Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 15 Issue 1(2024), Pages 23-30
https://doi.org/10.54379/jiasf-2024-1-3

THE FAN–TAUSSKY–TODD INEQUALITIES AND THE LUMER–PHILLIPS THEOREM

BENEDICT BAUER, STEFAN GERHOLD

Abstract. We argue that a classical inequality due to Fan, Taussky and Todd (1955) is equivalent to the dissipativity of a Jordan block. As the latter can be characterised via the zeros of Chebyshev polynomials, we obtain a short new proof of the inequality. Three other inequalities of Fan–Taussky–Todd are reproven similarly. By the Lumer–Phillips theorem, the matrix semigroup generated by the Jordan block is contractive. This yields new extensions of the classical Fan–Taussky–Todd inequalities. As applications, we give an estimate for the partial sums of a Bessel function, and a contribution to the classification of self-similar Gaussian Markov processes.

1. Introduction

In 1955, Fan, Taussky and Todd proved the following two theorems. To obtain the precise form stated in [7]), use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

Theorem 1.1. [7, Theorem 9] *For real numbers $a_1, \dots, a_n$, with $a_0 := a_{n+1} := 0$, we have*

$$\sum_{k=1}^{n+1}(a_k - a_{k-1})^2 \geq 2\Big(1 - \cos\frac{\pi}{n+1}\Big)\sum_{k=1}^{n} a_k^2. \tag{1.1}$$

Theorem 1.2. [7, Theorem 8] *For real numbers $a_1, \dots, a_n$, with $a_0 := 0$, we have*

$$\sum_{k=1}^{n}(a_k - a_{k-1})^2 \geq 2\Big(1 - \cos\frac{\pi}{2n+1}\Big)\sum_{k=1}^{n} a_k^2. \tag{1.2}$$

The following converse inequalities are special cases of results by Milovanović and Milovanović [9].

1991 *Mathematics Subject Classification.* 26D15, 33C45, 15A16, 33C10, 60G15.

Key words and phrases. an-Taussky-Todd inequalities; Jordan block; Chebyshev polynomials; Lumer-Phillips theorem; modified Bessel function of the first kind.

Submitted June 7, 2023. Published March 13, 2024.

Communicated by S. S. Dragomir.

Financial support from the Austrian Science Fund (FWF) under grants P 30750 and Y 1235 is gratefully acknowledged.

Theorem 1.3. *For real numbers $a_1, \dots, a_n$, with $a_0 := a_{n+1} := 0$, we have*

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \le 2\Big(1 + \cos\frac{\pi}{n+1}\Big) \sum_{k=1}^{n} a_k^2. \tag{1.3}$$

Theorem 1.4. *For real numbers $a_1, \dots, a_n$, with $a_0 := 0$, we have*

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \le 2\Big(1 - \cos\frac{2\pi}{2n+1}\Big) \sum_{k=1}^{n} a_k^2. \tag{1.4}$$

Alzer [1] gave short proofs of Theorems 1.3 and 1.4. In this note we give new short proofs of Theorems 1.1 and 1.3, and analogous proofs of Theorems 1.2 and 1.4. There are several proofs in the literature [8, 11, 12]. Our approach is symmetric in the sense that our proofs of Theorems 1.1 and 1.3 are trivial modifications of each other, and the same holds for Theorems 1.2 and 1.4. The proofs are based on the dissipativity of Jordan blocks, which can be checked in a straightforward way, using the well-known zeros of Chebyshev polynomials. Besides reproving inequalities which are already known (Section 2), we note in Section 3 that dissipativity characterizes contractiveness of the matrix semigroup generated by the Jordan block, by the Lumer–Phillips theorem. This leads to new generalizations of Theorems 1.1 and 1.2. These generalizations are then applied to estimate partial sums of the modified Bessel function of the first kind of order zero. In Section 4, we prove strict versions of the new inequalities. Section 5 presents an application of dissipativity of Jordan blocks to the classification of self-similar Gaussian Markov processes.

