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SOME INEQUALITIES FOR THE HYPERBOLIC TANGENT RELATED $\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^p}\right)}$

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Abstract. In this paper, we prove the following inequality with hyperbolic tangent: for $x>0$, we have

$$\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\alpha}}\right)}<\tanh x<\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\beta}}\right)},$$

where the constants $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{6}$ are the best possible.

1. Introduction

In [1] Problem 51, Ivády proposed the following problem. Show that, for $x>0$,

$$\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{2}}}\right)}<\tanh x<\sqrt[3]{1-exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)} \tag{1.1}$$

holds. A few years later, in [2], the proposer himself gave the proof. In [2], Ivády's proof of the left-hand side of the inequality (1.1) is correct, but the proof of the right-hand side is not correct. Recently, Zhang and Chen [5] gave the correct proof for the right-hand side of the inequality (1.1). The inequality (1.1) is an interesting inequality that evaluates $\tanh x$ from both the left and right sides. In this, paper, we show the two theorems and one conjecture related the inequality (1.1).

Theorem 1.1. *For $x>0$, we have*

$$\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\alpha}}\right)}<\tanh x<\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\beta}}\right)}, \tag{1.2}$$

where the constants $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{6}$ are the best possible.

From the following theorem, it can be seen that the right-hand side of the inequality (1.2) is a stronger evaluation formula than the inequality (1.1) near $x=0$.

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Theorem 1.2. *For $0 < x < 1$, we have*

$$\sqrt{1 - exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{6}}}\right)} < \sqrt[3]{1 - exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)}.$$

2. Some Lemmas to prove main theorems

Lemma 2.1. *For $0 < x \le \sqrt{2}$, we have $4x + 8x^3 > e^{2x} - e^{-2x}$.*

Proof. By the inequality 3.6.6 [3] in pp 269, we have

$$e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{2(3-x)}$$

for $0 < x < 2$ and

$$\begin{aligned}
&4x + 8x^3 - e^{2x} + e^{-2x} \\
>&4x + 8x^3 - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2(3-x)}\right)^2 + \frac{1}{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{2(3-x)}\right)^2} \\
=&\frac{x^3(6912 + 4536x - 720x^2 - 1624x^3 - 352x^4 + 63x^5 + 32x^6)}{4(-3+x)^2(6+4x+x^2)^2} \\
=&\frac{x^3 f(x)}{4(-3+x)^2(6+4x+x^2)^2}.
\end{aligned}$$

Since we have $f(x) > 6912+4536\cdot0-720\cdot1-1624\cdot1-352\cdot1+63\cdot0+32\cdot0 = 4216 > 0$ for $0 < x < 1$ and

$$\begin{aligned}
f(x) > &6912 + 4536\cdot1 - 720\cdot\left(\sqrt{2}\right)^2 - 1624\cdot\left(\sqrt{2}\right)^3 - 352\cdot\left(\sqrt{2}\right)^4 \\
&+ 63\cdot1 + 32\cdot1 = 8695 - 3248\sqrt{2} \cong 4101.63 > 0
\end{aligned}$$

for $1 < x \le \sqrt{2}$, we obtain $f(x) > 0$ and $4x + 8x^3 - e^{2x} + e^{-2x} > 0$ for $0 < x \le \sqrt{2}$. □

Lemma 2.2. *For $x > 0$, we have $-x - \ln 2 + \ln(1 + e^{2x}) > 0$.*

Proof. By $(1 - e^x)^2 > 0$ for $x > 0$, we have $\frac{1+e^{2x}}{2e^x} > 1$ and $\ln\left(\frac{1+e^{2x}}{2e^x}\right) > 0$. Hence, $-x - \ln 2 + \ln(1 + e^{2x}) > 0$ for $x > 0$. □

Lemma 2.3. *For $x > 0$, we have $e^{2x} - e^{-2x} > 4x + \frac{8x^3}{3}$.*

Proof. By $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ for $x > 0$, we have

$$\begin{aligned}
&e^{2x} - e^{-2x} - 4x - \frac{8x^3}{3} \\
>&\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)^2 + \frac{1}{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)^2} - 4x - \frac{8x^3}{3} \\
=&\frac{x^5 f(x)}{576(24 + 24x + 12x^2 + 4x^3 + x^4)^2},
\end{aligned}$$

where $f(x) = 165888 + 340992x + 340992x^2 + 240768x^3 + 129024x^4 + 53760x^5 + 17664x^6 + 4672x^7 + 928x^8 + 144x^9 + 16x^{10} + x^{11}$. From $f(x) > 0$ for $x > 0$, we have $e^{2x} - e^{-2x} - 4x - \frac{8x^3}{3}$ for $x > 0$. □

