Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 15 Issue 1(2024), Pages 1-10
https://doi.org/10.54379/jiasf-2024-1-1

A CLASS OF MORE ACCURATE HILBERT-TYPE INEQUALITIES INVOLVING PARTIAL SUMS

LAITH EMIL AZAR, TSERENDORJ BATBOLD, NIZAR KH. AL-OUSHOUSH, MARIO KRNIĆ

Abstract. The main goal of this paper is a study of a class of more accurate Hilbert-type inequalities involving partial sums. We first derive a discrete Hilbert-type inequality involving two partial sums and the kernel $1/\max(m^\lambda, n^\lambda)$. Then, by virtue of the Hardy inequality, we establish a weaker version of the latter relation involving only one partial sum, as well as the corresponding equivalent form. In addition, we prove that the constants appearing on the right-hand sides of the established inequalities are the best possible. Finally, we also give integral analogues of the corresponding discrete results.

1. Introduction

Let $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be positive real sequences such that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$. Then, there holds the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}} \tag{1.1}$$

and its equivalent form

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2\sum_{n=1}^{\infty} a_n^2. \tag{1.2}$$

Here, the constants π and π^2 are the best possible. This means that they can not be replaced with a smaller constants so that (1.1) and (1.2) remain valid. Inequality (1.1) is known as the Hilbert inequality (see, e.g. [5]) and it is important in analysis and its applications. On the other hand, the equivalent form (1.2) is often referred to as the Hardy-Hilbert inequality.

Although classical, the above pair of relations is nowadays also interesting topic to numerous mathematicians. Roughly speaking, the study of Hilbert-type inequalities can be divided into two directions. The first direction includes relations with

2000 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Hilbert inequality; Hölder inequality; the best possible constant; Abel summation formula; convexity.

Submitted October 21, 2023. Published January 24, 2024.
Communicated by S.S. Dragomir.

more general kernels and weight functions, as well as the extensions to more general spaces. The second direction refers to refinements of the existing Hilbert-type inequalities. For a historical overview of the Hilbert inequality, including various proofs and diverse applications, the reader is referred to [5]. On the other hand, the most relevant recent results about the Hilbert-type inequalities are collected in monographs [3] and [11].

The basic step in our research is the well-known Hilbert-type inequality (see [5], Theorem 341)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m,n)} < 4\left(\sum_{n=1}^{\infty}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}, \tag{1.3}$$

and its equivalent form

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max(m,n)}\right)^2 < 4^2\sum_{n=1}^{\infty}a_n^2, \tag{1.4}$$

which are valid under the same conditions as inequalities (1.1) and (1.2). In addition, the constants appearing on the right-hand sides of these inequalities are the best possible. Furthermore, the integral form of (1.3) asserts that

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max(x,y)}dxdy < 4\left(\int_0^{\infty}f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^{\infty}g^2(x)dx\right)^{\frac{1}{2}}, \tag{1.5}$$

where f and g are non-negative integrable functions on $(0,\infty)$ such that $0 < \int_0^{\infty}f^2(x)dx < \infty$ and $0 < \int_0^{\infty}g^2(x)dx < \infty$. Of course, by replacing the sum with an integral and the sequence with a non-negative integrable function, one obtains integral form of (1.4). The constants appearing on the right-hand sides of these integral inequalities are also the best possible.

In the literature, inequalities (1.2) and (1.4) are usually known as the Hardy-Hilbert-type inequalities, since their general form implies the famous Hardy inequality (for more details, see [11] and [10]). Recall that if $p > 1$ and $F(x) = \int_0^x f(t)dt$, where f is a non-negative integrable function, then there holds the inequality

$$\int_0^{\infty}\left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)dx, \tag{1.6}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. This inequality has been discovered by Hardy while he was trying to introduce a simple proof of the Hilbert inequality. The weighted form of (1.6) is given by

$$\int_0^{\infty}x^{\alpha}\left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p\int_0^{\infty}x^{\alpha}f^p(x)dx, \tag{1.7}$$

where $\alpha < p-1$ and the constant $\left(\frac{p}{p-1-\alpha}\right)^p$ is the best possible (for more details, see [5]). If $0 \le \alpha < p-1$ and $A_n = \sum_{k=1}^{n}a_m$, where $(a_n)_{n\in\mathbb{N}}$ is a positive sequence, then the discrete analogue of (1.7) reads

$$\sum_{n=1}^{\infty}n^{\alpha}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-\alpha-1}\right)^p\sum_{n=1}^{\infty}n^{\alpha}a_n^p, \tag{1.8}$$

where $\left(\frac{p}{p-1-\alpha}\right)^p$ is also the best possible constant (see [4]). For more details about the Hardy inequality and its development, we refer the reader to [8] and [9], as well as to the references cited therein.

