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REFINEMENTS OF THE ARITHMETIC - GEOMETRIC MEAN INEQUALITY ASSOCIATED WITH THE NUMBER e

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Abstract. Let A and G denote, respectively, the arithmetic and geometric means of the positive numbers real positive numbers $a_1, a_2, \ldots, a_n$, and for any real c, let

$$\mathcal{R}_c = \frac{G}{ne} \ln\left(1 + \frac{e^{cn\frac{A}{G}}}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right).$$

In this paper we show that $A - G \geq \mathcal{R}_2$. This gives a refinement of the Arithmetic mean – Geometric mean inequality, which asserts that $A - G \geq 0$. Also, for any real c, let

$$\mathcal{B}_c = \frac{1}{ne} \sum_{i=1}^{n} \ln\left(\left(\frac{ea_i}{G}\right)^e + e^{\frac{ca_i}{G}}\left(\frac{a_i}{G} - 1\right)^2\right).$$

We show that the double sided inequality $\mathcal{B}_\alpha \leq \frac{A}{G} \leq \mathcal{B}_\beta$ holds, with the best possible integer constants $\alpha = 2$ and $\beta = 5$.

1. Introduction

For real positive numbers $a_1, a_2, \ldots, a_n$, we define the arithmetic mean A and the geometric mean G by

$$A = \frac{1}{n} \sum_{i=1}^{n} a_i \quad \text{and} \quad G = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$

The Arithmetic mean – Geometric mean inequality asserts that $A \geq G$ with equality if and only if $a_1 = a_2 = \cdots = a_n$. In [2, 3, 8] many proofs of the inequality are given and refined version of the inequality are introduced. Even nowadays, many mathematicians proved the many refinements of the Arithmetic mean – Geometric mean inequality (see [1, 4, 5, 6, 7] and the references given there).

The following argument for obtaining the refined Arithmetic mean-Geometric mean inequality is due to the first author in [5]. We consider the function

$$F(x) = e^x - x^e - \frac{1}{e^2}(x - e)^2$$

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for $x > 0$, then the function $F(x) \geq 0$ with equality if and only if $x = e$. For $1 < i < n$, we put $x = \frac{a_i e}{G}$ in $F(x) \geq 0$ to get

$$e^{\frac{e}{G}nA} \geq \left(\prod_{i=1}^{n} \frac{a_i e}{G}\right)^e + \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 \geq e^{ne} \left(1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right).$$

Then, we take logarithm and we divide the resulting inequality by ne to obtain

$$\frac{A}{G} \geq 1 + \frac{1}{ne} \ln \left(1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right).$$

Therefore, we can get the inequality $A \geq G + \mathcal{R} \geq G$, where

$$\mathcal{R} = \frac{G}{ne} \ln \left(1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right) \geq 0,$$

with equality if and only if $a_1 = \cdots = a_n$. The aim of the present paper is to improve on this result. To this end, for any real c, we define $\mathcal{R}_c$ by

$$\mathcal{R}_c = \frac{G}{ne} \ln \left(1 + \frac{e^{cn\frac{A}{G}}}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right). \tag{1.1}$$

Note that the above mentioned inequality of Hassani [5] asserts that $A - G \geq \mathcal{R}_0$. In this paper, we obtain the following refinement.

Theorem 1.1. *Let $\mathcal{R}_2$ be the expression defined in (1.1) with $c = 2$. For any real positive numbers $a_1, a_2, \ldots a_n$, the following inequality holds*

$$A - G \geq \mathcal{R}_2.$$

Remark. *Note that*

$$\frac{d}{dc}\mathcal{R}_c = \frac{e^{cn\frac{A}{G} - ne - 1} A \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2}{1 + e^{cn\frac{A}{G} - ne} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2} > 0.$$

Thus, the function $c \mapsto \mathcal{R}_c$ is strictly increasing, and $\mathcal{R}_2 > \mathcal{R}_0$. Moreover, by using Taylor expansion of the logarithmic function, we observe that as $c \to \infty$ the following asymptotic expansion holds

$$\mathcal{R}_c \sim \frac{A}{e} c + \left(\frac{G}{ne} \ln \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 - G\right) + \frac{G}{ne} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^{2j} e^{n\frac{A}{G}jc - nej}}.$$

This shows that $\mathcal{R}_c$ has a linear growth at rate $\frac{A}{e}$, as $c \to \infty$.