Lemma 2.4. *For $0<t<1$, we have*

$$\ln\left(e^{\frac{t}{(1+t)^{\frac{1}{2}}}}-1\right)>\ln t+\frac{t^2}{24}-\frac{t^3}{48}.$$

Proof. We consider the function

$$f(t)=e^{\frac{t}{(1+t)^{\frac{1}{2}}}}-1-e^{\ln t+\frac{t^2}{24}-\frac{t^3}{48}}$$

and the derivative of $f(t)$ is

$$f'(t)=\frac{2+t}{2(1+t)^{\frac{3}{2}}}e^{\frac{t}{(1+t)^{\frac{1}{2}}}}-\frac{48+4t^2-3t^3}{48}e^{\frac{t^2}{24}-\frac{t^3}{48}}.$$

Here, we consider the logarithm function

$$\begin{aligned}g(t)=&\ln\left(\frac{2+t}{2(1+t)^{\frac{3}{2}}}e^{\frac{t}{(1+t)^{\frac{1}{2}}}}\right)-\ln\left(\frac{48+4t^2-3t^3}{48}e^{\frac{t^2}{24}-\frac{t^3}{48}}\right)\\ =&\frac{t}{(1+t)^{\frac{1}{2}}}+\ln(2+t)-\ln 2-\frac{3}{2}\ln(1+t)\\ &+\ln 48-\left(\frac{t^2}{24}-\frac{t^3}{48}\right)-\ln(48+4t^2-3t^3)\end{aligned}$$

and the derivative of $g(t)$ is

$$\begin{aligned}g'(t)=&-\frac{t}{12}+\frac{t^2}{16}-\frac{t}{2(1+t)^{\frac{3}{2}}}-\frac{3}{2(1+t)}+\frac{1}{(1+t)^{\frac{1}{2}}}+\frac{1}{2+t}-\frac{8t-9t^2}{48+4t^2-3t^3}\\ =&\frac{2+t}{2(1+t)^{\frac{3}{2}}}-\frac{4608+2304t+960t^2-1312t^3-648t^4-38t^5+3t^6+9t^7}{48(1+t)(2+t)(48+4t^2-3t^3)}.\end{aligned}$$

We consider the logarithm function

$$\begin{aligned}h(t)=&\ln\left(\frac{2+t}{2(1+t)^{\frac{3}{2}}}\right)\\ &-\ln\left(\frac{4608+2304t+960t^2-1312t^3-648t^4-38t^5+3t^6+9t^7}{48(1+t)(2+t)(48+4t^2-3t^3)}\right)\\ =&\ln(2+t)-\ln 2-\frac{3}{2}\ln(1+t)+\ln(1+t)+\ln(2+t)+\ln(48+4t^2-3t^3)\\ &-\ln(4608+2304t+960t^2-1312t^3-648t^4-38t^5+3t^6+9t^7)+\ln 48\end{aligned}$$

and the derivative of $h(t)$ is

$$h'(t)=\frac{t^2 j(t)}{2(1+t)(2+t)(48+4t^2-3t^3)k(t)},$$

where $j(t)=534528+1330176t+751744t^2+184960t^3+3968t^4-10624t^5-4368t^6-606t^7+369t^8+135t^9$ and $k(t)=4608+2304t+960t^2-1312t^3-648t^4-38t^5+3t^6+9t^7$. From $j(t)>534528-10624t^5-4368t^6-606t^7>534528-10624-4368-606=518930>0$ and $k(t)>4608-1312t^3-648t^4-38t^5>4608-1312-648-38=2610>0$ for $0<t<1$, we have $h'(t)>0$ and $h(t)$ is strictly increasing for $0<t<1$. By $\lim_{t\to 0}h(t)=0$ and $g'(t)>0$, $g(t)$ is strictly increasing for $0<t<1$. From $\lim_{t\to 0}g(t)=0$ and $f'(t)>0$, $f(t)$ is strictly increasing for $0<t<1$. By $\lim_{t\to 0}f(t)=0$, we can get $f(t)>0$ for $0<t<1$. □