In the last decade, an interesting topic in connection with the Hilbert-type inequalities is a study of related inequalities where functions and sequences are replaced by certain integral or discrete operators, as in the Hardy inequality. For example, Adiyasuren e.t al. [2], proved that if $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $\lambda>0$, $\alpha_1\in\left[-\frac{1}{q},0\right)$, $\alpha_2\in\left[-\frac{1}{p},0\right)$, $A_m=\sum_{k=1}^m a_k$, $B_n=\sum_{k=1}^n b_k$, where $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are positive sequences, then there holds the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<C\left(\sum_{m=1}^{\infty}m^{pq\alpha_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{pq\alpha_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{1.9}$$

where $p\alpha_2+q\alpha_1=-\lambda$ and $C=pq\alpha_1\alpha_2B(-p\alpha_2,-q\alpha_1)$. Here, B stands for the usual beta function defined by $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a,b>0$, and C is the best possible constant in (1.9). For some related Hilbert-type inequalities including partial sums and upper limit function, the reader is referred to [1, 6, 7, 12, 13, 15, 16] and the references cited therein.

We aim here to establish several new Hilbert-type inequalities with partial sums, closely related to inequality (1.3). In fact, we obtain several inequalities which are sharper than the Hilbert-type inequality (1.3). The paper's outline is as follows: after this Introduction, Section 2 is devoted to discrete Hilbert-type inequalities. We first derive the Hilbert-type inequality involving the kernel $1/\max(m^{\lambda},n^{\lambda})$ and two partial sums. Then, bearing in mind the weighted Hardy inequality, we establish a weaker form of the latter inequality involving only one partial sum, as well as the corresponding equivalent form. In addition, we prove that the constant factors included in the derived inequalities are the best possible. Finally, we show that these inequalities refine the Hilbert-type inequality (1.3). Finally, in Section 3, we derive integral analogues of the corresponding discrete results.

2. Discrete results

In this section, we establish several discrete Hilbert-type inequalities involving partial sums instead of the corresponding sequences. It turns out that these new forms of inequalities are sharper than the Hilbert-type inequality (1.3). In addition, these inequalities are closely related to the Hardy inequality. Our first result, which includes two partial sums, reads as follows:

Theorem 2.1. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $0<\lambda\le\min(p,q)$, and let $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be positive sequences such that $\sum_{m=1}^{\infty}a_m<\infty$, $\sum_{n=1}^{\infty}b_n<\infty$. If $A_m=\sum_{k=1}^m a_k$ and $B_n=\sum_{k=1}^n b_k$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max(m^{\lambda},n^{\lambda})}<\lambda\left(\sum_{m=1}^{\infty}m^{-\lambda-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}}, \tag{2.1}$$

provided that $\sum_{m=1}^{\infty}m^{-\lambda-1}A_m^p<\infty$ and $\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q<\infty$. In addition, constant λ is the best possible.

Proof. The starting point in this proof is a suitable transformation of a double series on the left-hand side of (2.1). More precisely, we have that

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda,n^\lambda)} &= \sum_{m=1}^{\infty} a_m\left(\sum_{n=1}^{\infty}\frac{b_n}{\max(m^\lambda,n^\lambda)}\right)\\ &= \sum_{m=1}^{\infty} a_m\left(\sum_{n=1}^{m}\frac{b_n}{\max(m^\lambda,n^\lambda)}+\sum_{n=m+1}^{\infty}\frac{b_n}{\max(m^\lambda,n^\lambda)}\right)\\ &= \sum_{m=1}^{\infty} a_m\left(\frac{B_m}{m^\lambda}+\sum_{n=m+1}^{\infty}\frac{b_n}{n^\lambda}\right). \qquad (2.2)\end{aligned}$$