Another refinement of the Arithmetic mean – Geometric mean inequality $A \geq G$ based on considering it in ratio form as $\frac{A}{G} \geq 1$. For $1 < i < n$, we put $x = \frac{a_i e}{G}$ in $F(x) \geq 0$, then we have

$$e^{\frac{a_i e}{G}} \geq \left(\frac{a_i e}{G}\right)^e + \left(\frac{a_i}{G} - 1\right)^2,$$

and

$$e^{\frac{e}{G}nA} = \prod_{i=1}^{n} e^{\frac{a_i e}{G}} \geq \prod_{i=1}^{n} \left(\left(\frac{a_i e}{G}\right)^e + \left(\frac{a_i}{G} - 1\right)^2\right).$$

By taking the logarithm, the formula is transformed as follows.

$$\frac{e}{G}nA \geq \ln \prod_{i=1}^{n} \left(\left(\frac{a_i e}{G} \right)^e + \left(\frac{a_i}{G} - 1 \right)^2 \right) = \sum_{i=1}^{n} \ln \left(\left(\frac{e a_i}{G} \right)^e + \left(\frac{a_i}{G} - 1 \right)^2 \right),$$

and consequently

$$\frac{A}{G} \geq \frac{1}{ne} \sum_{i=1}^{n} \ln \left(\left(\frac{e a_i}{G} \right)^e + \left(\frac{a_i}{G} - 1 \right)^2 \right).$$

Let us define $\mathcal{B}_c$, for any real c, by

$$\mathcal{B}_c = \frac{1}{ne} \sum_{i=1}^{n} \ln \left(\left(\frac{e a_i}{G} \right)^e + e^{\frac{c a_i}{G}} \left(\frac{a_i}{G} - 1 \right)^2 \right). \tag{1.2}$$

Then, the last inequality reads as $\frac{A}{G} \geq \mathcal{B}_0$. In the following result we improve on this and obtain a double sided refinement with best possible integers.

Theorem 1.2. *Let $\mathcal{B}_c$ be the expression defined in* (1.2). *For any real positive numbers $a_1, a_2, \ldots a_n$, the following double sided inequality holds*

$$\mathcal{B}_\alpha \leq \frac{A}{G} \leq \mathcal{B}_\beta.$$

The constants $\alpha = 2$ and $\beta = 5$ are the best possible integers.

The proof of the above results strongly based on the study of the family of functions

$$F_\eta(x) = e^x - x^e - e^{\eta x/e - 2}(x - e)^2, \tag{1.3}$$

where $\eta = 2, 3, 4, 5$. Their graphs partially pictured in Figures $1 - 5$. More precisely, the proof of Theorem 1.1 is based on the following auxiliary lemma.

Lemma 1.3. *Let $F_2(x)$ be the function defined in* (1.3) *with $\eta = 2$. The inequality $F_2(x) \geq 0$ holds for $x > 0$, with equality if and only if $x = e$.*

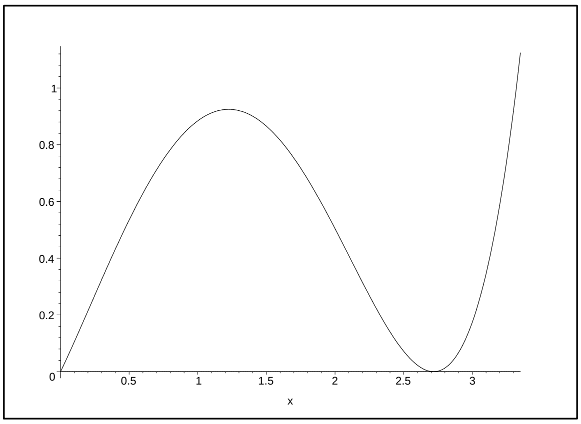

FIGURE 1. Graph of the function $x \mapsto e^x - x^e - e^{2x/e-2}(x - e)^2$

Also, the proof of Theorem 1.2 is based on the following auxiliary lemmas.

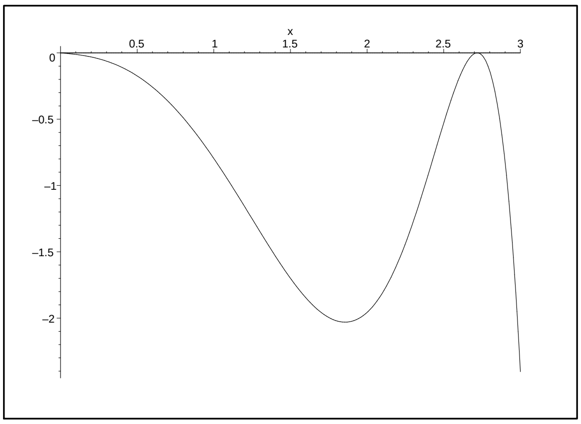

FIGURE 2. Graph of the function $x \mapsto e^x - x^e - e^{5x/e-2}(x-e)^2$

Lemma 1.4. *Let $F_5(x)$ be the function defined in* (1.3) *with $\eta = 5$. The inequality $F_5(x) \le 0$ holds for $x > 0$, with equality if and only if $x = e$.*

Lemma 1.5. *For $x > 0$ let $F_3(x)$ be the function defined in* (1.3) *with $\eta = 3$. Then, there exists $e < x_0 < 3$ such that $F_3(x) > 0$ for $e < x < x_0$ and $F_3(x) < 0$ for $x_0 < x < 3$.*