2. Proofs

Define the Jordan block

$$J_n(x) := \begin{pmatrix} x & 1 & & 0 \\ & x & 1 & \\ & & \ddots & 1 \\ 0 & & & x \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}.$$

We denote the standard scalar product on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and the identity matrix by I, and use the following terminology from operator theory:

Definition 2.1. *(see* [4, p. 52]*) A matrix $A \in \mathbb{R}^{n\times n}$, not necessarily symmetric, is* dissipative*, if $\langle Aa, a\rangle \le 0$ for all $a \in \mathbb{R}^n$.*

Since

$$\langle J_n(\alpha)a, a\rangle = \alpha \sum_{k=1}^{n} a_k^2 + \sum_{k=2}^{n} a_k a_{k-1}, \quad a = (a_1, \dots, a_n)^{\mathrm{T}} \in \mathbb{R}^n,$$

it is clear that (1.1) and (1.3) are immediate consequences of the following lemma.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$.*
(i) $J_n(\alpha)$ is dissipative if and only if $\alpha \le -\cos(\pi/(n+1))$.
(ii) $-J_n(\alpha)$ is dissipative if and only if $\alpha \ge \cos(\pi/(n+1))$.

Proof. Since $2a^{\mathrm{T}} J_n(\alpha)a = a^{\mathrm{T}} J_n(\alpha)^{\mathrm{T}} a + a^{\mathrm{T}} J_n(\alpha)a$, $a \in \mathbb{R}^n$, the first condition in (i) is equivalent to $B_n(\alpha) := J_n(\alpha)^{\mathrm{T}} + J_n(\alpha)$ being negative semidefinite, i.e. all its eigenvalues being non-positive. It is well-known [3, pp. 25-26] that $\det B_n(x) =$

$U_n(x)$, $x \in \mathbb{R}$, where $U_n(x)$ is the nth Chebyshev polynomial of the second kind. Thus, any eigenvalue μ of $B_n(\alpha)$ must satisfy

$$0 = \det(B_n(\alpha) - \mu I) = \det B_n(\alpha - \tfrac{1}{2}\mu) = U_n(\alpha - \tfrac{1}{2}\mu).$$

Now note that all such μ are ≤ 0 if and only if

$$\alpha \le \min_{1\le k\le n} \cos\frac{k\pi}{n+1} = \cos\frac{n\pi}{n+1} = -\cos\frac{\pi}{n+1},$$

where $\cos(k\pi/(n+1))$, $1 \le k \le n$, are the zeros of U_n. The proof of (ii) is analogous; now α must satisfy

$$\alpha \ge \max_{1\le k\le n} \cos\frac{k\pi}{n+1} = \cos\frac{\pi}{n+1}. \qquad \square$$

To prove Theorems 1.2 and 1.4, define

$$\tilde{J}_n(x) := \begin{pmatrix} x & 1 & & & 0 \\ & x & 1 & & \\ & & \ddots & 1 & \\ & & & x & 1 \\ 0 & & & & x - \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}.$$

Lemma 2.3. *For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have the determinant evaluation*

$$\det\left(\tilde{J}_n(x)^{\mathrm{T}} + \tilde{J}_n(x)\right) = U_n(x) - U_{n-1}(x).$$

As above, $U_n(x)$ denotes the nth Chebyshev polynomial of the second kind.

Proof. By a classical result on tridiagonal matrices [10, Chapter XIII], this determinant is $b_n = b_n(x)$, where the sequence $(b_k)_{-1\le k\le n}$ is defined by $b_{-1} = 0$, $b_0 = 1$, and

$$\begin{aligned} b_k &= 2x b_{k-1} - b_{k-2}, \quad 1 \le k < n, \\ b_n &= (2x-1)b_{n-1} - b_{n-2}. \end{aligned}$$

For $k < n$, this is the recurrence for the Chebyshev polynomials of the second kind, and so $b_k = U_k$ for $k < n$. Finally,

$$b_n = (2x-1)U_{n-1} - U_{n-2} = U_n - U_{n-1}. \qquad \square$$

Lemma 2.4. *For $n \in \mathbb{N}$, the zeros of $U_n(x) - U_{n-1}(x)$ are $(-1)^{k+1}\cos(k\pi/(2n+1))$, $1 \le k \le n$.*