Further, application of the Abel summation by parts formula to the series $\sum_{n=m+1}^{\infty}\frac{b_n}{n^\lambda}$ yields

$$\begin{aligned}\sum_{n=m+1}^{\infty}\frac{b_n}{n^\lambda} &= \lim_{n\to\infty}\frac{B_{n+1}}{(n+1)^\lambda}-\frac{B_m}{m^\lambda}-\sum_{n=m}^{\infty}B_n\left(\frac{1}{(n+1)^\lambda}-\frac{1}{n^\lambda}\right)\\ &= -\frac{B_m}{m^\lambda}+\sum_{n=m}^{\infty}B_n\left(\frac{1}{n^\lambda}-\frac{1}{(n+1)^\lambda}\right).\end{aligned}$$

On the other hand, the function $h(x)=\frac{1}{x^\lambda}$ is strictly convex, so utilizing the well-known relation $h(x)>h(y)+h'(y)(x-y)$ (see, e.g. [14]), we arrive at the inequality

$$\frac{1}{n^\lambda}-\frac{1}{(n+1)^\lambda}<\frac{\lambda}{n^{\lambda+1}},$$

which yields,

$$\sum_{n=m+1}^{\infty}\frac{b_n}{n^\lambda}<-\frac{B_m}{m^\lambda}+\lambda\sum_{n=m}^{\infty}\frac{B_n}{n^{\lambda+1}}.$$

Now, combining the last estimate with (2.2), as well as changing the order of summation, we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda,n^\lambda)}<\lambda\sum_{m=1}^{\infty}a_m\sum_{n=m}^{\infty}\frac{B_n}{n^{\lambda+1}}=\lambda\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{a_m B_n}{n^{\lambda+1}}=\lambda\sum_{n=1}^{\infty}\frac{A_n B_n}{n^{\lambda+1}}.$$

Finally, applying the Hölder inequality to the right-hand side of the above inequality, we derive (2.1), as claimed.

Now, it remains to prove that λ is the best possible value. To do this, let us suppose that there exists a positive constant $K<\lambda$ such that (2.1) holds when λ is replaced by K. Furthermore, consider the sequences $\tilde{a}_m=m^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}$ and $\tilde{b}_n=n^{\frac{\lambda}{q}-\frac{\varepsilon}{q}-1}$, where $\varepsilon>0$ is sufficiently small number. Then, the corresponding partial sums can be bounded from the above as follows:

$$\tilde{A}_m=\sum_{k=1}^{m}\tilde{a}_k=\sum_{k=1}^{m}k^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}\le\int_0^m x^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}dx=\frac{m^{\frac{\lambda}{p}-\frac{\varepsilon}{p}}}{\frac{\lambda}{p}-\frac{\varepsilon}{p}},$$

and similarly,

$$\tilde{B}_n=\sum_{k=1}^{n}\tilde{b}_k\le\frac{n^{\frac{\lambda}{q}-\frac{\varepsilon}{q}}}{\frac{\lambda}{q}-\frac{\varepsilon}{q}}.$$

By virtue of the above estimates, we can derive the corresponding upper bound for the right-hand side of (2.1). Namely, we have that

$$\begin{aligned}
&K\left(\sum_{m=1}^{\infty} m^{-\lambda-1}\tilde{A}_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\lambda-1}\tilde{B}_n^q\right)^{\frac{1}{q}}\\
&\le K\left(\sum_{m=1}^{\infty} m^{-\lambda-1}\frac{m^{\lambda-\varepsilon}}{(\frac{\lambda}{p}-\frac{\varepsilon}{p})^p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\lambda-1}\frac{n^{\lambda-\varepsilon}}{(\frac{\lambda}{q}-\frac{\varepsilon}{q})^q}\right)^{\frac{1}{q}}\\
&=\frac{pqK}{(\lambda-\varepsilon)^2}\left(\sum_{m=1}^{\infty} m^{-1-\varepsilon}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\right)^{\frac{1}{q}}\\
&=\frac{pqK}{(\lambda-\varepsilon)^2}\left(1+\sum_{m=2}^{\infty} m^{-1-\varepsilon}\right).
\end{aligned}$$

Moreover, since $\sum_{m=2}^{\infty} m^{-1-\varepsilon}\le\int_1^{\infty} x^{-1-\varepsilon}dx=\frac{1}{\varepsilon}$, we arrive at the following estimate

$$K\left(\sum_{m=1}^{\infty} m^{-\lambda-1}\tilde{A}_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\lambda-1}\tilde{B}_n^q\right)^{\frac{1}{q}}\le\frac{pqK(1+\varepsilon)}{\varepsilon(\lambda-\varepsilon)^2}. \tag{2.3}$$