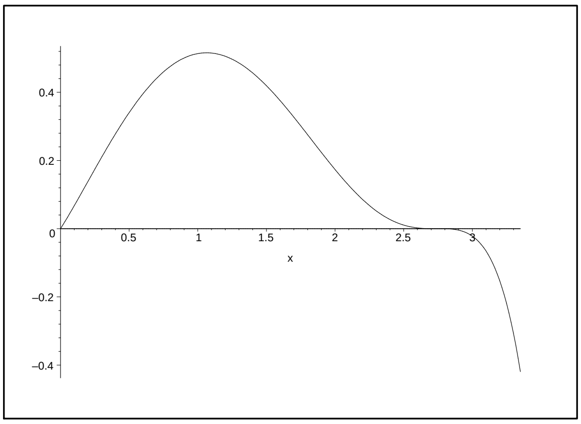

FIGURE 3. Graph of the function $x \mapsto e^x - x^e - e^{3x/e-2}(x-e)^2$ for $x \in [0, 3.5]$

Lemma 1.6. *For $x > 0$ let $F_4(x)$ be the function defined in* (1.3) *with $\eta = 4$. Then, there exists $\frac{95}{100} < x_0 < \frac{96}{100}$ such that $F_4(x) > 0$ for $\frac{95}{100} < x < x_0$ and $F_4(x) < 0$ for $x_0 < x < \frac{96}{100}$.*

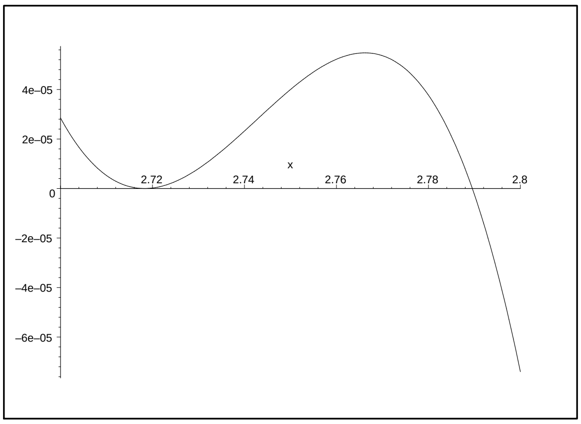

FIGURE 4. Graph of the function $x \mapsto e^x - x^e - e^{3x/e-2}(x-e)^2$ for $x \in [2.7, 2.8]$, showing the point it changes the sign

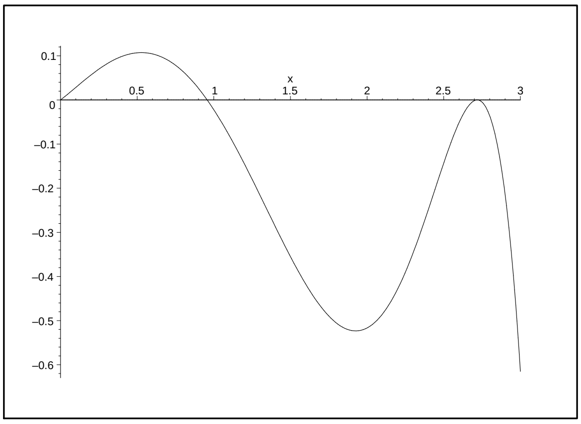

FIGURE 5. Graph of the function $x \mapsto e^x - x^e - e^{4x/e-2}(x-e)^2$

2. Proof of Lemmas

Proof of Lemma 1.3. The proof of the lemma can be done through the graph of function $F_2(x)$ (see Figure 1). We set

$$f(x) = \ln(e^x - x^e) - \ln\left((e-x)^2\right) + 2 - \frac{2x}{e}$$

for $x > 0$, then the derivative of $f(x)$ is

$$f'(x) = \frac{g(x)}{e(e-x)x(e^x - x^e)},$$

where $g(x) = e^x x\left(-ex + 2x + e^2\right) - x^e\left(2x^2 - e^2 x + e^3\right)$. First, we consider the case of $x > \frac{e^2}{e-2} \approxeq 10.2871$. Since we have

$$2x^2 - e^2 x + e^3 \geq 2\left(\frac{e^2}{4}\right)^2 - e^2\left(\frac{e^2}{4}\right) + e^3 = \frac{e^3(8-e)}{8} > 0$$

for $x > 0$ and $-ex + 2x + e^2 < 0$ for $x > \frac{e^2}{e-2}$, we get $g(x) < 0$ and $f'(x) > 0$ for $x > \frac{e^2}{e-2}$. Thus, $f(x)$ is strictly increasing for $x > \frac{e^2}{e-2}$. This gives

$$\begin{aligned} f(x) > f\left(\frac{e^2}{e-2}\right) &= -\frac{4}{e-2} - 2\ln\left(\frac{2e}{e-2}\right) + \ln\left(e^{\frac{e^2}{e-2}} - (e-2)^{-e}e^{2e}\right) \\ &= -\ln e^{\frac{4}{e-2}} - \ln\left(\frac{2e}{e-2}\right)^2 + \ln\left(e^{\frac{e^2}{e-2}} - \left(\frac{e^2}{e-2}\right)^e\right) = \ln\frac{e^{\frac{e^2}{e-2}} - \left(\frac{e^2}{e-2}\right)^e}{e^{\frac{4}{e-2}}\left(\frac{2e}{e-2}\right)^2}. \end{aligned}$$

