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ON A NOTABLE INEQUALITY

VASILE CIRTOAJE

Abstract. In this paper, we give a proof of the inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\geq\frac{n}{2}$$

for $n\leq 8$ and nonnegative real numbers $a_1,a_2,\ldots,a_n$ such that

$$\sum_{1\leq i<j\leq n}a_ia_j=\frac{n(n-1)}{2}.$$

In addition, some more general results that might help to prove the inequality for $n\geq 9$ are presented.

1. Introduction

The following inequality was proposed and proved in [2] in 2005: If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}.$$

Problem 3 given at the Olympic Revenge Contest from Brazil-2013 [3] has the following statement: If a,b,c,d are nonnegative real numbers such that $ab+ac+ad+bc+bd+cd=6$, then

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1}\geq 2.$$

In the same year, Henrique Vaz posted this inequality on the website Art of Problem Solving [4], where three known readers have posted distinct proofs. Note that all methods applied here for four variables fail for more variables. Moreover, as far as we know, no proof for $n\geq 5$ has been published anywhere for the inequality

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_n^2+1}\geq\frac{n}{2},$$

where $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that

$$\sum_{1\leq i<j\leq n}a_ia_j=\frac{n(n-1)}{2}.$$

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In this paper, a proof for $n \le 8$ is given. Sections 2 and 3 provide two useful lemmas and two propositions, respectively, while Section 4 provides the main result of the paper, materialized in a theorem easily applicable to our inequality for $n \le 8$. Section 5 provides the proof of the inequality for $n = 3, 4, \ldots, 8$.

2. Two helpful lemmas

Lemma 2.1. *Let x_1 and y_1 be fixed nonnegative real variables such that $x_1 \ge y_1 \ge 0$, $x_1 y_1 \le 1$ and $x_1 y_1 + a(x_1 + y_1) = b$, where $a > 0$, $b \ge 0$ and $2a + 1 \ge b$. If x and y are nonnegative real variables such that*

$$x \ge y \ge 0, \qquad xy \le 1, \qquad xy + a(x + y) = b,$$

then the expression

$$E = \frac{1}{x^2 + 1} + \frac{1}{y^2 + 1}$$

has the minimum value for $y = 0$ or $x = y$.

Proof. If $b = 0$, then $x = y = 0$ and $E = 2$. Assume now that $b > 0$ and x is a function of y. From $xy + a(x + y) = b$, we get $x = f(y)$, where

$$f(y) = \frac{b - ay}{y + a}, \qquad y \in [0, M], \quad M = \sqrt{a^2 + b} - a \le 1.$$

Note that the maximum value of y follows from

$$b = xy + a(x + y) \ge y^2 + 2ay,$$

and occurs when $x = y$. By deriving the equation $xy + a(x + y) = b$, we obtain

$$x'y + x + a(x' + 1) = 0, \qquad x' = \frac{-(x + a)}{y + a},$$

therefore

$$\frac{1}{2}E'(y) = \frac{-xx'}{(x^2 + 1)^2} - \frac{y}{(y^2 + 1)^2} = \frac{x(x + a)}{(y + a)(x^2 + 1)^2} - \frac{y}{(y^2 + 1)^2}$$

$$= \frac{a[x(y^2 + 1)^2 - y(x^2 + 1)^2] + x^2(y^2 + 1)^2 - y^2(x^2 + 1)^2}{(y + a)(x^2 + 1)^2(y^2 + 1)^2}$$

$$= \frac{(x - y)A}{(y + a)(x^2 + 1)^2(y^2 + 1)^2},$$

where

$$A = a[(1 - xy)^2 - xy(x + y)^2] + (x + y)(1 - x^2y^2).$$

Having in view that

$$x + y = \frac{b - xy}{a},$$

we get

$$A = \frac{xyF}{a}, \qquad F = (a^2 + b)\left(xy + \frac{1}{xy}\right) - 2a^2 - b^2 - 1.$$