Our next intention is to establish the corresponding estimate for the left-hand side of (2.1), equipped with the above sequences $(\tilde{a}_m)_{m\in\mathbb{N}}$ and $(\tilde{b}_n)_{n\in\mathbb{N}}$. Obviously, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{\max(m^\lambda,n^\lambda)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}n^{\frac{\lambda}{q}-\frac{\varepsilon}{q}-1}}{\max(m^\lambda,n^\lambda)}\ge\int_1^{\infty}\int_1^{\infty}\frac{x^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}y^{\frac{\lambda}{q}-\frac{\varepsilon}{q}-1}}{\max(x^\lambda,y^\lambda)}dxdy.$$

Moreover, since

$$\begin{aligned}
&\int_1^{\infty}\int_1^{\infty}\frac{x^{\frac{\lambda}{p}-\frac{\varepsilon}{p}-1}y^{\frac{\lambda}{q}-\frac{\varepsilon}{q}-1}}{\max(x^\lambda,y^\lambda)}dxdy\\
&=\int_1^{\infty} x^{-\varepsilon-1}\left(\int_{\frac{1}{x}}^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{\max(1,u^\lambda)}du\right)dx\\
&=\int_1^{\infty} x^{-\varepsilon-1}\left(\int_0^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{\max(1,u^\lambda)}du-\int_0^{\frac{1}{x}}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{\max(1,u^\lambda)}du\right)dx\\
&=\frac{1}{\varepsilon}\left(\int_0^1 u^{\frac{\lambda-\varepsilon}{q}-1}du+\int_1^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{u^\lambda}du\right)-\int_1^{\infty} x^{-\varepsilon-1}\left(\int_0^{\frac{1}{x}} u^{\frac{\lambda-\varepsilon}{q}-1}\right)dudx\\
&=\frac{1}{\varepsilon}\left(\frac{q}{\lambda-\varepsilon}+\frac{1}{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\right)-\frac{q}{\lambda-\varepsilon}\int_1^{\infty} x^{-\varepsilon-1-\frac{\lambda}{q}+\frac{\varepsilon}{q}}dx\\
&=\frac{1}{\varepsilon}\left(\frac{q}{\lambda-\varepsilon}+\frac{1}{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\right)-O(1),
\end{aligned}$$

we obtain the relation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{\max(m^\lambda,n^\lambda)}\ge\frac{1}{\varepsilon}\left(\frac{q}{\lambda-\varepsilon}+\frac{1}{\frac{\lambda}{p}-\frac{\varepsilon}{q}}\right)-O(1). \tag{2.4}$$

Now, combining relations (2.3) and (2.4), we arrive at the inequality

$$\frac{q}{\lambda-\varepsilon}+\frac{1}{\frac{\lambda}{p}-\frac{\varepsilon}{q}}-\varepsilon O(1)\leq\frac{pqK(1+\varepsilon)}{(\lambda-\varepsilon)^2}.$$

Finally, letting $\varepsilon\to 0+$, the above inequality reduces to $\lambda\leq K$, which contradicts with our assumption that $K<\lambda$. Consequently, the constant λ is the best possible value. The proof is now completed. □

We now give a weaker version of inequality (2.1), involving only one partial sum. In fact, we give two equivalent forms of this inequality. Here, the Hardy inequality plays a crucial role.

Theorem 2.2. *Suppose that the assumptions of Theorem* 2.1 *are satisfied. Then hold the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max(m^\lambda,n^\lambda)}<p\left(\sum_{m=1}^{\infty}m^{p-\lambda-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}},\tag{2.5}$$

$$\left[\sum_{m=1}^{\infty}m^{\lambda(q-1)-1}\left(\sum_{n=1}^{\infty}\frac{b_n}{\max(m^\lambda,n^\lambda)}\right)^q\right]^{\frac{1}{q}}<p\left(\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}},\tag{2.6}$$

and they are equivalent. Moreover, p *is the best possible constant factor in both relations.*

Proof. Inequality (2.5) is a consequence of (2.1). Namely, having in mind the weighted Hardy inequality (1.8), we have that

$$\sum_{m=1}^{\infty}m^{-\lambda-1}A_m^p<\left(\frac{p}{\lambda}\right)^p\sum_{m=1}^{\infty}m^{p-\lambda-1}a_m^p,$$

so (2.5) holds. The proof that p is the best possible constant follows the lines of the corresponding part of the proof of Theorem 2.1 and it is omitted here.