By $e^{\frac{e^2}{e-2}} \approxeq 29352.3$, $\left(\frac{e^2}{e-2}\right)^e \approxeq 564.548$, $e^{\frac{4}{e-2}} \approxeq 262.131$, and $\frac{2e}{e-2} \approxeq 7.56884$, we obtain

$$f(x) > \ln\frac{29350 - 565}{263 \cdot 8^2} = \ln\frac{28785}{16832} > 0.$$

Next, we consider the case of $0 < x < \frac{e^2}{e-2}$. From $-ex+2x+e^2 > 0$ for $0 < x < \frac{e^2}{e-2}$, we can take logarithm and set

$$\begin{aligned} h(x) &= \ln\left(e^x x\left(-ex + 2x + e^2\right)\right) - \ln\left(x^e\left(2x^2 - e^2x + e^3\right)\right) \\ &= x + \ln x + \ln\left(-ex + 2x + e^2\right) - e\ln x - \ln\left(2x^2 - e^2x + e^3\right) \end{aligned}$$

for $0 < x < \frac{e^2}{e-2}$. The derivative of $h(x)$ is

$$h'(x) = \frac{(x-e)^2 j(x)}{x(-ex + 2x + e^2)(2x^2 - e^2x + e^3)},$$

where $j(x) = -2ex^2 + 4x^2 + e^3x - 2e^2x + 4ex - e^4 + e^3$. The derivative of $j(x)$ is

$$j'(x) = -4ex + 8x + e^3 - 2e^2 + 4e.$$

Here, we have $j'(x) > 0$ for $0 < x < x_0$ and $j'(x) < 0$ for $x_0 < x < \frac{e^2}{e-2}$, where

$$x_0 = \frac{4e - 2e^2 + e^3}{4(e-2)} \approxeq 5.63169.$$

Therefore, $j(x)$ is strictly increasing for $0 < x < x_0$ and $j(x)$ is strictly decreasing for $x_0 < x < \frac{e^2}{e-2}$. From

$$\begin{aligned} j(e) &= 8e^2 - 3e^3 \approxeq -1.14416 < 0, \\ j(3) &= 36 - 6e - 6e^2 + 4e^3 - e^4 \approxeq 1.09997 > 0, \\ j(8) &= 256 - 96e - 16e^2 + 9e^3 - e^4 \approxeq 2.99173 > 0 \end{aligned}$$

and

$$j(10) = 400 - 160e - 20e^2 + 11e^3 - e^4 \approxeq -16.3635 < 0,$$

there exists the real numbers x_1 and x_2 with

$$e < x_1 < 3 \quad \text{and} \quad 8 < x_2 < 10$$

such that $j(x) > 0$ for $x_1 < x < x_2$, $j(x) < 0$ for $0 < x < x_1$ and $x_2 < x < \frac{e^2}{e-2}$. From $j(e) < 0$, we have $j(x) < 0$ for $0 < x < e$. Hence, $h'(x) < 0$ and $h(x)$ is strictly decreasing for $0 < x < e$. By $h(x) > h(e) = 0$ for $0 < x < e$, $g(x) > 0$

and $f'(x) > 0$ for $0 < x < e$. Since $f(x)$ is strictly increasing for $0 < x < e$ and $f(0) = 0$, we have $f(x) > 0$ for $0 < x < e$. For $e < x < 3$ we obtain

$$f(x) > \ln(e^x - x^e) - \ln\left((e-x)^2\right) + \left(2 - \frac{6}{e}\right) = k(x).$$

The derivative of $k(x)$ is

$$k'(x) = \frac{x^e\left(-ex + 2x + e^2\right) - e^x x\left(-x + e + 2\right)}{(x-e)x(e^x - x^e)}.$$

Since $-ex + 2x + e^2 > 0$ and $-x + e + 2 > 0$ for $e < x < 3$, we can take logarithm and set

$$\begin{aligned} l(x) &= \ln(x^e(-ex + 2x + e^2)) - \ln(e^x x(-x + e + 2)) \\ &= e\ln x + \ln(-ex + 2x + e^2) - x - \ln x - \ln(-x + e + 2). \end{aligned}$$

The derivative of $l(x)$ is

$$l'(x) = \frac{(e-x)^2(-ex + 2x + e^2 + e - 2)}{(-x + e + 2)x(-ex + 2x + e^2)}.$$

From

$$-ex + 2x + e^2 + e - 2 > -3e + 2e + e^2 + e - 2 = e^2 - 2 > 0$$

for $e < x < 3$, we have $l'(x) > 0$ and $l(x)$ is strictly increasing for $e < x < 3$. By $l(e) = 0$, $l(x) > 0$ and $k'(x) > 0$ for $e < x < 3$. Since $k(x)$ is strictly increasing for $e < x < 3$ and

$$k(e) = \frac{e^2 + e - 6 - e\ln 2}{e} \approxeq \frac{2.22317}{e} > 0,$$

$k(x) > 0$ and $f(x) > 0$ for $e < x < 3$. For $3 < x < 8$, we have $j(8) > 0$ and $j(3) > 0$. Hence, we have $j(x) > 0$ and $h'(x) > 0$ for $3 < x < 8$. Therefore, $h(x)$ is strictly increasing for $3 < x < 8$ and

$$\begin{aligned} h(x) > h(3) &= 3 - e\ln 3 + \ln 3 + \ln(6 - 3e + e^2) - \ln(18 - 3e^2 + e^3) \\ &= \ln\frac{e^3(6 - 3e + e^2)}{3^{e-1}(18 - 3e^2 + e^3)}. \end{aligned}$$

