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A GENERALIZED L^p-BOUNDEDNESS CONDITION FOR THE LITTLEWOOD-PALEY g-FUNCTION IN q-CALCULUS

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Abstract. Recently, the authors prove an L^p-boundedness condition of the so-called q-Littlewood-Paley g-function for $p \in (1, 2]$. In this paper, we shall generalize this condition for every $p \in (1, \infty)$, under an admissible condition on the parameter $q \in (0, 1)$.

1. Introduction.

As a tool for decomposing functions into different frequency components. The Littlewood-Paley g-function is a smooth, rapidly decaying function that is localized in frequency space and has a scaling property that makes it well-suited for analyzing functions at different scales. More specifically, the Littlewood-Paley g-function [22] is defined as the Fourier transform of a smooth, compactly supported function with integral equal to 1. It is then multiplied by a sequence of dyadic numbers that control the frequency localization and scaling of the function. The resulting functions form a partition of unity in frequency space, meaning that they cover the entire frequency domain and sum up to the original function.

The Littlewood-Paley theory has been used extensively in the study of various functional spaces in harmonic analysis [22], including the Hardy space [21], the Lipschitz space, and the BMO space [23]. It has also found applications in other areas of mathematics and physics, such as number theory [1], probability theory [2] and quantum mechanics [17].

One of the interesting fields of extensions of calculus is the so-called q-calculus which is an important sub-field in harmonic analysis and which provides some discrete and some refinement of the continuous harmonic analysis in sub-spaces $\mathbb{R}_q := \{\pm q^k, k \in \mathbb{Z}\}$, $q \in (0, 1)$. Note that, for all nonzero real number x, there exists a unique $k \in \mathbb{Z}$ such that $q^{k+1} < x \leq q^k$, this guarantees the density of the set $\mathbb{R}_q$ in $\mathbb{R}$.

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Many special functions have been generalized to a base q, and are usually reported as q-special functions. Interest in such q-functions is motivated by the recent and increasing relevance of q-analysis in exactly solvable models in statistical mechanics. Basic analogues of integral, derivative, Bessel function have been introduced by Jackson [13] and Gasper [11] as q-generalizations of the power series expansions.

Recently, there has been interest in generalizing the Littlewood-Paley theory to higher dimensions and to more general settings [20], such as non-Euclidean spaces and fractals. These developments have led to new insights and techniques in the study of various problems in analysis and geometry.

The q-Littlewood-Paley g-function [17] is defined in the one dimensional Euclidean space $\mathbb{R}_{q,+} := \{q^k, k \in \mathbb{Z}\}$, by the virtue of the q-integral [13] and the q-derivative [11]:

$$g(f)(x;q^2) := \left(\int_0^\infty \left|\nabla_q \mathcal{P}^t f(x)\right|^2 t d_q t\right)^{1/2},$$

where $\nabla_q := (D_{q,x}, D_{q,t})$ is the q-analogue of the gradient and $\mathcal{P}^t f(x)$ is the q-analogue of the Poisson integral studied in [18].

Our interest in this paper is to generalized the result given in [17], for any $p \in (1,2]$ by

$$\| g(f)(x;q^2) \|_{L^p(\mathbb{R}_{q,+})} \asymp \| f \|_{L^p(\mathbb{R}_{q,+})}, \; f \in L^p(\mathbb{R}_{q,+}),$$

where $L^p(\mathbb{R}_{q,+})$ is the space of functions f such that

$$\| f \|_{L^p(\mathbb{R}_{q,+})} := \left[(1-q)^p \sum_{k=-\infty}^{\infty} |f(q^k)|^p q^{kp}\right]^{\frac{1}{p}} < \infty$$

to any number $p \in (1,\infty)$, under an admissible condition on the parameter q

$$\ln(1+q)/\ln q \in \mathbb{Z}. \tag{1.1}$$

This major result will be proved in the last section of this paper.

This work is organized as follows. In the second section, a brief review on some q-harmonic analysis results related to q-calculus is developed. In the third section, we present our main result by the virtue of the q-Poisson kernel and q-Poisson integral and we present some technical lemmas that will be useful for the proof of the main result of this paper. The last section will be devoted with the proof of our main result, using the q-Hardy-Littlewood maximal function $\mathcal{M}_q(f)$.