Since

$$(xy)' = x'y + x = \frac{a(x - y)}{y + a},$$

it follows

$$F'(y) = (a^2 + b)\left(1 - \frac{1}{x^2y^2}\right)(xy)' = \frac{a(a^2 + b)(x - y)}{y + a}\left(1 - \frac{1}{x^2y^2}\right) \le 0.$$

Therefore, $F(y)$ is a decreasing function. Since $F(0+) = \infty$, two cases are possible: $F(y) \ge 0$ for all $y \in [0, M]$, or $F(y) \ge 0$ for $y \in [0, y_1]$ and $F(y) \le 0$ for $y \in [y_1, M]$. Because $E'(y)$ has the same sign as $F(y)$, it follows that $E(y)$ is a strictly increasing function, or is strictly increasing on $[0, y_1]$ and strictly decreasing on $[y_1, M]$. In both cases, $E(y)$ has the minimum value for an extreme value of y, i.e. for $y = 0$ or $y = x$. □

Lemma 2.2. *Let x_1 and y_1 be fixed nonnegative real variables such that $x_1 \ge y_1$, $x_1 y_1 \ge 1$ and $a(a-1)x_1^2 + 2ax_1y_1 + 2b(ax_1 + y_1) = 2c$, where $a \ge 1$ and $b, c \ge 0$. If x and y are nonnegative real variables such that*

$$x \ge y, \qquad xy \ge 1, \qquad a(a-1)x^2 + 2axy + 2b(ax + y) = 2c,$$

then the expression

$$E = \frac{a}{x^2+1} + \frac{1}{y^2+1}$$

has the minimum value for $x = y$.

Proof. First we show that

$$2c \ge (a+1)(a+2b).$$

Indeed, we have

$$\begin{aligned} 2c - (a+1)(a+2b) &= a(a-1)x_1^2 + 2ax_1y_1 + 2b(ax_1 + y_1) - (a+1)(a+2b) \\ &\ge a(a-1)x_1^2 + 2a + 2b\left(ax_1 + \frac{1}{x_1}\right) - (a+1)(a+2b) \\ &= a(a-1)(x_1^2 - 1) + 2ab(x_1 - 1) - \frac{2b(x_1-1)}{x_1} \\ &= (x_1 - 1)\left[a(a-1)(x_1+1) + 2ab - \frac{2b}{x_1}\right] \\ &\ge (x_1 - 1)\left[2a(a-1) + 2ab - 2b\right] = 2(x_1-1)(a-1)(a+b) \ge 0. \end{aligned}$$

Assume now that y is a function of x. By deriving the given constraint, we get

$$a(a-1)x + a(y + xy') + b(a + y') = 0, \qquad y' = -a \cdot \frac{(a-1)x + y + b}{ax + b},$$

hence

$$\frac{1}{2a}E'(x) = \frac{-x}{(x^2+1)^2} - \frac{yy'}{a(y^2+1)^2} = \frac{-x}{(x^2+1)^2} + \frac{y}{(y^2+1)^2} \cdot \frac{(a-1)x + y + b}{ax + b}.$$

Since

$$\frac{(a-1)x + y + b}{ax + b} - \frac{(a-1)x + y}{ax} = \frac{b(x-y)}{ax(ax+b)} \ge 0,$$

we have

$$\begin{aligned} \frac{1}{2a}E'(x) &\ge \frac{-x}{(x^2+1)^2} + \frac{y}{(y^2+1)^2} \cdot \frac{(a-1)x + y}{ax} \\ &= \frac{(x^2-y^2)(x^2y^2-1) + (a-1)x(x-y)[xy(x^2+xy+y^2) + 2xy - 1]}{ax(x^2+1)^2(y^2+1)^2} \ge 0. \end{aligned}$$

Therefore, $E(x)$ is strictly increasing and $E(x) \ge E(y)$. To complete the proof, we need to show that $x = y$ involves $xy \ge 1$, which means $x \ge 1$. Indeed, for $x = y$, from the given constraint $a(a-1)x^2 + 2axy + 2b(ax + y) = 2c$, we get

$$a(a+1)x^2 + 2b(a+1)x = 2c,$$

$$a(a+1)x^2+2b(a+1)x \geq (a+1)(a+2b),$$

$$(a+1)(x-1)(ax+a+2b) \geq 0,$$

hence $x \geq 1$.