By $e^3 \approxeq 20.0855369$, $6 - 3e + e^2 \approxeq 5.23421061$, $3^{e-1} \approxeq 6.60433024$ and $18 - 3e^2 + e^3 \approxeq 15.9183686$, we have

$$h(x) > h(3) > \ln\frac{20.0855 \cdot 5.23421}{6.604331 \cdot 15.9184} = \ln\frac{525658624775}{525651912952} > 0$$

and $g(x) > 0$ for $3 < x < 8$. Thus, we have $f'(x) < 0$ for $3 < x < 8$. Since $f(x)$ is strictly decreasing for $3 < x < 8$ we get

$$f(x) > f(8) = -\frac{16}{e} + 2 - 2\ln(8 - e) + \ln(e^8 - 8^e) = \ln\frac{e^{2-\frac{16}{e}}(e^8 - 8^e)}{(8-e)^2}.$$

By $e^{2-\frac{16}{e}} \approxeq 0.0205258$, $e^8 - 8^e \approxeq 2695.95$ and $8 - e \approxeq 5.28172$, we have

$$f(x) > f(8) > \ln\frac{0.02 \cdot 2695}{(5.3)^2} = \ln\frac{5390}{2809} > 0$$

for $3 < x < 8$. For $8 < x < 10$, we can get

$$f(x) > \ln(e^x - x^e) + \left(2 - \frac{2x}{e}\right) - 2\ln(10 - e) = m(x)\,.$$

The derivative of $m(x)$ is

$$m'(x) = \frac{(e-2)e^x x + (2x - e^2)x^e}{ex(e^x - x^e)} > 0$$

for $8 < x < 10$. Thus, $m(x)$ is strictly increasing for $8 < x < 10$. This implies that

$$m(x) > m(8) = -\frac{16}{e} + 2 - 2\ln(10-e) + \ln(e^8 - 8^e) = \ln\frac{e^{2-\frac{16}{e}}(e^8 - 8^e)}{(10-e)^2}.$$

By $e^{2-\frac{16}{e}} \approxeq 0.0205258$, $e^8 - 8^e \approxeq 2695.95$ and $10 - e \approxeq 7.28172$, we have

$$f(x) > m(8) > \ln\frac{0.02 \cdot 2695}{(7.3)^2} = \ln\frac{5390}{5329} > 0$$

for $8 < x < 10$. For $10 < x < \frac{e^2}{e-2}$, we have $j(10) < 0$ and $j(x) < 0$ for $10 < x < \frac{e^2}{e-2}$. Hence, $h'(x) < 0$ and $h(x)$ is strictly decreasing for $10 < x < \frac{e^2}{e-2}$,

$$\begin{aligned} h(x) < h(10) &= 10 - e\ln 10 + \ln 10 + \ln(20 - 10e + e^2) - \ln(200 - 10e^2 + e^3) \\ &= 10 - (e-1)\ln 10 + \ln\frac{20 - 10e + e^2}{200 - 10e^2 + e^3} = \ln\frac{e^{10}(20 - 10e + e^2)}{10^{e-1}(200 - 10e^2 + e^3)}. \end{aligned}$$

By $e^{10} \approxeq 22026.5$, $20 - 10^e + e^2 \approxeq 0.206238$, $10^{e-1} \approxeq 52.2735$ and $200 - 10e^2 + e^3 \approxeq 146.195$, we have

$$h(x) < h(10) < \ln\frac{22027 \cdot 0.207}{52 \cdot 146} = \ln\frac{4559589}{7592000} < 0$$

and $f'(x) > 0$ for $10 < x < \frac{e^2}{e-2}$. Since $f(x)$ is strictly increasing for $10 < x < \frac{e^2}{e-2}$,

$$f(x) > f(10) = -\frac{20}{e} + 2 - 2\ln(10-e) + \ln(e^{10} - 10^e) = \ln\frac{e^{2-\frac{20}{e}}(e^{10} - 10^e)}{(10-e)^2}.$$

By $e^{2-\frac{20}{e}} \approxeq 0.00471225$, $e^{10} - 10^e \approxeq 21503.7$ and $10 - e \approxeq 7.28172$, we obtain

$$f(x) > f(10) > \ln\frac{0.0047 \cdot 21503}{(7.3)^2} = \ln\frac{1010641}{532900} > 0$$

for $10 < x < \frac{e^2}{e-2}$. Hence, we can obtain $f(x) > 0$ and $F(x) \geq 0$ for $x > 0$. □