2. Some q-calculus Toolkit.

The aim of this section is to recall some notions of q-calculus. For $q \in (0,1)$, denote

$$\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^k, k \in \mathbb{Z}\}, \quad \widetilde{\mathbb{R}}_{q,+} = \{q^k, k \in \mathbb{Z}\} \cup \{0\}.$$

For $a, b \in \mathbb{R}_{q,+}$ $(a < b)$ and a function f given on $[a, b]$, the q-integral [13] is defined by

$$\int_a^b f(x) d_q x := (1-q)\Big(\sum_{n=0}^{\infty}(bf(bq^n) - af(aq^n))q^n\Big).$$

The improper integral is defined in the following way

$$\int_0^{\infty} f(x) d_q x := (1-q) \sum_{k=-\infty}^{+\infty} f\left(q^k\right) q^k. \tag{2.1}$$

Note that for $a, b \in \mathbb{R}_{q,+}$, we may also have an analogue of the change variables theorem

$$\int_0^{\infty} f(ax) d_q x = a^{-1} \int_0^{\infty} f(x) d_q x, \qquad \int_0^b f(ax) d_q x = a^{-1} \int_0^{ab} f(x) d_q x. \tag{2.2}$$

We denote by μ the measure on $\mathbb{R}_{q,+}$ given by

$$\mathrm{d}_q\mu(y) = \Big(\frac{1+q}{1-q}\Big)^{-1/2} \Gamma_{q^2}^{-1}(1/2) \mathrm{d}_q y = c_q \mathrm{d}_q y, \tag{2.3}$$

where Γ_{q^2} is the q-gamma function [11].

The q-derivative of any function f, $D_{q,x} f$ [11] is defined by

$$D_{q,x} f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x, q \neq 0,$$

and the second derivative operator $\Delta_{q,x} := \Lambda_{q,x}^{-1} D_{q,x}^2$, where the q-shift operators is $(\Lambda_{q,x}^{-1} f)(x) := f(q^{-1}x)$ and

$$\Delta_q := \Delta_{q,x} + \Delta_{q,t}, (x, t) \in \mathbb{R}_q \times \mathbb{R}_{q,+}. \tag{2.4}$$

Notice that using the definition of the q-derivative, for any $k = 0, 1, 2, ...$

$$D_{q,x}^k f(x) = \frac{(-1)^k}{x^k(1-q)^k} \sum_{i=0}^{k} (-1)^i \frac{(q;q)_k}{(q;q)_i (q;q)_{k-i}} q^{\binom{k-i}{2}} f(q^{k-i}x), \ \binom{k}{2} = k(k-1)/2, \tag{2.5}$$

and

$$\Delta_{q,x}^n f(x) = \frac{q^{(2-n)n}(q;q)_{2n}}{(1-q)^{2n}} \sum_{k=-n}^{n} (-1)^{n-k} \frac{q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}} f(q^k x), \tag{2.6}$$

where $(a;q)_n$ is the q-shifted factorials are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Moreover, in [17] the author proves for all $n \in \mathbb{N}$, that

$$D_{q,x}(f^n(x)) = \frac{f^n(x) - f^n(qx)}{f(x) - f(qx)} D_{q,x} f(x) = \Big[\sum_{k=0}^{n-1} f^k(x) f^{-k}(qx)\Big] f^{n-1}(qx) D_{q,x} f(x), \tag{2.7}$$

$$\begin{aligned}
D^2_{q,x}(f^n(x)) &= q\Big[\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(x)f^{-i}(qx).\Big(D_{q,x}f(qx)\Big)\Big(D_{q,x}f(x)\Big)\\
&+ q\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-2} f^i(q^2x)f^{-i}(qx)\times\Big(D_{q,x}f(qx)\Big)^2\Big]f^{n-2}(qx) \quad (2.8)\\
&+ \Big[\sum_{k=0}^{n-1} f^k(x)f^{-k}(qx)\Big]D^2_{q,x}f(x)f^{n-1}(qx). \quad (2.9)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Delta_{q,x}(f^n(x)) &= q\Big[\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(q^{-1}x)f^{-i}(x).\Big(D_{q,x}f(x)\Big)\Big(D_{q,x}f(q^{-1}x)\Big)\Big)\\
&+ q\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-2} f^i(qx)f^{-i}(x)\\
&\times \Big(D_{q,x}f(x)\Big)^2\Big]f^{n-2}(x)+\Big[\sum_{k=0}^{n-1} f^k(q^{-1}x)f^{-k}(x)\Big]f^{n-1}(x)\Delta_{q,x}f(x).
\end{aligned}$$

Note that when $q\uparrow 1^-$, equation (2.7) tends to $nf^{n-1}(x)f'(x)$ and (2.8) to $nf^{n-1}(x)f''(x)+n(n-1)f^{n-2}(x)f'(x)$.

We introduce some q-functional spaces which we need in this work.