Our next step is to show that (2.5) implies inequality (2.6). In order to summarize our further discussion, let I and J stand for the left-hand sides of (2.5) and (2.6), respectively. Now, consider the sequence

$$a_m=m^{\lambda(q-1)-1}\left(\sum_{n=1}^{\infty}\frac{b_n}{\max(m^\lambda,n^\lambda)}\right)^{q-1},\ m\geq 1.$$

Then, taking into account (2.5), we have that

$$\begin{aligned}J^q=\sum_{m=1}^{\infty}m^{p-\lambda-1}a_m^p=I&<p\left(\sum_{m=1}^{\infty}m^{p-\lambda-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}}\\&=pJ^{q-1}\left(\sum_{n=1}^{\infty}n^{-\lambda-1}B_n^q\right)^{\frac{1}{q}},\end{aligned}$$

so (2.6) holds. On the other hand, assuming that (2.6) is valid, we have that

$$\begin{aligned} I &= \sum_{m=1}^{\infty} m^{\frac{1}{q}-\frac{\lambda}{p}} a_m \left(m^{-\frac{1}{q}+\frac{\lambda}{p}} \sum_{n=1}^{\infty} \frac{b_m}{\max(m^\lambda, n^\lambda)} \right) \\ &\leq \left(\sum_{m=1}^{\infty} m^{p-\lambda-1} a_m^p \right)^{\frac{1}{p}} \cdot J, \end{aligned}$$

due to the Hölder inequality. In conclusion, inequalities (2.5) and (2.6) are equivalent. Clearly, this equivalence also provides the best possible constant in (2.5). The proof is now completed. □

Both inequalities (2.1) and (2.5) are more accurate than the Hilbert-type inequality (1.3), which we discuss in the sequel.

Remark. *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be positive sequences such that $\sum_{m=1}^{\infty} a_m < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Clearly, the series $\sum_{m=1}^{\infty} \left(\frac{A_m}{m}\right)^2$ and $\sum_{n=1}^{\infty} \left(\frac{B_n}{n}\right)^2$ converge due to the weighted Hardy inequality (1.8). Now, let $p = q = 2$ and $\lambda = 1$. Then, inequalities (2.1) and (2.5) yield the following interpolating series:*

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\max(m,n)} &< \left(\sum_{m=1}^{\infty} m^{-2} A_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^{-2} B_n^2 \right)^{\frac{1}{2}} \\ &< 2 \left(\sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^{-2} B_n^2 \right)^{\frac{1}{2}} \\ &< 4 \left(\sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This means that inequalities (2.1) and (2.5) represent refinements of the Hilbert-type inequality (1.3). More generally, applying (1.8) to the right-hand side of (2.5), we arrive at the inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\max(m^\lambda, n^\lambda)} < \frac{pq}{\lambda} \left(\sum_{m=1}^{\infty} m^{p-\lambda-1} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{q-\lambda-1} b_n^q \right)^{\frac{1}{q}}.$$

Of course, (2.1) also represents refinement of the above inequality.

3. Integral analogues

We aim here to establish integral versions of the Hilbert-type inequalities from Section 2. Our first result is an integral analogue of Theorem 2.1.

Theorem 3.1. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $\lambda > 0$, and let the functions f, g be non-negative integrable on $(0, \infty)$. If $F(x) = \int_0^x f(u)du$ and $G(x) = \int_0^x g(u)du$, then holds the inequality*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max(x^\lambda, y^\lambda)} dxdy \\ &\leq \lambda \left(\int_0^\infty x^{-\lambda-1} F^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\lambda-1} G^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

provided that $\int_0^\infty x^{-\lambda-1}F^p(x)dx < \infty$ *and* $\int_0^\infty y^{-\lambda-1}G^q(y)dy < \infty$. *In addition, constant* λ *is the best possible in* (3.1).