Proof. Let $k \in \{1, \ldots, n\}$ be even. We have

$$-\cos\frac{k\pi}{2n+1} = \cos\Big(\pi - \frac{k\pi}{2n+1}\Big).$$

The assertion now follows from the representation

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad \theta \in \mathbb{R},$$

because

$$\begin{aligned} n\Big(\pi - \frac{k\pi}{2n+1}\Big) &= \Big(n - \frac{k}{2} + \frac{1}{2}\Big)\pi - \frac{(2n-k+1)\pi}{4n+2}, \\ (n+1)\Big(\pi - \frac{k\pi}{2n+1}\Big) &= \Big(n - \frac{k}{2} + \frac{1}{2}\Big)\pi + \frac{(2n-k+1)\pi}{4n+2}. \end{aligned}$$

The proof for odd k is analogous. $\square$

Since

$$\min_{1\le k\le n} (-1)^{k+1} \cos \frac{k\pi}{2n+1} = -\cos \frac{2\pi}{2n+1}$$

and

$$\max_{1\le k\le n} (-1)^{k+1} \cos \frac{k\pi}{2n+1} = \cos \frac{\pi}{2n+1},$$

Lemmas 2.3 and 2.4 show that the following result can be proven analogously to Lemma 2.2.

Lemma 2.5. *Let $\alpha \in \mathbb{R}$.*
(i) $\tilde{J}_n(\alpha)$ is dissipative if and only if $\alpha \le -\cos(2\pi/(2n+1))$.
(ii) $-\tilde{J}_n(\alpha)$ is dissipative if and only if $\alpha \ge \cos(\pi/(2n+1))$.

For $a \in \mathbb{R}^n$, we have

$$a^{\mathrm{T}} \tilde{J}_n(\alpha) a = \alpha \sum_{k=1}^{n} a_k^2 - \frac{a_n^2}{2} + \sum_{k=2}^{n} a_k a_{k-1}.$$

Now (1.2) and (1.4) easily follow, by applying Lemma 2.5 (i), (ii) with the respective optimal values of α.

3. A generalization

In what follows, we consider the Euclidean norm on $\mathbb{R}^n$, and write

$$\|A\|_{\mathrm{op}} = \sup_{0\ne a\in\mathbb{R}^n} \frac{\|Aa\|}{\|a\|}$$

for the operator norm of an $n\times n$ matrix A. The contractivity of matrix semigroups w.r.t. this norm is characterized by the Lumer–Phillips theorem [4, p. 52].

Theorem 3.1. *[Lumer–Phillips] For any real $n \times n$ matrix Q, the following are equivalent:*
(i) $\|\exp(Qx)\|_{\mathrm{op}} \le 1$ for all $x \ge 0$,
(ii) Q is dissipative.

This yields a new generalization of Theorem 1.1, involving an extra parameter $x \ge 0$.

Theorem 3.2. *For real numbers $a_1, \dots, a_n$ and $x \ge 0$, we have*

$$\sum_{j=0}^{n-1} \left(\sum_{k=0}^{j} \frac{x^k}{k!} a_{n-j+k} \right)^2 \le \exp\left(2x \cos \frac{\pi}{n+1}\right) \sum_{j=1}^{n} a_j^2. \tag{3.1}$$

Proof. By Lemma 2.2, the Jordan block $J_n(\alpha)$ satisfies condition (ii) of Theorem 3.1 for $\alpha = -\cos(\pi/(n+1))$, and so

$$\|\exp(J_n(\alpha)x)a\|^2 \le \|a\|^2, \quad x \ge 0,\ a \in \mathbb{R}^n. \tag{3.2}$$

By calculating the matrix exponential of a nilpotent matrix (see [6, p. 9]),

$$\exp\big(x(J_n(1)-I)\big) = \exp \begin{pmatrix} 0 & x & & 0 \\ & 0 & x & \\ & & \ddots & x \\ 0 & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & \frac{x^2}{2} & \cdots & \frac{x^{n-1}}{(n-1)!} \\ & 1 & x & \cdots & \frac{x^{n-2}}{(n-2)!} \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}, \quad x \in \mathbb{R},$$

we find

$$\exp(J_n(\alpha)x) = \exp\big(x(J_n(1)-I)+\alpha x I\big) = e^{\alpha x}\begin{pmatrix} 1 & x & \frac{x^2}{2} & \dots & \frac{x^{n-1}}{(n-1)!} \\ & 1 & x & \dots & \frac{x^{n-2}}{(n-2)!} \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}. \tag{3.3}$$