Proof of Lemma 1.4. The proof of the lemma can be done through the graph of function $F_5(x)$ (see Figure 2). We set

$$f(x) = \frac{5x}{e} - 2 + \ln\left((e-x)^2\right) - \ln\left(e^x - x^e\right).$$

The derivative of $f(x)$ is

$$f'(x) = \frac{g(x)}{e(e-x)x(e^x - x^e)},$$

where $g(x) = x^e(5x^2 - e^2x - 3ex + e^3) - e^x x(-ex + 5x + e^2 - 3e)$. First, we consider the case of $0 < x < \frac{3e - e^2}{5-e} \approxeq 0.33562$. From $5x^2 - e^2x - 3ex + e^3 > 0$ for $x > 0$ and

$$-ex + 5x + e^2 - 3e < 0$$

for $0 < x < \frac{3e-e^2}{5-e}$, we have $g(x) > 0$ for $0 < x < \frac{3e-e^2}{5-e}$. Therefore, $f'(x) > 0$ for $0 < x < \frac{3e-e^2}{5-e}$. Since $f(x)$ is strictly increasing for $0 < x < \frac{3e-e^2}{5-e}$, we have

$$f(x) > f(0) = 0.$$

Next, we consider the case of $x > \frac{3e-e^2}{5-e}$. By $-ex + 5x + e^2 - 3e > 0$, we can take logarithm and set

$$\begin{aligned} h(x) &= \ln\left(x^e(5x^2 - e^2x - 3ex + e^3)\right) - \ln\left(e^x x(-ex + 5x + e^2 - 3e)\right) \\ &= e\ln x + \ln(5x^2 - e^2x - 3ex + e^3) - x - \ln x - \ln(-ex + 5x + e^2 - 3e) \end{aligned}$$

for $x > \frac{3e-e^2}{5-e}$. The derivative of $h(x)$ is

$$h'(x) = \frac{(x-e)^2 j(x)}{x(-ex + 5x + e^2 - 3e)(5x^2 - e^2x - 3ex + e^3)},$$

where $j(x) = 5ex^2 - 25x^2 - e^3x + 2e^2x + 5ex + e^4 - 4e^3 + 3e^2$. Since we have

$$j(x) \le j\left(\frac{5e + 2e^2 - e^3}{10(5-e)}\right) = \frac{e^2(325 - 440e + 174e^2 - 24e^3 + e^4)}{20(5-e)}$$

for $x > 0$ and

$$325 - 440e + 174e^2 - 24e^3 + e^4 \approxeq -12.803 < 0,$$

we conclude that $j(x) < 0$ for $x > 0$ and $h(x)$ is strictly decreasing for $x > \frac{3e-e^2}{5-e}$. By $h(e) = 0$, $h(x) > 0$ for $\frac{3e-e^2}{5-e} < x < e$ and $h(x) < 0$ for $x > e$. Since we can get $g(x) > 0$ for $\frac{3e-e^2}{5-e} < x < e$ and $g(x) < 0$ for $x > e$, $f'(x) > 0$ for $x > \frac{3e-e^2}{5-e}$ and $f(x)$ is strictly increasing for $x > 0$. From $f(0) = 0$, we have $f(x) > 0$ and $G(x) \ge 0$ for $x > 0$. □

Proof of Lemma 1.5. The proof of the lemma can be done through the graph of function $F_3(x)$ (see Figures 3 and 4). We set

$$f(x) = \ln(e^x - x^e) - 2\ \ln(x-e) + 2 - \frac{3x}{e}$$

for $e < x < 3$. The derivative of $f(x)$ is

$$f'(x) = \frac{g(x)}{e(x-e)x(e^x - x^e)},$$

where $g(x) = x^e(3x^2 - e^2x - ex + e^3) - e^x x(-ex + 3x + e^2 - e)$. From $3x^2 - e^2x - ex + e^3 > 0$ and $-ex + 3x + e^2 - e > 0$ for $e < x < 3$, we can take logarithm and set

$$\begin{aligned} h(x) &= \ln\left(x^e(3x^2 - e^2x - ex + e^3)\right) - \ln\left(e^x x(-ex + 3x + e^2 - e)\right) \\ &= e\ln x + \ln(3x^2 - e^2x - ex + e^3) - x - \ln x - \ln(-ex + 3x + e^2 - e) \end{aligned}$$

for $e < x < 3$. The derivative of $h(x)$ is

$$h'(x) = \frac{(x-e)^2 j(x)}{x(-ex + 3x + e^2 - e)(3x^2 - e^2x - ex + e^3)},$$

where $j(x) = 3ex^2 - 9x^2 - e^3x + 2e^2x - 3ex + e^4 - 2e^3 + e^2$. Since we have

$$\begin{aligned} j(x) &< 3e \cdot 3^2 - 9 \cdot e^2 - e^3 \cdot e + 2e^2 \cdot 3 - 3e \cdot e + e^4 - 2e^3 + e^2 \\ &= 27e - 5e^2 - 2e^3 \approxeq -3.72274, \end{aligned}$$

we obtain $h'(x) < 0$ and $h(x)$ is strictly decreasing for $e < x < 3$. By $h(e) = 0$, we get $h(x) < 0$ and $f'(x) < 0$ for $e < x < 3$. Hence, $f(x)$ is strictly decreasing for $e < x < 3$. Here, we have

$$f(e) = e - 2 - \ln 2 \approxeq 0.0251346$$

and

$$f(3) = -\frac{9}{e} + 2 - 2\ln(3-e) + \ln(e^3 - 3^e) = \ln \frac{e^{2-\frac{9}{e}}(e^3 - 3^e)}{(3-e)^2}.$$