Lemma 2.5. *For $0 < t < 1$, we have*

$$\ln t + \frac{t}{3} + \frac{t^2}{24} > \ln\left(e^{\frac{t}{(1+t)^{\frac{1}{6}}}} - 1\right).$$

Proof. We consider the function

$$f(t) = e^{\ln t + \frac{t}{3} + \frac{t^2}{24}} - e^{\frac{t}{(1+t)^{\frac{1}{6}}}} + 1$$

and the derivative of $f(t)$ is

$$f'(t) = \frac{12+4t+t^2}{12} e^{\frac{t}{3}+\frac{t^2}{24}} - \frac{6+5t}{6(1+t)^{\frac{7}{6}}} e^{\frac{t}{(1+t)^{\frac{1}{6}}}}.$$

Here, we consider the logarithm function

$$\begin{aligned} g(t) =& \ln\left(\frac{12+4t+t^2}{12} e^{\frac{t}{3}+\frac{t^2}{24}}\right) - \ln\left(\frac{6+5t}{6(1+t)^{\frac{7}{6}}} e^{\frac{t}{(1+t)^{\frac{1}{6}}}}\right) \\ =& -\ln 2 + \frac{t}{3} + \frac{t^2}{24} + \ln(12+4t+t^2) - \frac{t}{(1+t)^{\frac{1}{6}}} - \ln(6+5t) + \frac{7}{6}\ln(1+t) \end{aligned}$$

and the derivative of $g(t)$ is

$$\begin{aligned} g'(t) =& \frac{1}{3} + \frac{t}{12} + \frac{t}{6(1+t)^{\frac{7}{6}}} + \frac{7}{6(1+t)} - \frac{1}{(1+t)^{\frac{1}{6}}} - \frac{5}{6+5t} + \frac{4+2t}{12+4t+t^2} \\ =& \frac{864+1584t+1164t^2+364t^3+51t^4+5t^5}{12(1+t)(6+5t)(12+4t+t^2)} - \frac{6+5t}{6(1+t)^{\frac{7}{6}}}. \end{aligned}$$

We consider the logarithm function

$$\begin{aligned} h(t) =& \ln\left(\frac{864+1584t+1164t^2+364t^3+51t^4+5t^5}{12(1+t)(6+5t)(12+4t+t^2)}\right) - \ln\left(\frac{6+5t}{6(1+t)^{\frac{7}{6}}}\right) \\ =& \ln(864+1584t+1164t^2+364t^3+51t^4+5t^5) - \ln 12 - \ln(1+t) \\ & - \ln(6+5t) - \ln(12+4t+t^2) - \ln(6+5t) + \ln 6 + \frac{7}{6}\ln(1+t) \end{aligned}$$

and the derivative of $h(t)$ is

$$h'(t) = \frac{t\, j(t)}{6(1+t)(6+5t)(12+4t+t^2)\, k(t)},$$

where $j(t) = 186624 + 400032t + 309744t^2 + 127008t^3 + 39896t^4 + 12458t^5 + 2275t^6 + 175t^7 > 0$ and $k(t) = 864 + 1584t + 1164t^2 + 364t^3 + 51t^4 + 5t^5 > 0$. Since we have $h'(t) > 0$ and $h(t)$ is strictly increasing for $0 < t < 1$. By $\lim_{t\to 0} h(t) = 0$ and $g'(t) > 0$, $g(t)$ is strictly increasing for $0 < t < 1$. From $\lim_{t\to 0} g(t) = 0$ and $f'(t) > 0$, $f(t)$ is strictly increasing for $0 < t < 1$. By $\lim_{t\to 0} f(t) = 0$, we can get $f(t) > 0$ for $0 < t < 1$. □

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Consider the equation

$$\tanh x = \sqrt{1 - exp\left(-\frac{x^2}{(x^2+1)^n}\right)}.$$

Using the equation $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, the equation becomes

$$n = \frac{2\ln x - \ln 2 - \ln\left(\ln(1+e^{2x}) - \ln 2 - x\right)}{\ln(1+x^2)} = f(x)\,.$$