- ▷ $\mathcal{D}_{*,q}(\mathbb{R}_q)$, the space of even functions infinitely q-differentiable on $\mathbb{R}_q$ with compact support in $\mathbb{R}_q$. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.
- ▷ $L^p(\mathbb{R}_{q,+})$, $p\in[1,+\infty]$, the space of functions f such that $\parallel f\parallel_{L^p(\mathbb{R}_{q,+})}<+\infty$, where

$$\parallel f\parallel_{L^p(\mathbb{R}_{q,+})}:=\left[\int_0^\infty |f(x)|^p\mathrm{d}_q\mu(x)\right]^{\frac{1}{p}},\ \ \text{for } p<\infty,$$

with $\mathrm{d}_q\mu(x)$ given by (2.3), and

$$\parallel f\parallel_{L^\infty(\mathbb{R}_{q,+})}=\sup_{x\in\mathbb{R}_{q,+}} |\ f(x)\ |,\ \text{for } p=\infty.$$

Note that, in [9] the authors prove that

$$\parallel f\parallel_{L^p(\mathbb{R}_{q,+})}=\sup_{\{h\in\mathcal{D}_{*,q}(\mathbb{R}_q);\|h\|_{L^m(\mathbb{R}_{q,+})}=1\}}\Big|\int_0^\infty f(x)h(x)\mathrm{d}_q\mu(x)\Big|,\ 1/p+1/m=1. \tag{2.10}$$

The one-parameter family of q-exponential functions with $\alpha\in\mathbb{R}$ has been considered in [10]

$$E_q^{(\alpha)}(x):=\sum_{n=0}^\infty q^{\alpha n^2/4}\frac{(1-q)^n}{(q;q)_n}x^n,\quad x\in\mathbb{R}. \tag{2.11}$$

Two particular cases of this family with $\alpha=0$ and $\alpha=1$ are well known: they are the q-exponential

$$e_q(x)=E_q^{(0)}(x):=\frac{1}{((1-q)x;q)_\infty}=\sum_{n=0}^\infty\frac{(1-q)^n}{(q;q)_n}x^n,$$

and its reciprocal

$$E_q(x) = e_q^{-1}(x) = E_q^{(1)}(-q^{-1/2}x) := (-(1-q)x;q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{(1-q)^n}{(q;q)_n}x^n.$$

In [7], A. Fitouhi et al. study the q-analogue of the well-known heat kernel of one dimensional space $K(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}; x \in \mathbb{R}, t > 0$ denoted $G(x,t;q^2)$ and given by

$$G(x,t;q^2) := \frac{1}{A(t,q^2)}e_{q^2}(-\frac{x^2}{q(1+q)t}),$$

where $A(t,q^2) = q^{-\frac{1}{2}}(1-q)^{\frac{1}{2}}\frac{(-\frac{1-q}{1+q}\frac{1}{t}, -\frac{1+q}{1-q}q^2t;q^2)_\infty}{(-\frac{1-q}{1+q}\frac{1}{qt}, -\frac{1+q}{1-q}q^3t;q^2)_\infty}$, $t > 0$.

3. Main result

In this section, we define and study the L^p-boundedness of the Littlewood-Paley g-function, for $p \in (1,\infty)$. For more backgrounds on the q-Littlewood Paley g-function, q-Poisson kernel, q-Hardy-Littlewood maximal $\mathcal{M}_q(f)$ function the reader may be referred to [17, 18].

The q-Poisson kernel $P_t(x;q^2)$ and the q-Poisson integral $u(f)(x,t;q^2)$ of any function $f \in L^p(\mathbb{R}_{q,+})$ have been developed and studied

$$P_t(x;q^2) := P(t,x;q^2) = d_q\frac{t}{t^2+x^2},\ d_q = \frac{1}{\Gamma_{q^2}(\frac{1}{2})A(\frac{1}{q(1+q)^2};q^2)}, \tag{3.1}$$

$$u(f)(x,t;q^2) = \mathcal{P}^t f(x) = \int_0^\infty f(y)T_{q,x}P_t(y;q^2)\mathrm{d}_q\mu(y), \tag{3.2}$$

here, $T_{q,x}$ is the q-even translation operators [8] are defined by

$$T_{q,x}f(y) := \int_0^\infty f(z)D_q(x,y,z)\mathrm{d}_q\mu(z),$$

where $D_q(x,y,z)$ is defined for x and y in $\mathbb{R}_{q,+}$ by

$$D_q(x,y,z) := \int_0^\infty \cos(xt;q^2)\cos(yt;q^2)\cos(zt;q^2)\mathrm{d}_q\mu(t),$$

satisfying the commutativity property [6],

$$\int_0^\infty T_{q,x}f(y)g(y)\mathrm{d}_q\mu(y) = \int_0^\infty f(y)T_{q,x}g(y)\mathrm{d}_q\mu(y). \tag{3.3}$$