Using this in (3.2) yields (3.1), because

$$\begin{aligned}\|\exp(J_n(\alpha)x)a\|^2 &= e^{2\alpha x}\sum_{m=1}^{n}\left(\sum_{k=m}^{n} a_k\frac{x^{k-m}}{(k-m)!}\right)^2 \\ &= e^{2\alpha x}\sum_{j=0}^{n-1}\left(\sum_{k=n-j}^{n} a_k\frac{x^{k+j-n}}{(k+j-n)!}\right)^2 \\ &= e^{2\alpha x}\sum_{j=0}^{n-1}\left(\sum_{k=0}^{j} a_{n-j+k}\frac{x^{k}}{k!}\right)^2. \qquad \square\end{aligned}$$

As (3.1) is obviously sharp for $x \downarrow 0$, the inequality must hold after taking the derivative w.r.t. x at zero on both sides. An easy calculation shows that this yields (1.1), and so Theorem 3.2 can be viewed as a generalization of Theorem 1.1. Another special case might be worth mentioning: Putting $a = (0, \dots, 0, 1)$ in (3.1) yields

$$\sum_{j=0}^{n-1}\frac{x^{2j}}{j!^2} \le \exp\Big(2x\cos\frac{\pi}{n+1}\Big), \quad n\in\mathbb{N},\ x\ge 0. \tag{3.4}$$

This inequality for the partial sums of the modified Bessel function of the first kind $I_0(2x) = \sum_{j=0}^{\infty}\frac{x^{2j}}{j!^2}$ seems to be new. Analogously to Theorem 3.2, we can generalize Theorem 1.2 as follows.

Theorem 3.3. *For real numbers $a_1, \dots, a_n$ and $x \ge 0$, we have*

$$\sum_{j=1}^{n-1}\left(\sum_{k=0}^{j}\frac{x^k}{k!}a_{n-j+k}\right)^2 + e^{-x}a_n^2 \le \exp\Big(2x\cos\frac{2\pi}{2n+1}\Big)\sum_{j=1}^{n}a_j^2.$$

Proof. The proof is very similar to that of Theorem 3.2. We use Lemma 2.5, with $\alpha = -\cos(2\pi/(2n+1))$, and calculate $\|\exp(\tilde{J}_n(\alpha)x)a\|^2$ in order to apply Theorem 3.1 . $\square$

Taking the derivative of (3.1) at zero yields (1.2) (cf. the remark after Theorem 3.2). For $a = (0, \dots, 0, 1)$, as above, (3.1) yields

$$\sum_{j=0}^{n-1}\frac{x^{2j}}{j!^2} \le 1 - e^{-x} + \exp\Big(2x\cos\frac{2\pi}{2n+1})\Big), \quad n\in\mathbb{N},\ x\ge 0. \tag{3.5}$$

For fixed n, the estimate (3.5) is sharper than (3.4) for large x. We conjecture that there is a threshold $x_0 = x_0(n) > 0$ such that (3.4) gives a better bound for $0 < x < x_0$.

4. Strict inequalities

We were not able the find the following strict version of the Lumer–Phillips theorem in the literature:

Theorem 4.1. *For any real $n \times n$ matrix Q, the following are equivalent:*
(i) $\|\exp(Qx)\|_{\mathrm{op}} < 1$ *for all* $x > 0$,
(ii) $\langle Qa, a\rangle < 0$ *for all* $a \in \mathbb{R}^n \setminus \{0\}$.

Analogously to Theorem 3.2, Theorem 4.1 implies the following statement, and an analogous strict variant of Theorem 3.3.

Theorem 4.2. *For real numbers $a_1, \dots, a_n$, not all zero, $\alpha > \cos(\pi/(n+1))$, and $x > 0$, we have*

$$\sum_{j=0}^{n-1} \left(\sum_{k=0}^{j} \frac{x^k}{k!} a_{n-j+k} \right)^2 < e^{2x\alpha} \sum_{j=1}^{n} a_j^2.$$

It remains to prove Theorem 4.1.