□

3. Two helpful propositions

Proposition 3.1. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that*

$$a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n \geq 0, \qquad \sum_{1 \leq i < j \leq n} a_i a_j = \frac{n(n-1)}{2}.$$

For fixed $a_1, a_2, \ldots, a_{n-2}$, the expression

$$E = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1}$$

has the minimum value when $a_n = 0$ or $a_{n-1} = a_n$.

Proof. For fixed $a_1, a_2, \ldots, a_{n-2}$, denoting

$$x = a_{n-1}, \quad y = a_n \qquad (x \geq y \geq 0),$$

$$a = \sum_{i=1}^{n-2} a_i, \qquad b = \frac{n(n-1)}{2} - \sum_{1 \leq i < j \leq n-2} a_i a_j,$$

we have $a > 0$, $b \geq 0$, $xy \leq 1$ and $a(x+y)+xy = b$. In addition, since

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt{\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j} = 1,$$

$$\sum_{i=1}^{n-2} a_i \geq \frac{n-2}{n} \sum_{i=1}^{n} a_i \geq n-2,$$

$$\sum_{1 \leq i < j \leq n-2} a_i a_j \geq \frac{(n-2)(n-3)}{n(n-1)} \sum_{1 \leq i < j \leq n} a_i a_j = \frac{(n-2)(n-3)}{2},$$

we have

$$2a+1-b = 2\sum_{i=1}^{n-2} a_i + \sum_{1 \leq i < j \leq n-2} a_i a_j - \frac{n^2-n-2}{2}$$

$$\geq 2(n-2) + \frac{(n-2)(n-3)}{2} - \frac{n^2-n-2}{2} = 0.$$

By Lemma 2.1, the expression $\frac{1}{x^2+1} + \frac{1}{y^2+1}$ has the minimum value when $y = 0$ or $x = y$. As a consequence, the expression

$$E = \frac{1}{a_1^2+1} + \cdots + \frac{1}{a_{n-2}^2+1} + \left(\frac{1}{x^2+1} + \frac{1}{y^2+1}\right)$$

has the minimum value when $y = 0$ or $x = y$, that is when $a_n = 0$ or $a_{n-1} = a_n$. □

Proposition 3.2. *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad \sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2}.$$

For fixed $a_{k+2}, \dots, a_n$ and

$$a_1 = a_2 = \cdots = a_k, \qquad a_k a_{k+1} \ge 1, \qquad k \in \{1, 2, \dots, n-2\},$$

the expression

$$E = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1}$$

has the minimum value when $a_1 = a_2 = \cdots = a_{k+1}$.

Proof. Denoting

$$x = a_k, \quad y = a_{k+1} \qquad (x \ge y, \quad xy \ge 1),$$

$$a = k, \qquad b = a_{k+2} + \cdots + a_n, \qquad c = \frac{n(n-1)}{2} - \sum_{k+2 \le i < j \le n} a_i a_j,$$

we have

$$a(a-1)x^2 + 2axy + 2b(ax+y) = 2c.$$

By Lemma 2.2, the expression $\frac{a}{x^2+1} + \frac{1}{y^2+1}$ has the minimum value when $x = y$. As a consequence, the expression

$$E = \left(\frac{a}{x^2+1} + \frac{1}{y^2+1} \right) + \frac{1}{a_{k+1}^2+1} + \cdots + \frac{1}{a_n^2+1}$$

has the minimum value when $x = y$, that means $a_1 = a_2 = \cdots = a_{k+1}$. □

4. The main result

Based on Propositions 3.1 and 3.2, we can state and prove the following theorem:

Theorem 4.1. *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad \sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2}.$$

Let $k = \left\lfloor \frac{n}{2} \right\rfloor$. If the inequality

$$\frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1} \ge \frac{n}{2}$$

holds for both cases

(a) $a_1 = a_2 = \cdots = a_{k+1}$ *and* $a_n = 0$,

(b) $a_1 = a_2 = \cdots = a_{k+1}$ *and* $a_{n-1} = a_n$,

then it holds for all $a_1, a_2, \dots, a_n$.