The q-cosine function is given in [14] as a series of functions

$$\cos(x;q^2) := \sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q;q)_{2n}}x^{2n} = \sum_{n=0}^{\infty}(-1)^n b_n(x;q^2).$$

Recalling that $u(f)(x,t;q^2)$ can be written [18] as

$$u(f)(x,t;q^2) = \int_0^{+\infty}\frac{E_{q^2}(-q^2y)}{\sqrt{y}}T^{\frac{t^2}{y}}f(x)\mathrm{d}_{q^2}\mu(y),$$

where

$$T^t f(x) = (G(.,t/q(1+q)^2;q^2) *_q f)(x), \tag{3.4}$$

"$*_q$" is the q-convolution product [8] defined by

$$f *_q g(x) = \int_0^\infty T_{q,x} f(y) g(y) \mathrm{d}_q\mu(y).$$

The following lemmas hold.

Lemma 3.1. [17]

(i) $D_{q,t}P_t(x;q^2) = d_q \dfrac{x^2 - qt^2}{(t^2+x^2)(q^2t^2+x^2)}$ *and* $D_{q,x}P_t(x;q^2) = -d_q \frac{(1+q)tx}{(t^2+x^2)(t^2+q^2x^2)}$.

(ii)

$$\Delta_{q,t}P_t(x;q^2) = d_q \frac{q^2t}{1-q} \frac{(q^3+q^2-q^{-1}-1)x^2 + (1-q^2)t^2}{(t^2+x^2)(t^2+q^2x^2)(q^2t^2+x^2)},$$

$$\Delta_{q,x}P_t(x;q^2) = -d_q \frac{(1+q)q^2t}{1-q} \frac{(1-q)t^2 + (q^2-q^{-1})x^2}{(t^2+x^2)(t^2+q^2x^2)(q^2t^2+x^2)},$$

$$D_{q,x}D_{q,t}P_t(x;q^2) = D_{q,t}D_{q,x}P_t(x;q^2) = -d_q \frac{1}{1-q} \frac{(1-q^2)x^4 + (q^3+q^2-q^{-1}-1)x^2t^2}{(t^2+x^2)(t^2+q^2x^2)(q^2t^2+x^2)}.$$

(iii) *For all* $k \in \mathbb{N}$, $\|D^k_{q,t}P_t(.;q^2)\|_{L^\infty(\mathbb{R}_{q,+})} \le C_q t^{-(k+1)}$.

Lemma 3.2. [17] *Let* $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ *be a positive function and* $p \in (1,\infty)$. *Then*

(i) $u(f)(x,t;q^2) \ge 0$.
(ii) $\Delta_q u(f)(x,t;q^2) = \Delta_{q,x}u(f)(x,t;q^2) + \Delta_{q,t}u(f)(x,t;q^2) = 0$.
(iii) *For all* $k \in \mathbb{N}$, *there exists* $C_q > 0$ *such that* $\left|D^k_{q,t}u(f)(x,t;q^2)\right| \le C_q t^{-(k+1)}$.

Lemma 3.3. [17] *Let* $f, h \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ *be positive functions and* $p \in (1,\infty)$. *Then*

(i) $\lim\limits_{R\to\infty} \int_0^R \int_0^R \Delta_{q,t}(u^p(f)(x,t)) t d_q t d_q\mu(x) = \int_0^\infty f^p(x) d_q\mu(x)$.
(ii) $\lim\limits_{R\to\infty} \int_0^R \int_0^R \Delta_{q,x}(u^p(f)(x,t)) t d_q\mu(x) d_q t = 0$.
(iii) $\int_0^\infty \int_0^\infty \Delta_q(u^p(f)(x,t)) t d_q t d_q\mu(x) = \| f \|_{L^p(\mathbb{R}_{q,+})}$.
(iv) $\int_0^\infty \int_0^\infty \Delta_q(u^p(f)(x,t)u(h)(x,t)) t d_q t d_q\mu(x) = \int_0^\infty f^p(x)h(x) d_q\mu(x)$.