Lemma 4.3. *Let Q be a real $n \times n$ matrix satisfying $\|\exp(Qx)\|_{\mathrm{op}} \le 1$ for all $x \ge 0$. Assume there exists $x_0 > 0$ such that $\|\exp(Qx_0)\|_{\mathrm{op}} = 1$. Then $\|\exp(Qx)\|_{\mathrm{op}} = 1$ for all $x \ge 0$.*

Proof. Choose $a \in \mathbb{R}^n$ with unit length such that $\|\exp(Qx_0)a\| = 1$. Clearly, we have $\|\exp(Qx)a\| = 1$ for all $x \in [0, x_0]$, since otherwise, by submultiplicativity,

$$\begin{aligned} \|\exp(Qx_0)a\| &= \|\exp(Q(x_0 - x))\exp(Qx)a\| \\ &\le \underbrace{\|\exp(Q(x_0 - x))\|}_{\le 1} \underbrace{\|\exp(Qx)a\|}_{<1} < 1. \end{aligned}$$

Now $\langle \exp(Qx)a, \exp(Qx)a\rangle - 1$ is analytic in x and vanishes on $[0, x_0]$. Hence it must vanish everywhere, and we infer $\|\exp(Qx)a\| = 1$ for all x. □

Lemma 4.4. *Let Q be a real $n \times n$ matrix satisfying $\|\exp(Qx)\|_{\mathrm{op}} \le 1$ for all $x \ge 0$. The set*

$$W_Q := \big\{ a \in \mathbb{R}^n : \|\exp(Qx)a\| = \|a\|,\ x \ge 0 \big\}$$

is a subspace. Moreover, W_Q and $W_Q^\perp$ are invariant under $\exp(Qx)$.

Proof. For $a \in \mathbb{R}^n$ and $x \ge 0$, we have

$$\|\exp(Qx)a\| = \|a\| \quad \Longleftrightarrow \quad \big\langle \big(\mathrm{id} - \exp(Q^{\mathrm{T}}x)\exp(Qx)\big)a, a \big\rangle = 0.$$

Our assumption implies that $\mathrm{id} - \exp(Q^{\mathrm{T}}x)\exp(Qx)$ is positive semidefinite, and thus the condition $\|\exp(Qx)a\| = \|a\|$ is equivalent to

$$a \in \ker(\mathrm{id} - \exp(Q^{\mathrm{T}}x)\exp(Qx)).$$

By Lemma 4.3, this kernel is the same for every $x > 0$. Let $u \in W_Q^\perp$. Then, for any $w \in W_Q$ we have

$$\langle \exp(Qx)u, w\rangle = \langle u, \exp(Qx)^{\mathrm{T}} w\rangle = \langle u, \exp(-Qx)w\rangle = 0,$$

since $\exp(-Qx)w \in W_Q$. Hence $W_Q^\perp$ is invariant under $\exp(Qx)$. Invariance of W_Q is clear. □

Proof of Theorem 4.1. If (ii) holds, then $\|\exp(Qx)\|_{\mathrm{op}} \le 1$, $x \ge 0$, by Theorem 3.1. Assume for the sake of contradiction that $\|\exp(Qx)\|_{\mathrm{op}} = 1$ for some $x > 0$. By Lemmas 4.3 and 4.4, there exists $0 \ne a \in \mathbb{R}^n$ such that $\langle \exp(Qx)a, \exp(Qx)a\rangle$ is constant in x. Hence the derivative at $x = 0$ has to be zero, and thus $\langle Qa, a\rangle = 0$, contradicting our assumption.