Proof. Because the domain

$$D = \left\{ (a_1, \dots, a_n) \in \mathbb{R}_+^n : \sum_{1 \le i < j \le n} a_i a_j = \frac{n(n-1)}{2} \right\}$$

is a non-empty compact set in $\mathbb{R}_+^n$, the expression

$$E = \frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1}$$

achieves its minimum. Since $a_{n-1}a_n \le 1$, by Proposition 3.1 it follows that E has the minimum value for $a_n = 0$ or $a_{n-1} = a_n$. Next, we will show by induction that it suffices to prove the desired inequality $E \ge \frac{n}{2}$ for $a_1 = a_2 = \cdots = a_{k+1}$, where $k \le \left\lfloor \frac{n}{2} \right\rfloor$. For $k = 1$, we have $a_k a_{k+1} = a_1 a_2 \ge 1$. Thus, from Proposition 3.2, it follows that E has the minimum value when $a_1 = a_2$. Assume now that the following statement is true: "If the inequality $E \ge \frac{n}{2}$ holds for $a_1 = a_2 = \cdots = a_k$, then it holds for all a_i" and show that "If the inequality $E \ge \frac{n}{2}$ holds for $a_1 = a_2 = \cdots = a_{k+1}$, then it holds for all a_i". There are two cases to consider: $a_k a_{k+1} \ge 1$ and $a_k a_{k+1} \le 1$.

Case 1: $a_k a_{k+1} \ge 1$. According to Proposition 3.2, the expression E has the minimum value when $a_1 = a_2 = \cdots = a_{k+1}$. Therefore, the inequality $E \ge \frac{n}{2}$ holds for all a_i if it holds for $a_1 = a_2 = \cdots = a_k = a_{k+1}$.

Case 2: $a_k a_{k+1} \le 1$. For $a_1 = a_2 = \cdots = a_k$, the inequality $E \ge \frac{n}{2}$ has the form

$$\frac{k}{a_k^2+1} + \frac{1}{a_{k+1}^2+1} + \frac{1}{a_{k+2}^2+1} + \cdots + \frac{1}{a_n^2+1} \ge \frac{n}{2}.$$

Since $a_{k+1} \ge a_{k+2} \ge \cdots \ge a_n$, it suffices to show that

$$\frac{k}{a_k^2+1} + \frac{n-k}{a_{k+1}^2+1} \ge \frac{n}{2},$$

which is equivalent to the obvious inequality

$$(n-2k)(a_k^2 - a_{k+1}^2) + n(1 - a_k^2 a_{k+1}^2) \ge 0.$$

□

5. Proving the inequality for $n = 3, 4, \ldots, 8$

Application 1: $n = 3$**.** *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3,$$

then

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \ge \frac{3}{2}.$$

Proof. Assume that $a \ge b \ge c$. By Theorem 1, it suffices to consider the cases $a = b$, $b = c$ (hence $a = b = c$) and $a = b$, $c = 0$. The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$ (or any cyclic permutation). □

Application 2: $n = 4$**.** *If a, b, c, d are nonnegative real numbers such that*

$$ab + ac + ad + bc + bd + cd = 6,$$

then

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \ge 2.$$

Proof. Assume that $a \ge b \ge c \ge d$. By Theorem 1, it suffices to consider the cases $a = b = c$, $c = d$ (hence $a = b = c = d$) and $a = b = c$, $d = 0$. The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = \sqrt{2}$ and $d = 0$ (or any cyclic permutation). □

Application 3: $n = 5$**.** *If* a, b, c, d, e *are nonnegative real numbers such that*

$$a(b+c+d+e)+b(c+d+e)+c(d+e)+de = 10,$$

then

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1}+\frac{1}{e^2+1} \ge \frac{5}{2}.$$

Proof. Assume that $a \ge b \ge c \ge d \ge e$. By Theorem 1, it suffices to consider two cases: $a = b = c$, $d = e$ and $a = b = c$, $e = 0$.