Lemma 3.4. [17] *For any* $f_1, f_2 \in \mathcal{D}_{*,q}(\mathbb{R}_q)$, *there exists* $A_q > 0$, *such that*

$$\int_0^\infty \int_0^\infty tD_{q,t}u(f_1)(x,t;q^2)D_{q,t}u(f_2)(x,t;q^2) d_q t d_q\mu(x) = A_q \int_0^\infty f_1(x)f_2(x) d_q\mu(x).$$

Moreover

$$\Delta_q(u(f)^p(x,t;q^2)) = q\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{k-1} u(f)^i(q^{-1}x,t;q^2) f^{-i}(x,t;q^2)\cdot$$

$$\Big(D_{q,x}u(f)(x,t;q^2).D_{q,x}u(f)(q^{-1}x,t;q^2)$$

$$+D_{q,t}u(f)(x,t;q^2)\times D_{q,t}u(f)(x,q^{-1}t;q^2)\Big) + q\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2)$$

$$\times\Big(\nabla_q u(f)(x,t;q^2)\Big)^2\Big]u(f)^{p-2}(x,t;q^2), \tag{$\bigstar$}$$

and

$$\Big|\nabla_{q,x}u(f)(x,t;q^2)\Big|^2 \leq \frac{2q^{-2}}{p(p-1)}u(f)^{2-p}(x,t;q^2)\Delta_q(u(f)^p(x,t;q^2)). \tag{3.5}$$

Furthermore, $T^t f(x)$ satisfies

$$\| T^t f(x) \|_{L^p(\mathbb{R}_{q,+})} \leq \| f \|_{L^p(\mathbb{R}_{q,+})} . \tag{3.6}$$

Definition 3.5. *Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$. The q-Hardy-Littlewood maximal $\mathcal{M}_q(f)$ function is defined by*

$$\mathcal{M}_q(f)(x) := \sup_{t\in\mathbb{R}_{q,+}} \mid u(f)(x,t) \mid, \ x \in \mathbb{R}_q.$$

Proposition 3.6. [17] *Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ and $p \in (1,\infty)$. Then there exists $C_{p,q} > 0$ such that*

$$\| \mathcal{M}_q(f)(x) \|_{L^p(\mathbb{R}_{q,+})} \leq C_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} .$$

Definition 3.7. *The q-Littlewood-Paley g-function for $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ is given by*

$$g(f)(x;q^2) := \left(\int_0^\infty \Big|\nabla_q u(f)(x,t;q^2)\Big|^2 t d_q t\right)^{1/2},$$

where $u(f)(x,t)$ is the q-Poisson integral and $\nabla_q u(f) := \Big(D_{q,x}u(f), D_{q,t}u(f)\Big)$ it is q-gradient vector, verifying

$$\Big|\nabla_q u(f)(x,t;q^2)\Big|^2 := (D_{q,x}u(f)(x,t;q^2))^2 + (D_{q,t}u(f)(x,t;q^2))^2.$$

In the present paper, we consider and prove a general result developed in [17]. The following main theorem holds.

Theorem 3.8. *For $p \in (1,\infty)$ and $q \in (0,1)$ satisfying the admissible condition (1.1), there exist two constants $A_{p,q} > 0$ and $B_{p,q} > 0$, such that for $f \in L^p(\mathbb{R}_{q,+})$,*

$$B_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} \leq \| g(f)(x;q^2) \|_{L^p(\mathbb{R}_{q,+})} \leq A_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} .$$

4. Proof of main result

In the following, the proof of Theorem 3.8. will be done in three steps. The first one concerns the range $1 < p \leq 2$ and was proved in [17]. In the second step, using a maximum principle, we prove the result for $4 \leq p < \infty$. Finally the interpolation theorem permits to conclude for $2 < p < 4$. We need the following technical lemmas, which we need in the proof of the Theorem 3.8.

Lemma 4.1. *Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$. Then*

$$\Big|\nabla_q u(f)(x,(1+q)t;q^2)\Big|^2 \leq \mathcal{P}^t \Big|\nabla_q u(f)(x,t;q^2)\Big|^2,$$

where the operator $\mathcal{P}^t$ is given by (3.2).

Proof. Let $f \in \mathcal{D}_{*,q}(\mathbb{R}_q)$. Putting $F(x,t;q^2) = \Big(\nabla_q u(f)(x,t;q^2)\Big)^2$. We can verified easily that F is continuous, even with respect to the variable x, infinitely q-differentiable on $\mathbb{R}_q \times \mathbb{R}_{q,+}$. Moreover, from (★), (2.10) and Lemma 3.1.(ii)