By $e^{2-\frac{9}{e}} \approxeq 0.269573$, $e^3 - 3^e \approxeq 0.272546$ and $3 - e \approxeq 0.281718$, we have

$$f(3) < \ln \frac{0.27 \cdot 0.28}{(0.28)^2} = \ln \frac{27}{28} < 0.$$

Thus, there exists $e < x_0 < 3$ such that $f(x) > 0$ for $e < x < x_0$ and $f(x) < 0$ for $x_0 < x < 3$. Therefore we get $F_3(x) > 0$ for $e < x < x_0$ and $F_3(x) < 0$ for $x_0 < x < 3$. □

Proof of Lemma 1.6. The proof of the lemma can be done through the graph of function $F_4(x)$ (see Figure 5). We set

$$f(x) = \ln(e^x - x^e) - 2\ \ln(e - x) + 2 - \frac{4x}{e}$$

for $\frac{95}{100} < x < \frac{96}{100}$. The derivative of $f(x)$ is

$$f'(x) = \frac{g(x)}{e(e-x)x(e^x - x^e)},$$

where $g(x) = e^x x(-ex + 4x + e^2 - 2e) - x^e(4x^2 - e^2 x - 2ex + e^3)$. From $-ex + 4x + e^2 - 2e > 0$ and $4x^2 - e^2x - 2ex + e^3 > 0$ for $\frac{95}{100} < x < \frac{96}{100}$, we can take logarithm and set

$$\begin{aligned} h(x) &= \ln(e^x x(-ex + 4x + e^2 - 2e)) - \ln(x^e(4x^2 - e^2x - 2ex + e^3)) \\ &= x + \ln x + \ln(-ex + 4x + e^2 - 2e) - e\ln x - \ln(4x^2 - e^2x - 2ex + e^3) \end{aligned}$$

for $\frac{95}{100} < x < \frac{96}{100}$. The derivative of $h(x)$ is

$$h'(x) = \frac{(x-e)^2 j(x)}{x(-ex + 4x + e^2 - 2e)(4x^2 - e^2x - 2ex + e^3)},$$

where $j(x) = -4ex^2 + 16x^2 + e^3x - 2e^2x - e^4 + 3e^3 - 2e^2$. Since we have

$$\begin{aligned} j(x) &> -4e\left(\frac{96}{100}\right)^2 + 16\left(\frac{95}{100}\right)^2 + e^3\left(\frac{95}{100}\right) - 2e^2\left(\frac{96}{100}\right) - e^4 + 3e^3 - 2e^2 \\ &= \frac{36100 - 9216e - 9800e^2 + 9875e^3 - 2500e^4}{2500} \approxeq 0.193947 > 0, \end{aligned}$$

we obtain $h'(x) > 0$ and $h(x)$ is strictly increasing for $\frac{95}{100} < x < \frac{96}{100}$. Here, we have

$$\begin{aligned} h\left(\frac{96}{100}\right) =& \frac{24}{25} + \ln\frac{24}{25} - e\ln\frac{24}{25} + \ln\left(\frac{96}{25} - \frac{74e}{25} + e^2\right) \\ & - \ln\left(\frac{2304}{625} - \frac{48e}{25} - \frac{24e^2}{25} + e^3\right). \end{aligned}$$

By $e^{\frac{24}{25}} \approxeq 2.6117$, $(\frac{24}{25})^{e-1} \approxeq 0.93226$, $\frac{96}{25} - \frac{74e}{25} + e^2 \approxeq 3.18294$ and $\frac{2304}{625} - \frac{48e}{25} - \frac{24e^2}{25} + e^3 \approxeq 11.4593$, we get

$$h\left(\frac{96}{100}\right) < \frac{24}{25} + \ln\frac{24}{25} - e\ln\frac{24}{25} + \ln\frac{3183}{1000} - \ln\frac{1145}{100} = \ln\frac{e^{\frac{24}{25}} \cdot 3.183}{\left(\frac{24}{25}\right)^{e-1} \cdot 11.45}$$
$$< \ln\frac{2.62 \cdot 3.183}{0.93 \cdot 11.45} = \ln\frac{138991}{177475} < 0.$$