Case 1: $a = b = c$, $d = e$. We need to show that $3a^2 + 6ad + d^2 = 10$ involves

$$\frac{3}{a^2+1}+\frac{2}{d^2+1} \ge \frac{5}{2},$$

which can be written as follows:

$$\frac{3}{13a^2+6ad+d^2}+\frac{2}{3a^2+6ad+11d^2} \ge \frac{5}{2(3a^2+6ad+d^2)},$$

$$a^4+8a^3d-18a^2d^2+8ad^3+d^4 \ge 0, \qquad (a-d)^2(a^2+10ad+d^2) \ge 0.$$

Case 2: $a = b = c$, $e = 0$. We need to show that $3a^2 + 3ad = 10$ involves

$$\frac{3}{a^2+1}+\frac{1}{d^2+1} \ge \frac{3}{2},$$

which can be written as follows:

$$\frac{3}{a(13a+3d)}+\frac{1}{3a^2+3ad+10d^2} \ge \frac{1}{2a(a+d)},$$

$$\frac{1}{3a^2+3ad+10d^2} \ge \frac{7a-3d}{2a(a+d)(13a+3d)},$$

$$a^3+4a^2d-11ad^2+6d^3 \ge 0, \qquad (a-d)^2(a+6d) \ge 0.$$

The equality occurs for $a = b = c = d = e = 1$, and also for $a = b = c = d = \sqrt{\frac{5}{3}}$ and $e = 0$ (or any cyclic permutation). □

Application 4: $n = 6$**.** *If* a, b, c, d, e, f *are nonnegative real numbers such that*

$$a(b+c+d+e+f)+b(c+d+e+f)+c(d+e+f)+d(e+f)+ef = 15,$$

then

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}+\frac{1}{d^2+1}+\frac{1}{e^2+1}+\frac{1}{f^2+1} \ge 3.$$

Proof. Assume that $a \ge b \ge c \ge d \ge e \ge f$. By Theorem 1, it suffices to consider two cases: $a = b = c = d$, $e = f$ and $a = b = c = d$, $f = 0$.

Case 1: $a = b = c = d$, $e = f$. We need to show that

$$6a^2 + 8ae + e^2 = 15$$

involves

$$\frac{4}{a^2+1}+\frac{2}{e^2+1} \ge 3,$$

which is equivalent to

$$\frac{4}{21a^2+8ae+e^2}+\frac{1}{3a^2+4ae+8e^2}\geq\frac{3}{6a^2+8ae+e^2},$$

$$3a^4+28a^3d-62a^2d^2+28ad^3+3d^4\geq 0,$$

$$(a-d)^2(3a^2+34ad+3d^2)\geq 0.$$

Case 2: $a=b=c=d$, $f=0$. We need to show that

$$6a^2+4ae=15$$

involves

$$\frac{4}{a^2+1}+\frac{1}{e^2+1}\geq 2,$$

that is equivalent to

$$\frac{4}{a(21a+4e)}+\frac{1}{6a^2+4ae+15e^2}\geq\frac{1}{a(3a+2e)},$$

$$\frac{1}{6a^2+4ae+15e^2}\geq\frac{9a-4e}{a(3a+2e)(21a+4e)},$$

$$3a^3+14a^2e-37ae^2+20e^3\geq 0,\qquad (a-e)^2(3a+20e)\geq 0.$$

The equality occurs for $a=b=c=d=e=f=1$, and also for $a=b=c=d=e=\sqrt{\frac{3}{2}}$ and $e=0$ (or any cyclic permutation). □

Application 5: $n=7$. *If* $a_1,a_2,\ldots,a_7$ *are nonnegative real numbers such that*

$$\sum_{1\leq i<j\leq 7}a_ia_j=21,$$

then

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+\cdots+\frac{1}{a_7^2+1}\geq\frac{7}{2}.$$

Proof. Assume that $a_1\geq a_2\geq\ldots\geq a_7$. By Theorem 1, it suffices to consider two cases: $a_1=a_2=a_3=a_4$, $a_6=a_7$ and $a_1=a_2=a_3=a_4$, $a_7=0$.