Therefore, it is necessary to prove $\frac{1}{6} < f(x) < \frac{1}{2}$ and show the following two inequalities:

$$g(x) = \ln\left(\ln(1+e^{2x}) - \ln 2 - x\right) + \ln 2 - 2\ln x + \frac{\ln(1+x^2)}{2} > 0\,,$$
$$h(x) = -\ln 2 + 2\ln x - \frac{\ln(1+x^2)}{6} - \ln\left(\ln(1+e^{2x}) - \ln 2 - x\right) > 0\,.$$

First, we will prove $g(x) > 0$ for $x > 0$. The derivative of $g(x)$ is

$$\begin{aligned} g'(x) &= -\frac{2}{x} + \frac{x}{1+x^2} + \frac{-1 + \frac{2e^{2x}}{1+e^{2x}}}{-x - \ln 2 + \ln(1+e^{2x})} \\ &= \frac{(2+x^2)\left(x + \frac{(-1+e^x)(1+e^x)x(1+x^2)}{(1+e^{2x})(2+x^2)} + \ln 2 - \ln(1+e^{2x})\right)}{x(1+x^2)(-x - \ln 2 + \ln(1+e^{2x}))} \\ &= \frac{(2+x^2)j(x)}{x(1+x^2)(-x-\ln 2 + \ln(1+e^{2x}))} \end{aligned}$$

and the derivative of $j(x)$ is

$$\begin{aligned} j'(x) &= \frac{2 - 2e^{4x} + 8e^{2x}x - x^2 + e^{4x}x^2 + 12e^{2x}x^3 + 4e^{2x}x^5}{(1+e^{2x})^2(2+x^2)^2} \\ &= \frac{4e^{2x}x(1+x^2)(2+x^2) + (e^{4x}-1)(-2+x^2)}{(1+e^{2x})^2(2+x^2)^2}\,. \end{aligned}$$

In the case of $x \geq \sqrt{2}$, $j'(x) > 0$ is clearly established. Hence, we consider the case of $0 < x < \sqrt{2}$. From Lemma 2.1, we have

$$\begin{aligned} j'(x) &= \frac{4x(1+x^2)(2+x^2) + (e^{2x} - e^{-2x})(-2+x^2)}{e^{-2x}(1+e^{2x})^2(2+x^2)^2} \\ &\geq \frac{4x(1+x^2)(2+x^2) + (4x+8x^3)(-2+x^2)}{e^{-2x}(1+e^{2x})^2(2+x^2)^2} = \frac{12x^5}{e^{-2x}(1+e^{2x})^2(2+x^2)^2} \end{aligned}$$

for $0 < x < \sqrt{2}$. Therefore, we have $j'(x) > 0$ and $j(x)$ is strictly increasing for $x > 0$. By $\lim_{x\to 0} j(x) = 0$ and Lemma 2.2, $j(x) > 0$ and $g'(x) > 0$, hence, $g(x)$ is strictly increasing for $x > 0$. By l'Hopital's rule [4],

$$\lim_{x\to 0} \frac{\ln(1+e^{2x}) - \ln 2 - x}{x^2} = \lim_{x\to 0} \frac{e^{2x}-1}{2x(1+e^{2x})} = \lim_{x\to 0} \frac{e^{2x}}{1+e^{2x}+2e^{2x}x} = \frac{1}{2}\,.$$

Hence, we have $\lim_{x\to 0} g(x) = 0$ and $g(x) > 0$ for $x > 0$. Next, we will prove $h(x) > 0$ for $x > 0$. The derivative of $h(x)$ is

$$\begin{aligned} h'(x) &= \frac{2}{x} - \frac{x}{3(1+x^2)} - \frac{-1 + \frac{2e^{2x}}{1+e^{2x}}}{-x - \ln 2 + \ln(1+e^{2x})} \\ &= \frac{(6+5x^2)\left(-x - \ln 2 + \ln(1+e^{2x}) - \frac{3(-1+e^x)(1+e^x)x(1+x^2)}{(1+e^{2x})(6+5x^2)}\right)}{3x(1+x^2)(-x - \ln 2 + \ln(1+e^{2x}))} \\ &= \frac{(6+5x^2)k(x)}{3x(1+x^2)(-x - \ln 2 + \ln(1+e^{2x}))} \end{aligned}$$

and the derivative of $k(x)$ is

$$k'(x) = \frac{(e^{2x} - e^{-2x})(18 + 21x^2 + 10x^4) - 12x(1+x^2)(6+5x^2)}{e^{-2x}(1+e^{2x})^2(6+5x^2)^2}.$$

