha^{2n}(x)$ for $x \geq x_{2n}$. The proof of the lower bound in (2.3) is completely analogous: Now, (2.7) becomes

$$P_\alpha(x) - P_\alpha^{2n-1}(x) = \sum_{m=2n}^{\infty} (-1)^m a_m x^{2m}.$$

Again, the series on the right hand side is alternating and the ratios of absolute values of summands are less than 1 for $x < x_{2n-1}$. This implies $P_\alpha^{2n-1}(x) \leq P_\alpha(x)$. Further, (2.8) becomes

$$P_\alpha^{2n-1}(x) \leq -1, \quad x \geq x_{2n-1}.$$

To start the induction, we note that the function $P_\alpha^1(x) = 1 - x^2/(4\alpha + 4)$ falls below -1 for $x \geq x_1 = 2\sqrt{3(\alpha+1)}$. The induction step uses now (2.6) instead of (2.5). Since $P_\alpha(x) \geq -1$ by (1.2), it follows $P_\alpha^{2n-1}(x) \leq P_\alpha(x)$ also for $x \geq x_{2n-1}$.

Concerning (2.4), let

$$\bar{P}_\alpha^{2n-1}(x) := \max\bigl\{P_\alpha^{2n-1}(x), -1\bigr\}, \quad \bar{P}_\alpha^{2n}(x) := \min\bigl\{P_\alpha^{2n}(x), 1\bigr\}.$$

The upper bound $P(x) \leq \bar{P}_\alpha^{2n}(x)$ is just a combination of (2.3) and (1.2). So it remains to show that the sequence $\bar{P}_\alpha^{2n}(x)$ is monotone decreasing. We distinguish cases as before: For $x < x_{2n}$, (2.5) yields $P_\alpha^{2n+2}(x) \leq P_\alpha^{2n}(x)$, implying that $\bar{P}_\alpha^{2n+2}(x) \leq \bar{P}_\alpha^{2n}(x)$. For $x \geq x_{2n}$, (2.8) yields $P_\alpha^{2n+2}(x) \geq P_\alpha^{2n}(x) \geq 1$ and hence $\bar{P}_\alpha^{2n+2}(x) = P_\alpha^{2n}(x) = 1$. Again, the arguments for the lower bound are completely analogous. □

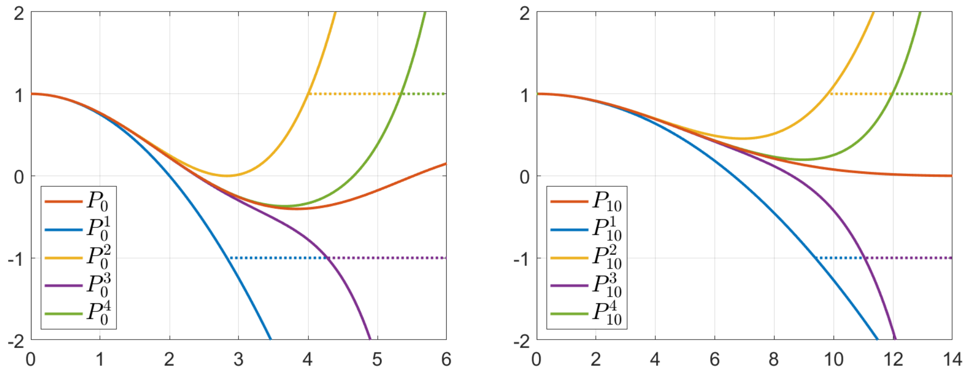

FIGURE 1. Bounds $P_\alpha^1, \ldots, P_\alpha^4$ and truncated variants (dotted) for $\alpha = 0$ and $\alpha = 10$.

3. COMPARISON

We briefly compare our findings with Neuman's bounds (1.4),(1.5) and Kapteyn's inequality (1.3). Proceeding from

$$J_\alpha(x) = J_\alpha^0(x) P_\alpha(x), \quad J_\alpha^n(x) = J_\alpha^0(x) P_\alpha^n(x),$$

we disregard the elementary factor $J_\alpha^0(x) = (x/2)^\alpha / \Gamma(\alpha + 1)$ and consider bounds on $P_\alpha(x)$. The basic estimate (1.2) now reads

$$-1 \le P_\alpha(x) \le 1. \tag{3.1}$$

Of course, for fixed x, Theorem 1 provides arbitrarily tight inclusions $P_\alpha(x) \in [P_\alpha^{2n-1}(x), P_\alpha^{2n}(x)]$ for large n. However, for practical purposes, low degrees may be the most interesting. The first few polynomials P_α^n are

$$\begin{aligned}
P_\alpha^1(x) &= 1 - \frac{(x/2)^2}{\alpha + 1} \\
P_\alpha^2(x) &= P_\alpha^1(x) + \frac{(x/2)^4}{2(\alpha+1)(\alpha+2)} \\
P_\alpha^3(x) &= P_\alpha^2(x) - \frac{(x/2)^6}{6(\alpha+1)(\alpha+2)(\alpha+3)} \\
P_\alpha^4(x) &= P_\alpha^3(x) + \frac{(x/2)^8}{24(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}.
\end{aligned}$$