Conversely, assume that there exists $a \ne 0$ with $\langle Qa, a\rangle = 0$. Then we have $a \in \ker(Q + Q^{\mathrm{T}})$, since $Q + Q^{\mathrm{T}}$ is negative semidefinite. Hence $\exp(Q^{\mathrm{T}}x)a = \exp(-Qx)a$, and we have

$$\begin{aligned}\langle \exp(Qx)a, \exp(Qx)a\rangle &= \langle a, \exp(Q^{\mathrm{T}}x)\exp(Qx)a\rangle \\ &= \langle a, \exp(-Qx)\exp(Qx)a\rangle = \langle a, a\rangle.\end{aligned}$$

Therefore, $\|\exp(Qx)\|_{\mathrm{op}} = 1$ for all $x \ge 0$. □

5. Application to Gaussian Stochastic Processes

In [2], all centered H-self-similar Gaussian Markov (SSGM) processes with values in $\mathbb{R}^n$ are characterized. In particular, it is shown that the matrix-valued covariance function

$$R(s,t) = t^{2H}\exp\big(M\log(t/s)\big), \quad H > 0,\ M \in \mathbb{R}^{n\times n},\ 0 \le s \le t, \tag{5.1}$$

yields such a process, in the sense of Definition 2.7 in [2], if and only if the matrix semigroup $(\exp(Mx))_{x\ge 0}$ satisfies the contractivity condition

$$\|\exp(Mx)\|_{\mathrm{op}} \le e^{-Hx}, \quad x \ge 0. \tag{5.2}$$

By Theorem 3.1, this is equivalent to the dissipativity of $M + H\,I$. In the special case where the matrix $M = J_n(\lambda)$ consists of a single Jordan block, we can thus apply Lemma 2.2 with $\alpha = \lambda + H$ to conclude:

Proposition 5.1. *The covariance function*

$$R(s,t) = t^{2H}\exp\big(J_n(\lambda)\log(t/s)\big), \quad H > 0,\ \lambda \in \mathbb{R},\ 0 \le s \le t, \tag{5.3}$$

defines an $\mathbb{R}^n$-valued SSGM process if and only if

$$\lambda + H \le -\cos\frac{\pi}{n+1}.$$

Of course, the matrix exponential in (5.3) can be evaluated by (3.3). For $n = 1$, we obtain the condition $\lambda + H \le 0$, and the covariance function 5.1 defines a two-dimensional family of rescaled Brownian motions, which is studied in [5] under the name *power Brownian motion*. We note that Theorem 4.1 was also applied in [2], in an auxiliary result leading to Volterra representations of SSGM processes for which (5.2) is strict.

References

[1] H. Alzer, *Converses of two inequalities of Ky Fan, O. Taussky, and J. Todd*, J. Math. Anal. Appl., 161 (1991), pp. 142–147.

[2] B. Bauer and S. Gerhold, *Self-similar Gaussian Markov processes*. Preprint, 2024.

[3] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[4] E. B. Davies, *One-parameter semigroups*, vol. 15 of London Mathematical Society Monographs, Academic Press, Inc., London-New York, 1980.

[5] I. Eliazar, *Power Brownian motion*, J. Phys. A, 57 (2024), pp. 1–12, Paper No. 03LT01.

[6] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[7] K. Fan, O. Taussky, and J. Todd, *Discrete analogs of inequalities of Wirtinger*, Monatsh. Math., 59 (1955), pp. 73–90.
[8] G. Lunter, *New proofs and a generalization of inequalities of Fan, Taussky, and Todd*, J. Math. Anal. Appl., 185 (1994), pp. 464–476.
[9] G. V. Milovanović and I. v. Milovanović, *On discrete inequalities of Wirtinger's type*, J. Math. Anal. Appl., 88 (1982), pp. 378–387.
[10] T. Muir, *A treatise on the theory of determinants*, Dover Publications, Inc., New York, 1960. Revised and enlarged by William H. Metzler.
[11] R. M. Redheffer, *Easy proofs of hard inequalities*, in General inequalities, 3 (Oberwolfach, 1981), vol. 64 of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, 1983, pp. 123–140.
[12] Z.-H. Zhang and Z.-G. Xiao, *A simple new proof of Fan-Taussky-Todd inequalities*, Aust. J. Math. Anal. Appl., 2 (2005), pp. 1–5.

Stefan Gerhold
TU Wien, Wiedner Hauptstr. 8-10/105-01, 1040 Vienna, Austria
E-mail address: `sgerhold@fam.tuwien.ac.at`

Benedict Bauer
Department of Statistics and Operations Research, University of Vienna, Kolingasse 14–16, 1090 Vienna, Austria
E-mail address: `benedict.bauer@univie.ac.at`