Case 1: $a_1=a_2=a_3=a_4:=a$, $a_6=a_7:=c$. Denoting $a_5=b$, we need to show that

$$\frac{4}{a^2+1}+\frac{1}{b^2+1}+\frac{2}{c^2+1}\geq\frac{7}{2}$$

for

$$6a^2+4a(b+2c)+2bc+c^2=21,\qquad a\geq b\geq c.$$

From

$$21=6a^2+4a(b+2c)+2bc+c^2\leq 21a^2,$$

we get $a\geq 1$. On the other hand, since

$$\frac{1}{b^2+1}=1-\frac{b^2}{b^2+1}\geq 1-\frac{b}{2}$$

and

$$\frac{1}{c^2+1}\geq 1-\frac{c}{2},$$

it suffices to show that

$$\frac{8}{a^2+1}+1\geq b+2c.$$

From

$$0 = 4[6a^2 + 4a(b + 2c) + 2bc + c^2 - 21] \le 3(a - b)(a - c) + (a - b)(b - c),$$

we get

$$4[6a^2 + 4a(b + 2c) + 2bc + c^2 - 21] \le 3a^2 - 2a(b + 2c) + 4bc - b^2,$$

$$(b + 2c)^2 + 18a(b + 2c) + 21a^2 - 84 \le 0,$$

$$b + 2c \le -9a + \sqrt{60a^2 + 84}.$$

Thus, we only need to show that

$$\frac{8}{a^2 + 1} + 1 \ge -9a + \sqrt{60a^2 + 84},$$

which is equivalent to

$$\left(\frac{9a^3 - a^2 + 9a + 7}{a^2 + 1}\right)^2 \ge 60a^2 + 84,$$

$$21a^6 - 18a^5 - 41a^4 + 108a^3 - 161a^2 + 126a - 35 \ge 0,$$

$$(a - 1)^2(21a^4 + 24a^3 - 14a^2 + 56a - 35) \ge 0.$$

Since $a \ge 1$, the last inequality is clearly true.

Case 2: $a_1 = a_2 = a_3 = a_4 := a$, $a_7 = 0$. Denoting $a_5 = b$ and $a_6 = c$, we need to show that

$$\frac{4}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \ge \frac{5}{2}$$

for

$$6a^2 + 4a(b + c) + bc = 21, \qquad a \ge b \ge c.$$

Sub-case 2-a: $bc \ge 1$. Since

$$\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} - \frac{2}{bc + 1} = \frac{(bc - 1)(b - c)^2}{(b^2 + 1)(c^2 + 1)(bc + 1)} \ge 0,$$

it is enough to show that

$$\frac{4}{a^2 + 1} + \frac{2}{bc + 1} \ge \frac{5}{2}.$$

We have

$$6a^2 + 8a\sqrt{bc} + bc \le 21.$$

Putting

$$a = x\sqrt{bc}, \qquad x \ge 1,$$

we need to show that

$$bc \le \frac{21}{6x^2 + 8x + 1}$$

involves

$$\frac{4}{x^2bc + 1} + \frac{2}{bc + 1} \ge \frac{5}{2}.$$

Clearly, it suffices to prove this inequality for $bc = \dfrac{21}{6x^2 + 8x + 1}$. So, we need to show that

$$\frac{4}{27x^2 + 8x + 1} + \frac{1}{3x^2 + 4x + 11} \ge \frac{5}{2(6x^2 + 8x + 1)},$$

that is equivalent to

$$9x^4 + 36x^3 - 94x^2 + 44x + 5 \ge 0,$$

$$(x-1)^2(9x^2+54x+5) \geq 0.$$

Sub-case 2-b: $bc \leq 1$. Since

$$\frac{1}{b^2+1} = 1 - \frac{b^2}{b^2+1} \geq 1 - \frac{b}{2}$$

and

$$\frac{1}{c^2+1} \geq 1 - \frac{c}{2},$$

it suffices to show that

$$\frac{8}{a^2+1} \geq b+c+1.$$

From

$$21 = 6a^2 + 4a(b+c) + bc \leq 6a^2 + 4a(b+c) + 1,$$

we get

$$b+c \geq \frac{10-3a^2}{2a}.$$

In addition, from $(a-b)(a-c) \geq 0$, which implies $bc \geq a(b+c) - a^2$, we get

$$21 = 6a^2 + 4a(b+c) + bc \geq 6a^2 + 4a(b+c) + a(b+c) - a^2,$$

hence

$$\frac{10-3a^2}{2a} \leq b+c \leq \frac{21-5a^2}{5a}, \qquad a^2 \leq \frac{21}{5}.$$