They, respectively their truncated variants $\bar{P}_\alpha^n$, are shown in Figure 1 for orders $\alpha = 0$ and $\alpha = 10$ together with the target function P_α. As expected, the approximation is highly accurate for small x.

Concerning the range of arguments where the bounds considered here are *effective* in the sense that they are tighter than (3.1), we observe the following: Denote the break point of $\bar{P}_\alpha^n$ by $x_\alpha^n := \inf\{x : |P_\alpha^n(x)| > 1\}$. That is, the polynomial $P_\alpha^n(x)$ is effective for $x < x_\alpha^n$. Figure 2 shows x_α^n for $n = 1, \ldots, 4$ as a function of α together with the limiting values $x_\alpha^{\mathrm{N}} = \pi/2$ of (1.4), (1.5) and $x_\alpha^{\mathrm{K}} = \alpha$ of (1.3). In all cases, x_α^n is larger than x_α^{N} and the gap is increasing with α. By contrast, for fixed n and α large enough, x_α^{K} exceeds x_α^n.

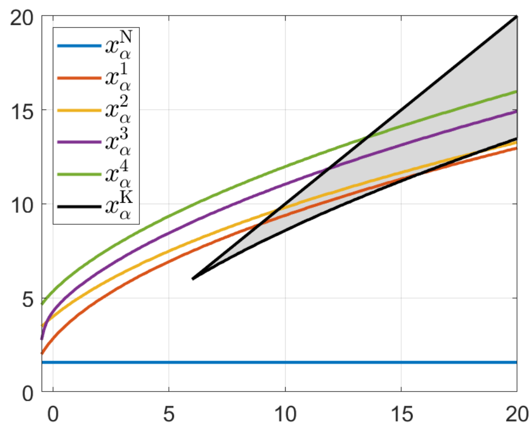

FIGURE 2. Neuman's bounds N_α^1, N_α^2 and the polynomial bounds $P_\alpha^1(x), \ldots, P_\alpha^4(x)$ are effective for x below the respective colored lines. Kapteyn's bound is effective in the shaded region.

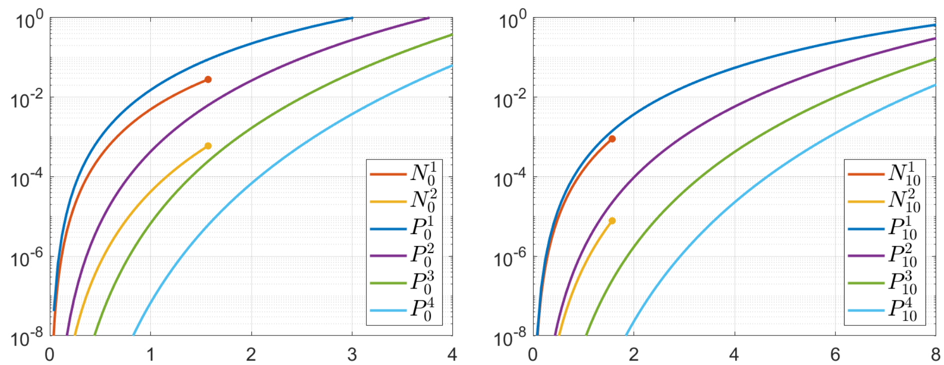

FIGURE 3. Deviations $|N_\alpha^i - P_\alpha|$ and $|P_\alpha^j - P_\alpha|$ for $\alpha = 0$ and $\alpha = 10$.

For a comparison with Neuman's bounds, denote by $N_\alpha^1(x)$ and $N_\alpha^2(x)$ the functions on the right hand side of (1.4) and (1.5) devided by $J_\alpha^0(x)$, respectively. Figure 3 shows some deviations $|N_\alpha^i - P_\alpha|$ and $|P_\alpha^j - P_\alpha|$ for $\alpha = 0$ and $\alpha = 10$. For $x \le \pi/2$, the lower bound $N_\alpha^1(x)$ is better than $P_\alpha^1(x)$, but worse than $P_\alpha^3(x)$. Equally, the upper bound $N_\alpha^2(x)$ is better than $P_\alpha^2(x)$, but worse than $P_\alpha^4(x)$.