Thus, $h(x) < 0$ and $f'(x) < 0$ for $\frac{95}{100} < x < \frac{96}{100}$. Consequently,

$$f\left(\frac{95}{100}\right) = -\frac{19}{5e} + 2 - 2\ln\left(e - \frac{19}{20}\right) + \ln\left(e^{\frac{19}{20}} - \left(\frac{19}{20}\right)^e\right)$$
$$= \ln\frac{e^{2-\frac{19}{5e}}\left(e^{\frac{19}{20}} - \left(\frac{19}{20}\right)^e\right)}{\left(e - \frac{19}{20}\right)^2}$$

and

$$f\left(\frac{96}{100}\right) = -\frac{96}{25e} + 2 - 2\ln\left(e - \frac{24}{25}\right) + \ln\left(e^{\frac{24}{25}} - \left(\frac{24}{25}\right)^e\right)$$
$$= \ln\frac{e^{2-\frac{96}{25e}}\left(e^{\frac{24}{25}} - \left(\frac{24}{25}\right)^e\right)}{\left(e - \frac{24}{25}\right)^2}.$$

By $e^{2-\frac{19}{5e}} \approxeq 1.82587$, $e^{\frac{19}{20}} - \left(\frac{19}{20}\right)^e \approxeq 1.71586$ and $e - \frac{19}{20} \approxeq 1.76828$, we have

$$f\left(\frac{95}{100}\right) > \ln\frac{1.825 \cdot 1.715}{(1.7683)^2} = \ln\frac{312987500}{312688489} > 0.$$

By $e^{2-\frac{96}{25e}} \approxeq 1.7992$, $e^{\frac{24}{25}} - \left(\frac{24}{25}\right)^e \approxeq 1.71673$ and $e - \frac{24}{25} \approxeq 1.75828$, we have

$$f\left(\frac{96}{100}\right) < \ln\frac{1.8 \cdot 1.7168}{(1.758)^2} = \ln\frac{85840}{85849} < 0.$$

Since there exists $\frac{95}{100} < x_0 < \frac{96}{100}$ such that $f(x) > 0$ for $\frac{95}{100} < x < x_0$ and $f(x) < 0$ for $x_0 < x < \frac{96}{100}$, we can get $F_4(x) > 0$ for $\frac{95}{100} < x < x_0$ and $F_4(x) < 0$ for $x_0 < x < \frac{96}{100}$. □

3. Proof of Theorems

Proof of Theorem 1.1. Put $x = \frac{ea_i}{G}$, $1 \le i \le n$, in the inequality $F_2(x) \ge 0$. Then we obtain

$$e^{\frac{ea_i}{G}} \ge \left(\frac{ea_i}{G}\right)^e + e^{\frac{2a_i}{G}}\left(\frac{a_i}{G} - 1\right)^2$$

and

$$e^{\frac{e}{G}nA} = e^{\frac{e}{G}\sum_{i=1}^n a_i} \ge \left(\prod_{i=1}^n \frac{ea_i}{G}\right) + \prod_{i=1}^n e^{\frac{2a_i}{G}}\left(\frac{a_i}{G} - 1\right)^2 = e^{ne} + e^{2n\frac{A}{G}}\prod_{i=1}^n\left(\frac{a_i}{G} - 1\right)^2.$$

Therefore, we have

$$e^{\frac{e}{G}nA} \ge e^{ne}\left(1 + \frac{e^{2n\frac{A}{G}}}{e^{ne}}\prod_{i=1}^n\left(\frac{a_i}{G} - 1\right)^2\right).$$

Now, we recall the definition of $\mathcal{R}_c$ in (1.1). Taking logarithm, we get

$$\frac{A}{G} \geq 1 + \frac{1}{ne} \ln \left(1 + \frac{e^{2n\frac{A}{G}}}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1 \right)^2 \right)$$

and

$$A - G \geq \frac{G}{ne} \ln \left(1 + \frac{e^{2n\frac{A}{G}}}{e^{ne}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1 \right)^2 \right) = \mathcal{R}_2.$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. From Lemmas 1.3, 1.4, 1.5 and 1.6, the ineqiality

$$x^e + e^{\frac{\alpha x}{e}} \left(\frac{x}{e} - 1 \right)^2 \leq e^x \leq x^e + e^{\frac{\beta x}{e}} \left(\frac{x}{e} - 1 \right)^2 \tag{3.1}$$

holds for $x > 0$, with equality if and only if $x = e$, where the constants $\alpha = 2$ and $\beta = 5$ are the best possible integers. Put $x = \frac{ea_i}{G}$, $1 \leq i \leq n$, in the inequality (3.1). Thus, we obtain

$$\prod_{i=1}^{n} \left(\left(\frac{ea_i}{G} \right)^e + e^{\frac{2a_i}{G}} \left(\frac{a_i}{G} - 1 \right)^2 \right) \leq e^{\frac{e}{G} nA} \leq \prod_{i=1}^{n} \left(\left(\frac{ea_i}{G} \right)^e + e^{\frac{5a_i}{G}} \left(\frac{a_i}{G} - 1 \right)^2 \right).$$

Now, we recall the definition of $\mathcal{B}_c$ in (1.2). Taking logarithm, we get

$$\mathcal{B}_\alpha \leq \frac{A}{G} \leq \mathcal{B}_\beta$$

with the best possible integer constants $\alpha = 2$ and $\beta = 5$. The proof of Theorem 1.2 is complete. □

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