$$\begin{aligned}
\Delta_q F(x,t;q^2) &= \Delta_q\Big[\Big(D_{q,x}u(f)(x,t;q^2)\Big)^2 + \Big(D_{q,t}u(f)(x,t;q^2)\Big)^2\Big] \\
&= q\Big[\Delta_{q,x}u(f)(q^{-1}x,t;q^2)\Delta_{q,x}u(f)(x,t;q^2) \\
&+ \Delta_{q,t}u(f)(x,q^{-1}t;q^2)\Delta_{q,t}u(f)(x,t;q^2)\Big] \\
&+ qD_{q,x}D_{q,t}u(f)(x,t;q^2)\Big[D_{q,x}D_{q,t}u(f)(q^{-1}x,t;q^2) \\
&+ D_{q,t}D_{q,x}u(f)(x,q^{-1}t;q^2)\Big] \\
&+ q^2\Big(\Big[\nabla_q(D_{q,x}u(f)(x,t;q^2))\Big]^2 + \Big[\nabla_q(D_{q,t}u(f)(x,t;q^2))\Big]^2\Big) \\
&\geq 0.
\end{aligned}$$

On the other hand, for $s \in \mathbb{R}_{q,+}$ let $H(x,t;q^2) = \mathcal{P}^t\Big|\nabla_q u(f)(x,t;q^2)\Big|^2 - F(x,s+qt;q^2)$, satisfies $H(x,0;q^2) = 0$ and from Lemma 3.2.,

$$\Delta_q H(x,t;q^2) = \Delta_q \mathcal{P}^t\Big|\nabla_q u(f)(x,t;q^2)\Big|^2 - \Delta_q F(x,s+qt;q^2) \leq 0.$$

So using the maximum principle of Hopf [24], the result follows by taking $s = t$. □

Lemma 4.2. *Let $f,h \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ and $q \in (0,1)$ satisfying the admissible condition (1.1). Then*

$$\int_0^\infty \Big(g(f)(x)\Big)^2 h(x)d_q\mu(x) \leq (1+q)^2\int_0^\infty\int_0^\infty t \mid \nabla_q u(f)(x,t;q^2) \mid^2 u(h)(x,t;q^2)d_q\mu(x)d_qt,$$

where the operator $\mathcal{P}^t$ is given by (3.2).

Proof. Before starting the proof of the lemma, noting that from property (3.3) the operator $\mathcal{P}^t$ is self-adjoint. Now, from Lemma 4.1., we have

$$\begin{aligned}
\int_0^\infty \Big(g(f)(x)\Big)^2 h(x)d_q\mu(x) &= \int_0^\infty\int_0^\infty t \mid \nabla_q u(f)(x,t;q^2) \mid^2 h(x)\mathrm{d}_q\mu(x)d_qt \\
&\leq \int_0^\infty\int_0^\infty t\mathcal{P}^{t(1+q)^{-1}} \mid \nabla_q u(f)(x,(1+q)^{-1}t;q^2) \mid^2 h(x)\mathrm{d}_q\mu(x)d_qt.
\end{aligned}$$

Now by (2.2), to make the change of variable $s = t(1+q)^{-1}$ a sense we need that s must be in $\mathbb{R}_{q,+}$. Hence $s = t(1+q)^{-1} = q^k$ and then for any $t \in \mathbb{R}_{q,+}$, there exist $i \in \mathbb{Z}$ such that $q^i(1+q)^{-1} = q^k$, which leads to $(1+q)^{-1} = q^{k-n}$ must be in $\mathbb{R}_{q,+}$. Then the admissible condition $\ln(1+q)/\ln q \in \mathbb{Z}$ given by (1.1) follows. So, we get easily the result. □

Lemma 4.3. *Let $f,h \in \mathcal{D}_{*,q}(\mathbb{R}_q)$ be positive functions. Then*

$$\int_0^\infty \Big(g(f)(x)\Big)^2 h(x)d_q\mu(x) \leq$$

$$q^{-2}(1+q)^3\Big(\int_0^\infty f^2(x)h(x)d_q\mu(x) + \int_0^\infty \mathcal{M}_q(f)(x)g(f)(x)g(h)(q^{-1}x)d_q\mu(x)\Big).$$

Proof. Using ($\bigstar$) for $p=2$, we get $\Delta_q(u^2(f)(x,t;q^2)) = 2q^2\Big(\nabla_q u(f)(x,t;q^2)\Big)^2$. By Lemma 4.2, we have

$$\int_0^\infty \Big(g(f)(x)\Big)^2 h(x)\mathrm{d}_q\mu(x) \leq$$
$$\frac{(1+q)^2}{2q^2}\int_0^\infty\int_0^\infty t\Delta_q(u^2(f)(x,t;q^2))u(h)(x,t;q^2)\mathrm{d}_q\mu(x)d_qt. \tag{4.1}$$