Proof. Similarly to Theorem 2.1, the starting point in this proof is a suitable transformation of the double integral on the left-hand side of (3.1). More precisely, we have that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max(x^\lambda, y^\lambda)}dxdy \\
&= \int_0^\infty f(x)\left[\int_0^\infty \frac{g(y)}{\max(x^\lambda, y^\lambda)}dy\right]dx \\
&= \int_0^\infty f(x)\left[\int_0^x \frac{g(y)}{\max(x^\lambda, y^\lambda)}dy + \int_x^\infty \frac{g(y)}{\max(x^\lambda, y^\lambda)}dy\right]dx \\
&= \int_0^\infty f(x)\left[\int_0^x \frac{g(y)}{x^\lambda}dy + \int_x^\infty \frac{g(y)}{y^\lambda}dy\right]dx \\
&= \int_0^\infty f(x)\left[\frac{G(x)}{x^\lambda} + \int_x^\infty \frac{g(y)}{y^\lambda}dy\right]dx.
\end{aligned}$$

Furthermore, using the integration by parts, as well as changing the order of integration, we find that

$$\begin{aligned}
\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max(x^\lambda, y^\lambda)}dxdy &= \int_0^\infty f(x)\left[\frac{G(x)}{x^\lambda} - \frac{G(x)}{x^\lambda} + \lambda\int_x^\infty \frac{G(y)}{y^{\lambda+1}}dy\right]dx \\
&= \lambda\int_0^\infty f(x)\left[\int_x^\infty \frac{G(y)}{y^{\lambda+1}}dy\right]dx \\
&= \lambda\int_0^\infty \frac{G(y)}{y^{\lambda+1}}\left[\int_0^y f(x)dx\right]dy \\
&= \lambda\int_0^\infty \frac{G(y)F(y)}{y^{\lambda+1}}dy.
\end{aligned} \tag{3.2}$$

Finally, applying the Hölder inequality to the right-hand side of (3.2), we obtain (3.1), as claimed.

In order to prove that λ is the best possible value in (3.1), suppose that there is $0 < K < \lambda$ such that (3.1) holds when λ is replaced by K. Further, let us define functions

$$f_\varepsilon(x) = \begin{cases} x^{\frac{\lambda-\varepsilon}{p}-1}, & x \geq 1, \\ 0, & 0 < x < 1, \end{cases} \quad \text{and} \quad g_\varepsilon(x) = \begin{cases} x^{\frac{\lambda-\varepsilon}{q}-1}, & x \geq 1, \\ 0, & 0 < x < 1, \end{cases}$$

where $0 < \varepsilon < \lambda$. Then, it follows that

$$F_\varepsilon(x) = \int_1^x t^{\frac{\lambda-\varepsilon}{p}-1}dt < \frac{p}{\lambda-\varepsilon}x^{\frac{\lambda-\varepsilon}{p}} \quad \text{and} \quad G_\varepsilon(x) = \int_1^x t^{\frac{\lambda-\varepsilon}{q}-1}dt < \frac{q}{\lambda-\varepsilon}x^{\frac{\lambda-\varepsilon}{q}},$$

for $x \geq 1$, while $F_\varepsilon(x) = G_\varepsilon(x) = 0$, for $0 < x < 1$. Now, with the above choice of functions, we obtain the following upper bound:

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{\max(x^\lambda, y^\lambda)} dxdy &\leq K \left(\int_0^\infty x^{-\lambda-1} F_\varepsilon^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\lambda-1} G_\varepsilon^q(y)dy \right)^{\frac{1}{q}} \\ &< K \frac{pq}{(\lambda-\varepsilon)^2} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\varepsilon-1} dy \right)^{\frac{1}{q}} \\ &= K \frac{pq}{\varepsilon(\lambda-\varepsilon)^2}.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{\max(x^\lambda, y^\lambda)} dxdy &= \int_1^\infty \int_1^\infty \frac{x^{\frac{\lambda-\varepsilon}{p}-1} y^{\frac{\lambda-\varepsilon}{q}-1}}{\max(x^\lambda, y^\lambda)} dxdy \\ &= \frac{1}{\varepsilon} \left(\frac{q}{\lambda-\varepsilon} + \frac{1}{\frac{\lambda}{p} - \frac{\varepsilon}{q}} \right) - O(1).\end{aligned}$$

Clearly, when $\varepsilon \to 0+$ the above two estimates imply that $\lambda \leq K$, which is a contradiction. This proves our assertion. $\square$

Remark. *It should be noticed here that if $f \equiv g$ and $\lambda = 1$, then (3.2) reduces to the identity*

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{\max(x, y)} dxdy = \int_0^\infty \left(\frac{F(x)}{x} \right)^2 dx.$$

Taking into account this identity, relation (1.5) implies the Hardy inequality (1.6) for $p = 2$. More precisely, we have that

$$\int_0^\infty \left(\frac{F(x)}{x} \right)^2 dx = \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{\max(x, y)} dxdy < 4 \int_0^\infty f^2(x)dx.$$

In order to complete our discussion, we also give an integral analogue of Theorem 2.2.