Also, from

$$\frac{10-3a^2}{2a} \leq \frac{21-5a^2}{5a},$$

we get $a^2 \geq \dfrac{8}{5}$. Thus, it suffices to show that

$$\frac{8}{5} \leq a^2 \leq \frac{21}{5}$$

involves

$$\frac{8}{a^2+1} \geq \frac{21-5a^2}{5a} + 1,$$

which is equivalent to

$$5a^4 - 5a^3 - 16a^2 + 35a - 21 \geq 0,$$

$$(5a^2-8)(3a^2-3a-1) + (3-a)(19a-24) + 1 \geq 0.$$

This is true since

$$5a^2 - 8 \geq 0, \qquad 3-a > 0, \qquad 19a-24 > 0, \qquad 3a^2-3a-1 > 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_7 = 1$, and also for $a_1 = a_2 = \cdots = a_6 = \sqrt{\dfrac{7}{5}}$ and $a_7 = 0$ (or any cyclic permutation). □

Application 6: $n = 8$**.** *If* $a_1, a_2, \ldots, a_8$ *are nonnegative real numbers such that*

$$\sum_{1 \leq i < j \leq 8} a_i a_j = 28,$$

then

$$\frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_8^2+1} \geq 4.$$

Proof. Assume that $a_1 \ge a_2 \ge \ldots \ge a_8$. By Theorem 1, it suffices to consider two cases: $a_1 = a_2 = a_3 = a_4 = a_5$, $a_7 = a_8$ and $a_1 = a_2 = a_3 = a_4 = a_5$, $a_8 = 0$.

Case 1: $a_1 = a_2 = a_3 = a_4 = a_5 := a$, $a_7 = a_8 := c$. Denoting $a_6 = b$, we need to show that

$$\frac{5}{a^2+1} + \frac{1}{b^2+1} + \frac{2}{c^2+1} \ge 4$$

for

$$10a^2 + 5a(b+2c) + 2bc + c^2 = 28, \qquad a \ge b \ge c.$$

From

$$28 = 10a^2 + 5a(b+2c) + 2bc + c^2 \le 28a^2,$$

we get $a \ge 1$. On the other hand, since

$$\frac{1}{b^2+1} = 1 - \frac{b^2}{b^2+1} \ge 1 - \frac{b}{2}$$

and

$$\frac{1}{c^2+1} \ge 1 - \frac{c}{2},$$

it suffices to show that

$$\frac{10}{a^2+1} \ge 2 + b + 2c.$$

From

$$0 = 4[10a^2 + 5a(b+2c) + 2bc + c^2 - 28] \le 3(a-b)(a-c) + (a-b)(b-c),$$

we get

$$4[10a^2 + 5a(b+2c) + 2bc + c^2 - 28] \le 3a^2 - 2a(b+2c) + 4bc - b^2,$$

$$(b+2c)^2 + 22a(b+2c) + 37a^2 - 112 \le 0,$$

$$b + 2c \le -11a + \sqrt{84a^2 + 112}.$$

Thus, we only need to show that

$$\frac{10}{a^2+1} \ge 2 - 11a + \sqrt{84a^2 + 112},$$

which is equivalent to

$$\left(\frac{11a^3 - 2a^2 + 11a + 8}{a^2+1}\right)^2 \ge 84a^2 + 112,$$

$$37a^6 - 44a^5 - 34a^4 + 132a^3 - 219a^2 + 176a - 48 \ge 0,$$

$$(a-1)^2(37a^4 + 30a^3 - 11a^2 + 80a - 48) \ge 0.$$

Since $a \ge 1$, the last inequality is clearly true.