Now, applying (2.4) we get easily the following identity

$$\Delta_q(u.v) = v(q^{-1}x,t)\Delta_q(u) + u(qx,t)\Delta_q(v)+$$
$$(1+q)\Big[D_{q,x}v(q^{-1}x,t)D_{q,x}u + D_{q,t}v(x,q^{-1}t)D_{q,t}u(x,t;q^2)\Big],$$

with $u = u^2(f)(x,t;q^2)$ and $v = u(h)(x,t;q^2)$, we deduce that

$$\Delta_q(u^2(f)(x,t;q^2))u(h)(x,t;q^2) = \Delta_q(u^2(f)(x,t;q^2)u(h)(x,t;q^2))-$$
$$(1+q)\Big[D_{q,x}u(h)(q^{-1}x,t;q^2)\times D_{q,x}(u^2(f)(x,t;q^2))+D_{q,t}u(h)(x,q^{-1}t)D_{q,t}(u^2(f)(x,t;q^2))\Big].$$

Then applying (2.8), Lemma 3.2.(ii), Lemma 3.3.(iv) and the elementary identity

$$(x_1x_2+y_1y_2)^2 \leq (x_1^2+y_1^2)(x_2^2+y_2^2)$$

to (4.1), we get respectively

$$\int_0^\infty\int_0^\infty t\Delta_q(u^2(f)(x,t;q^2)u(h)(x,t;q^2))d_q\mu(x)d_qt = \int_0^\infty f^2(x)h(x)d_q\mu(x),$$

and

$$\int_0^\infty\int_0^\infty t\Big[D_{q,x}u(h)(q^{-1}x,t;q^2)D_{q,x}(u^2(f)(x,t;q^2))+$$
$$D_{q,t}u(h)(x,q^{-1}t)D_{q,t}(u^2(f)(x,t;q^2))\Big]d_q\mu(x)d_qt$$
$$\leq 2\int_0^\infty \mathcal{M}_q(f)(x)\int_0^\infty t\Big[D_{q,x}u(h)(q^{-1}x,t;q^2)D_{q,x}(u(f)(x,t;q^2))+$$
$$D_{q,t}u(h)(x,q^{-1}t)D_{q,t}(u(f)(x,t;q^2))\Big]d_qtd_q\mu(x)$$
$$\leq 2\int_0^\infty \mathcal{M}_q(f)(x)\int_0^\infty t\Big|\nabla_q u(h)(x,t;q^2)\Big|\Big|\nabla_q(u(f)(x,t;q^2))\Big|d_qtd_q\mu(x).$$ □

Theorem 4.4. *For $p\in(1,\infty)$ and $q\in(0,1)$ satisfying the admissible condition (1.1), there exists a constant $A_{p,q}$ such that for all $f\in\mathcal{D}_{*,q}(\mathbb{R}_q)$, we have*

$$\| g(f)\|_{L^p(\mathbb{R}_{q,+})} \leq A_{p,q}\| f\|_{L^p(\mathbb{R}_{q,+})}.$$

Proof. The result will be proved in three steps. The first one is for $1<p\leq 2$ which was proved in [17]. The second steps using a maximum principle for $p\geq 4$. In order to prove this, we will use Lemma 4.1., Lemma 4.2. and Lemma 4.3., we can show for $4\leq p<\infty$, using the fact that

$$\| g(f)\|^2_{L^p(\mathbb{R}_{q,+})} = \sup_h \int_0^\infty g^2(f)(x)h(x)d_q\mu(x),$$

with $\| g(f) \|_{L^m(\mathbb{R}_{q,+})} \leq 1$, for $1 < m \leq 2$ and $2/p + 1/m = 1$. Applying property (3.3) to the inequality in Lemma 4.3., we obtain

$$\int_0^\infty \mathcal{M}_q(f)(x) g(f)(x) g(h)(q^{-1}x) d_q\mu(x) \leq$$
$$\| \mathcal{M}_q(f) \|_{L^p(\mathbb{R}_{q,+})} \| \mathcal{M}_q(f) \|_{L^p(\mathbb{R}_{q,+})} \| g(h) \|_{L^m(\mathbb{R}_{q,+})} .$$

So

$$\int_0^\infty \Big(g(f)(x) \Big)^2 h(x) d_q\mu(x) \leq$$
$$q^{-2}(1+q)^3 \Big(\| f \|_{L^p(\mathbb{R}_{q,+})} + \| \mathcal{M}_q(f) \|_{L^p(\mathbb{R}_{q,+})} \| g(f) \|_{L^p(\mathbb{R}_{q,+})} \| g(h) \|_{L^m(\mathbb{R}_{q,+})} \Big).$$