Theorem 3.2. *Assume that the conditions as in Theorem* 3.1 *are satisfied. Then hold the inequalities*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max(x^\lambda, y^\lambda)} dxdy < p \left(\int_0^\infty x^{p-\lambda-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{-\lambda-1} G^q(y)dy \right)^{\frac{1}{q}},$$

$$\left[\int_0^\infty x^{\lambda(q-1)-1} \left(\int_0^\infty \frac{g(y)}{\max(x^\lambda, y^\lambda)} dy \right)^q dx \right]^{\frac{1}{q}} < p \left(\int_0^\infty y^{-\lambda-1} G^q(y)dy \right)^{\frac{1}{q}},$$

and they are equivalent. Moreover, p is the best possible constant factor in both relations.

Proof. The proof follows the lines of the proof of Theorem 2.2, therefore it is omitted. $\square$

References

[1] L.E. Azar, *Two new forms of Hilbert–type integral inequality*, Math. Inequal. Appl., **17**(3) (2014), 937–946.

[2] V. Adiyasuren, Ts. Batbold, L. E. Azar, *A new discrete Hilbert-type inequality involving partial sums*, J. Inequal. Appl., 127 (2019), 6pp.

[3] Ts. Batbold, M. Krnić, J. Pečarić, P. Vuković, *Further development of Hilbert–type inequalities*, Element, Zagreb, 2017.

[4] G. Bennett, *Some elementary inequalities*, Quart. J. Math. Oxford Ser. (2) 38 (1987), no. 152, 401–425.

[5] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, second edition, Cambridge University Press, Cambridge, 1967.

[6] X. S. Huang, R. Luo, B. Yang, *On a new extended half-discrete Hilbert's inequality involving partial sums*, J. Inequal. Appl., 16 (2020), 14pp.

[7] Z. Gu, B.Yang, *An extended Hardy-Hilbert's inequality with parameters and applications*, J. Math. Inequal., **15**(4) (2021), 1375–1389.

[8] A. Kufner, L. Maligranda, L.-E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly, 113(2006), 715-732.

[9] A. Kufner, L. Maligranda, L.E. Persson, *The Hardy inequality–About its history and some related results*, Vydavatelský servis, Pilsen, 2007.

[10] M. Krnić, J. Pečarić, *General Hilbert's and Hardy's inequalities*, Math. Inequal. Appl. **8** (2005), 29–52.

[11] M. Krnić, J. Pečarić, I. Perić, P. Vuković, *Recent advances in Hilbert-type inequalities*, Element, Zagreb, 2012.

[12] H. Mo, B. Yang, *On a new Hilbert-type integral inequality involving the upper limit functions*, J. Inequal. Appl., 5 (2020), 12pp.

[13] A. O. K. Nizar, L. E. Azar, A. H. A. Bataineh, *A sharper form of half-discrete Hilbert inequality related to Hardy inequality*, Filomat **32**(19) (2018), 6733–6740.

[14] J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc, 1992.

[15] S. Wu, X. Huang, B. Yang, *A half-discrete Hardy-Mulholland-type inequality involving one multiple upper limit function*, J. South China Norm. Univ., Nat. Sci. Ed., **54**(1) (2022), 100–106.

[16] J. Zhong, B. Yang, Q. Chen, *A more accurate half-discrete hilbert-type inequality involving one higher-order derivative function*, J. Appl. Anal. Comput., **12**(1) (2022), 378–391.

Laith Emil Azar
Department of Mathematics, Al al-Bayt University, P.O. Box 130095, Mafraq, Jordan
E-mail address: azar1@aabu.edu.jo

Tserendorj Batbold
Department of Mathematics, National University of Mongolia, Ulaanbaatar 14201, Mongolia
E-mail address: tsbatbold@hotmail.com

Nizar Kh. Al-Oushoush
Department of Mathematics, Faculty of Science, Al-Balqa Applied University, P.O. Box 19117, Salt, Jordan
E-mail address: aloushoush@bau.edu.jo

Mario Krnić
Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia
E-mail address: mario.krnic@fer.hr