From Lemma 2.3, we have

$$\begin{aligned} k'(x) >& \frac{\left(4x + \frac{8x^3}{3}\right)(18 + 21x^2 + 10x^4) - 12x(1+x^2)(6+5x^2)}{e^{-2x}(1+e^{2x})^2(6+5x^2)^2} \\ =& \frac{4x^5(27+20x^2)}{3e^{-2x}(1+e^{2x})^2(6+5x^2)^2} \end{aligned}$$

for $x > 0$. Therefore, we have $k'(x) > 0$ and $k(x)$ is strictly increasing for $x > 0$. From $\lim_{x\to 0} k(x) = 0$ and Lemma 2.2, we have $k(x) > 0$ and $h'(x) > 0$. Thus, $h(x)$ is strictly increasing for $x > 0$ and by $\lim_{x\to 0} h(x) = 0$, we have $h(x) > 0$ for $x > 0$. Therefore, we obtain $\frac{1}{6} < f(x) < \frac{1}{2}$ for $x > 0$. Next, we consider that the constants $\frac{1}{6}$ and $\frac{1}{2}$ are the best possible. From l'Hopital's rule [4], we can get $\lim_{x\to\infty} \frac{\ln(1+e^{2x})}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1+e^{2x}} = 2$ and

$$\begin{aligned} &\lim_{x\to\infty} f(x) \\ &= \lim_{x\to\infty} \frac{(1+x^2)(x + 3e^{2x}x + 2\ln 2 + 2e^{2x}\ln 2 - 2\ln(1+e^{2x}) - 2e^{2x}\ln(1+e^{2x})}{2(1+e^{2x})x^2(x + \ln 2 - \ln(1+e^{2x}))} \\ &= \lim_{x\to\infty} \frac{\left(\frac{1}{x^2} + 1\right)\left(\frac{1}{e^{2x}} + 3 + \frac{2\ln 2}{e^{2x}x} + \frac{2\ln 2}{x} - \frac{2\ln(1+e^{2x})}{e^{2x}x} - \frac{2\ln(1+e^{2x})}{x}\right)}{2(1+\frac{1}{e^{2x}})\left(1 + \frac{\ln 2}{x} - \frac{\ln(1+e^{2x})}{x}\right)} = \frac{1}{2}. \end{aligned}$$

Also, by l'Hopital's rule [4], we have

$$\begin{aligned}
&\lim_{x\to 0}\frac{x+3e^{2x}x+2(1+e^{2x})(\ln 2-\ln(1+e^{2x}))}{x^4}\\
&=\lim_{x\to 0}\frac{1-e^{2x}+6e^{2x}x+4e^{2x}(\ln 2-\ln(1+e^{2x}))}{4x^3}\\
&=\lim_{x\to 0}\frac{4e^{2x}-\frac{8e^{4x}}{1+e^{2x}}+12e^{2x}x+8e^{2x}(\ln 2-\ln(1+e^{2x}))}{12x^2}\\
&=\lim_{x\to 0}\frac{20e^{2x}+\frac{16e^{6x}}{(1+e^{2x})^2}-\frac{48e^{4x}}{1+e^{2x}}+24e^{2x}x+16e^{2x}(\ln 2-\ln(1+e^{2x}))}{24x}\\
&=\lim_{x\to 0}\frac{64e^{2x}-\frac{64e^{8x}}{(1+e^{2x})^3}+\frac{192e^{6x}}{(1+e^{2x})^2}-\frac{224e^{4x}}{1+e^{2x}}+48e^{2x}x+32e^{2x}(\ln 2-\ln(1+e^{2x}))}{24}\\
&=-\frac{1}{3},
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to 0}f(x)\\
&=\lim_{x\to 0}\frac{(1+x^2)(x+3e^{2x}x+2\ln 2+2e^{2x}\ln 2-2\ln(1+e^{2x})-2e^{2x}\ln(1+e^{2x}))}{2(1+e^{2x})x^2(x+\ln 2-\ln(1+e^{2x}))}\\
&=\lim_{x\to 0}\frac{(1+x^2)(x+3e^{2x}x+2(1+e^{2x})(\ln 2-\ln(1+e^{2x})))}{2(1+e^{2x})x^2(x+\ln 2-\ln(1+e^{2x}))}\\
&=\lim_{x\to 0}\frac{1+x^2}{2(1+e^{2x})}\cdot\frac{x^2}{x+\ln 2-\ln(1+e^{2x})}\cdot\frac{x+3e^{2x}x+2(1+e^{2x})(\ln 2-\ln(1+e^{2x}))}{x^4}\\
&=\frac{1}{6}.
\end{aligned}$$