Now, using Proposition 3.6, we deduce

$$\int_0^\infty \Big(g(f)(x) \Big)^2 h(x) d_q\mu(x) \leq$$
$$q^{-2}(1+q)^3 \Big(\| f \|^2_{L^p(\mathbb{R}_{q,+})} + C_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} \| g(f) \|_{L^p(\mathbb{R}_{q,+})} \Big).$$

note that $m = p/(p-2) \in (1, 2]$ this implies that

$$\| g(f) \|^2_{L^p(\mathbb{R}_{q,+})} - q^{-2}(1+q)^3 C_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} \| g(f) \|_{L^p(\mathbb{R}_{q,+})} -$$
$$q^{-2}(1+q)^3 \| f \|^2_{L^p(\mathbb{R}_{q,+})} \leq 0.$$

The study of the sign of the elementary quadratic expression in $\| g(f) \|_{L^p(\mathbb{R}_{q,+})}$ shows that

$$\| g(f) \|_{L^p(\mathbb{R}_{q,+})} \leq \Big(q^{-2}(1+q)^3 C_{p,q} + q^{-1}(1+q)^{3/2} \sqrt{1 + q^{-2}(1+q)^3 C^2_{p,q}} \Big) \| f \|_{L^p(\mathbb{R}_{q,+})} .$$

Hence, for $p \geq 4$, the constant $A_{p,q}$ is given by

$$A_{p,q} = q^{-2}(1+q)^3 C_{p,q} + q^{-1}(1+q)^{3/2} \sqrt{1 + q^{-2}(1+q)^3 C^2_{p,q}}.$$

Finally, for the third steps for $2 < p < 4$ by applying the Marcinkiewicz interpolation theorem (for $\theta = 2 - 4/p$), we obtain that $g(f)$ is bounded from $L^p(\mathbb{R}_{q,+})$ into itself. Hence the constant $A_{p,q}$ is given by

$$A_{p,q} = A_{2,q}^{4/p-1} A_{4,q}^{2-4/p}.$$

This completes the proof of the theorem. □

Consequently, from the density of $\mathcal{D}_{*,q}(\mathbb{R}_q)$ in $L^p(\mathbb{R}_{q,+})$ ([9], Theorem 4.28), we have the following result.

Theorem 4.5. *For $p \in (1, \infty)$ and $q \in (0, 1)$ satisfying the admissible condition (1.1), there exists a constant $A_{p,q}$ such that for all $f \in L^p(\mathbb{R}_{q,+})$, we have*

$$\| g(f) \|_{L^p(\mathbb{R}_{q,+})} \leq A_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} .$$

To prove the left hand side of Theorem 3.8., we use the following theorem.

Theorem 4.6. *For $p \in (1, \infty)$ and $q \in (0, 1)$ satisfying the admissible condition (1.1), there exists a constant $B_{p,q}$ such that for all $f \in L^p(\mathbb{R}_{q,+})$, we have*

$$B_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})} \leq \| g(f) \|_{L^p(\mathbb{R}_{q,+})} .$$

Proof. To prove the result, we will use the function $g_1(f)$ given in [19] by

$$g_1 f(x;q^2) := \left(\int_0^\infty t\left|D_{q,t}u(f)(x,t;q^2)\right|^2 d_q t\right)^{1/2},\ f \in L^p(\mathbb{R}_{q,+}).$$

Obviously, we have

$$g_1(f)(x;q^2) \leq g(f)(x;q^2). \tag{4.2}$$

Now, computing relations (2.10), (4.2), Lemma 3.4. and Holder inequality, give that there exist $A_q > 0$ such that

$$\begin{aligned} \frac{1}{A_q}\left|\int_0^\infty f(x)h(x)\mathrm{d}_q\mu(x)\right| &\leq \int_0^\infty g_1(f)(x;q^2)g_1(h)(x;q^2)\mathrm{d}_q\mu(x) \\ &\leq \parallel g_1(f) \parallel_{L^p(\mathbb{R}_{q,+})}\parallel g_1(h) \parallel_{L^m(\mathbb{R}_{q,+})},\ 1/p+1/m=1 \\ &\leq C_{q,m} \parallel g_1(f) \parallel_{L^p(\mathbb{R}_{q,+})} \\ &\leq C_{q,m} \parallel g(f) \parallel_{L^p(\mathbb{R}_{q,+})}. \end{aligned}$$

So, by taking the supremum, we have

$$B_{q,p} \parallel f \parallel_{L^p(\mathbb{R}_{q,+})} \leq \parallel g(f) \parallel_{L^p(\mathbb{R}_{q,+})},\ B_{q,p} = 1/A_q C_{q,m}.$$

□

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