Case 2: $a_1 = a_2 = a_3 = a_4 = a_5 := a$, $a_8 = 0$. Denoting $a_6 = b$ and $a_7 = c$, we need to show that

$$\frac{5}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \ge 3$$

for

$$10a^2 + 5a(b+c) + bc = 28, \qquad a \ge b \ge c.$$

Sub-case 2-a: $bc \ge 1$. Since

$$\frac{1}{b^2+1} + \frac{1}{c^2+1} \ge \frac{2}{bc+1},$$

it is enough to show that

$$\frac{5}{a^2+1}+\frac{2}{bc+1}\geq 3.$$

We have

$$10a^2+10a\sqrt{bc}+bc\leq 28.$$

Putting

$$a=x\sqrt{bc},\qquad x\geq 1,$$

we need to show that

$$bc\leq\frac{28}{10x^2+10x+1}$$

involves

$$\frac{5}{x^2bc+1}+\frac{2}{bc+1}\geq 3.$$

Clearly, it suffices to prove this inequality for $bc=\frac{28}{10x^2+10x+1}$. So, we need to show that

$$\frac{5}{38x^2+10x+1}+\frac{2}{10x^2+10x+29}\geq\frac{3}{10x^2+10x+1},$$

which is equivalent to

$$6x^4+26x^3-67x^2+32x+3\geq 0,$$

$$(x-1)^2(6x^2+38x+3)\geq 0.$$

Sub-case 2-b: $bc\leq 1$. Since

$$\frac{1}{b^2+1}=1-\frac{b^2}{b^2+1}\geq 1-\frac{b}{2}$$

and

$$\frac{1}{c^2+1}\geq 1-\frac{c}{2},$$

it suffices to show that

$$\frac{10}{a^2+1}\geq 2+b+c.$$

From

$$28=10a^2+5a(b+c)+bc\leq 10a^2+5a(b+c)+1,$$

we get

$$b+c\geq\frac{27-10a^2}{5a}.$$

In addition, from $(a-b)(a-c)\geq 0$, that implies $bc\geq a(b+c)-a^2$, we get

$$28=10a^2+5a(b+c)+bc\geq 10a^2+5a(b+c)+bc+a(b+c)-a^2,$$

hence

$$\frac{27-10a^2}{5a}\leq b+c\leq\frac{28-9a^2}{6a},\qquad a^2\leq\frac{28}{9}.$$

From

$$\frac{27-10a^2}{5a}\leq\frac{28-9a^2}{6a},$$

we get $a^2\geq\frac{22}{15}$. Thus, it suffices to show that

$$\frac{22}{15}\leq a^2\leq\frac{28}{9}$$

involves
$$\frac{10}{a^2+1} \ge 2 + \frac{28-9a^2}{6a},$$
which is equivalent to
$$9a^4 - 12a^3 - 19a^2 + 48a - 28 \ge 0,$$
$$(3a^2-4)(3a^2-4a+1) + 2(2-a)(5a-6) \ge 0.$$
It is true since
$$3a^2 - 4 > 0, \quad 3a^2 - 4a + 1 > 0, \quad 2 - a > 0, \quad 5a - 6 > 0.$$
The equality occurs for $a_1 = a_2 = \cdots = a_8 = 1$, and also for $a_1 = a_2 = \cdots = a_7 = \frac{2}{\sqrt{3}}$ and $a_8 = 0$ (or any cyclic permutation). □

Remark. In our opinion, the following generalization holds:

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that
$$\sum_{1\le i<j\le n} a_i a_j = \frac{n(n-1)}{2},$$
then
$$\frac{1}{a_1^2+1} + \frac{1}{a_2^2+1} + \cdots + \frac{1}{a_n^2+1} \ge \frac{n}{2},$$
with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \sqrt{\frac{n}{n-2}}$ and $a_n = 0$ (or any cyclic permutation).

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Vasile Cirtoaje
University of Ploiesti, Department Automation and Computers, Bdul Bucuresti, No. 39, Ploiesti, Romania
E-mail address: vcirtoaje@upg-ploiesti.ro