Thus, the proof is completed. □

Proof of Theorem 1.2. We consider the logarithmic function

$$\ln\sqrt[3]{1-exp\left(-\frac{x^3}{(1+x^3)^{\frac{1}{2}}}\right)}-\ln\sqrt{1-exp\left(-\frac{x^2}{(1+x^2)^{\frac{1}{6}}}\right)}=\frac{f(x)}{6},$$

where

$$f(x)=2\ln\left(e^{\frac{x^3}{(1+x^3)^{\frac{1}{2}}}}-1\right)-\frac{2x^3}{(1+x^3)^{\frac{1}{2}}}-3\ln\left(e^{\frac{x^2}{(1+x^2)^{\frac{1}{6}}}}-1\right)+\frac{3x^2}{(1+x^2)^{\frac{1}{6}}}.$$

Here, we need to show $f(x)>0$ for $0<x<1$. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
f(x)&>2\left(3\ln x+\frac{x^6}{24}-\frac{x^9}{48}\right)-\frac{2x^3}{(1+x^3)^{\frac{1}{2}}}-3\left(2\ln x+\frac{x^2}{3}+\frac{x^4}{24}\right)+\frac{3x^2}{(1+x^2)^{\frac{1}{6}}}\\
&=x^2\left(-\frac{1}{24}(24+3x^2-2x^4+x^7)-\frac{2x}{(1+x^3)^{\frac{1}{2}}}+\frac{3}{(1+x^2)^{\frac{1}{6}}}\right)=x^2g(x).
\end{aligned}$$

The derivative of $g(x)$ is

$$\begin{aligned}
g'(x) &= -\frac{x}{(1+x^2)^{\frac{7}{6}}} + \frac{3x^3}{(1+x^3)^{\frac{3}{2}}} - \frac{2}{(1+x^3)^{\frac{1}{2}}} + \frac{1}{24}(-6x+8x^3-7x^6) \\
&< -\frac{x}{(1+1^2)^2} + \frac{3x^3}{(1+x^3)^{\frac{2}{2}}} - \frac{2}{(1+1^3)^{\frac{1}{2}}} + \frac{1}{24}(-6x+8x^3-7x^6) \\
&= \frac{-24\sqrt{2}-12x+80x^3-24\sqrt{2}x^3-12x^4+x^6-7x^9}{24(1+x)(1-x+x^2)} \\
&< \frac{-24\sqrt{2}+80x^3-24\sqrt{2}x^3-12x^4+x^6-7x^9}{24(1+x)(1-x+x^2)} = \frac{h(x)}{24(1+x)(1-x+x^2)}
\end{aligned}$$

and the derivative of $h(x)$ is $h'(x) = 3x^2(80-24\sqrt{2}-16x+2x^3-21x^6) > 3x^2(80-24\sqrt{2}-16\cdot 1-21\cdot 1) > 3x^2\cdot 9 > 0$ for $0<x<1$. Therefore, we have $h'(x)>0$ and $h(x)$ is strictly increasing for $0<x<1$. By $\lim_{x\to 1} h(x) = 62-48\sqrt{2} \cong -5.88225$, $h(x)<0$ and $g(x)$ is decreasing for $0<x<1$. From $\lim_{x\to 1} g(x) = -\frac{13}{12}+\frac{3}{2^{\frac{1}{6}}}-\sqrt{2} \cong 0.175149$, we obtain $f(x)>0$ for $0<x<1$ and the proof is completed. □

4. Conjecture

We shall pose a conjecture relate to the hyperbolic tangent and the function $\sqrt{1-exp\left(-\frac{x^q}{(x^q+1)^p}\right)}$.

Conjecture 4.1. *For $x>0$, we have*

$$\tanh x < \sqrt[e]{1-exp\left(-\frac{x^e}{(x^e+1)^{\frac{1}{2}}}\right)} < \sqrt[3]{1-exp\left(-\frac{x^3}{(x^3+1)^{\frac{1}{2}}}\right)}.$$

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