For a comparison with Kapteyn's inequality, we derive from (1.3) the function

$$K_\alpha(x) := \Gamma(\alpha+1)\left(\frac{(2/\alpha)e^{\sqrt{1-(x/\alpha)^2}}}{1+\sqrt{1-(x/\alpha)^2}}\right)^\alpha,$$

providing the bound $|P_\alpha(x)| \le K_\alpha(x)$ for $\alpha > 0$ and $0 \le x \le \alpha$. The gray-shaded region in Figure 2 shows pairs (α, x) for which $K_\alpha(x)$ improves the bound $|P_\alpha(x)| \le 1$. In particular, it shows that (1.3) is not effective if $\alpha \le 5$ or $x \le 5$. We combine P_α^{2n-1} and P_α^{2n} to obtain the bound

$$|P_\alpha(x)| \le Q_\alpha^n(x) := \max\{|P_\alpha^{2n-1}(x)|, |P_\alpha^{2n}(x)|\}$$

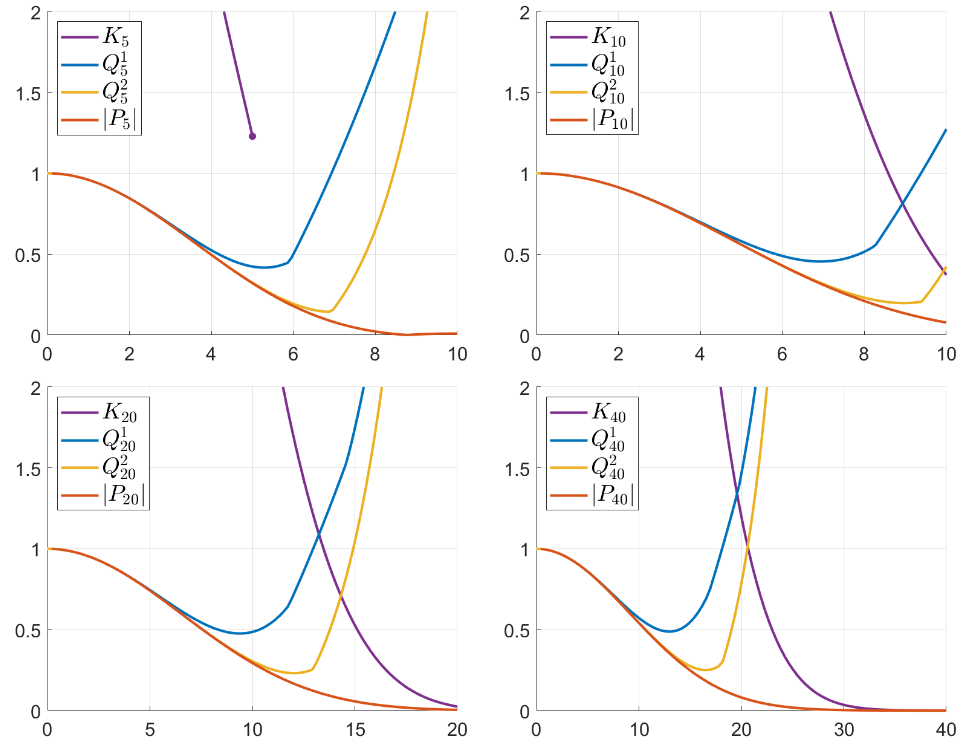

FIGURE 4. Bounds Q^1_α, Q^2_α, and K_α on $|P_\alpha|$ for $\alpha = 5, 10, 20$, and 40.

on the absolute value, which can now be compared with $K_\alpha(x)$. Numerical studies show that $K_\alpha(x)$ is weaker than $Q^1_\alpha(x)$ and $Q^2_\alpha(x)$ for all $x \le \alpha$ if α is less than approximately 7.5 and 10, respectively. Advantages of $K_\alpha(x)$ only show up for larger values of α and x. Figure 4 illustrates the situation for orders $\alpha = 5, 10, 20$, and 40.

REFERENCES

[DLMF] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.0 of 2024-03-15. F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, B.V. Saunders, H.S. Cohl, and M.A. McClain, eds.

[Neu04] Edward Neuman. Inequalities involving Bessel functions of the first kind. *Journal of Inequalities in Pure and Applied Mathematics [electronic only]*, 5(4):Paper No. 94, 4 p., 2004.

ULRICH REIF

TECHNICAL UNIVERSITY OF DARMSTADT, SCHLOSSGARTENSTR. 7, 64289 DARMSTADT, GERMANY

E-mail address: `reif@mathematik.tu-darmstadt.de`