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REDHEFFER-TYPE INEQUALITIES FOR GENERALIZED TRIGONOMETRIC FUNCTIONS

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Abstract. Famous Redheffer's inequality is generalized to a class of anti-periodic functions. We apply the novel inequality to the generalized trigonometric functions and establish several Redheffer-type inequalities for these functions.

1. Introduction

Redheffer's inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \quad x \neq 0,$$

was proposed by R. Redheffer in 1968 as an advanced problem in Amer. Math. Monthly [3, No. 5642]. In another issue of the journal [7, No. 5642], the inequality was titled *A delightful inequality* and J.P. Williams gave a proof relying on the infinite product representation of the sine function. Since then, Redheffer's inequality has been widely studied in the area of inequality (see [8] and the references given there).

In 2015, Sándor and Bhayo [8] presented a novel interesting proof of Redheffer's inequality, which is based on the elementary calculus. Their proof has the potential to generalize Redheffer's inequality to more functions than just the sine function, as well to the cosine function, the hyperbolic sine and cosine functions ([2]), and to Bessel functions ([1]).

In this paper, inspired by the proof of [8], we will generalize Redheffer's inequality so that it can be applied to a class of anti-periodic functions including the sine function. To be precise, we establish a Redheffer-type inequality for a function S that satisfies the following conditions: there exist an $a \in (0, \infty)$ and a finite subset $P \subset (0, a)$ such that

(S1) $S(-x) = -S(x)$ and $S(a + x) = -S(x)$ for $x \in [0, \infty)$;
(S2) $0 < S(x) < x$ for $x \in (0, a)$;
(S3) $S \in C([0, a]) \cap C^1([0, a)) \cap C^2([0, a) \setminus P)$;

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(S4) $S'(x)^2 - S''(x)S(x) \geq 1$ for $x \in [0,a) \setminus P$.

It is clear that $S(x)$ is odd, continuous, piecewise smooth in $\mathbb{R}$ and anti-periodic with period a in $[0,\infty)$; $S(na)=0$, $(-1)^n S(x) > 0$ for $x \in (na,(n+1)a)$ and $n \in \mathbb{Z}$; and $S(x) < x$ for $x \in (0,\infty)$ and $S \in C^1((-a,a))$ with $S'(0)=1$.

We begin with a general result on such a function S.

Theorem 1.1. *Let S be a function satisfying the conditions* (S1)-(S4). *Then,*

$$\frac{a^2-x^2}{a^2+x^2} \leq \frac{S(x)}{x}, \quad x \neq 0. \tag{1.1}$$

It is worth pointing out that Redheffer's inequality follows immediately from Theorem 1.1, since $S(x) = \sin x$ satisfies the conditions (S1)-(S4) with $a=\pi$ and $P=\emptyset$, especially $S'(x)^2 - S''(x)S(x) \equiv 1$.

Theorem 1.1 yields Redheffer-type inequalities for generalized functions of the trigonometric sine and cosine functions. Before stating the inequalities, we will define generalized trigonometric functions.

Let $1 < p,\ q < \infty$ and

$$F_{p,q}(x) := \int_0^x \frac{dt}{(1-t^q)^{1/p}}, \quad x \in [0,1].$$

We will denote by $\sin_{p,q}$ the inverse function of $F_{p,q}$, i.e.,

$$\sin_{p,q} x := F_{p,q}^{-1}(x).$$

Clealy, $\sin_{p,q} x$ is an increasing function in $[0,\pi_{p,q}/2]$ to $[0,1]$, where

$$\pi_{p,q} := 2F_{p,q}(1) = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}}.$$

We extend $\sin_{p,q} x$ to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q}-x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q} x \in C^1(\mathbb{R})$, we also define $\cos_{p,q} x$ by $\cos_{p,q} x := (\sin_{p,q} x)'$, where $' := d/dx$. Then, it follows that

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.$$

In case $(p,q)=(2,2)$, it is obvious that $\sin_{p,q} x$, $\cos_{p,q} x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x$, $\cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p,q)) and the *generalized* π, respectively.

The generalized trigonometric functions are well studied in the context of nonlinear differential equations as well as special functions (see [4, 10] and the references given there). Suppose that u is a solution of the initial value problem of p-Laplacian

$$-(|u'|^{p-2}u')' = \frac{(p-1)q}{p}|u|^{q-2}u, \quad u(0)=0,\ u'(0)=1,$$

which is reduced to the equation $-u''=u$ of simple harmonic motion for $u=\sin x$ in case $(p,q)=(2,2)$. Then,

$$(|u'|^p + |u|^q)' = \left(\frac{p}{p-1}(|u'|^{p-2}u')' + q|u|^{q-2}u\right)u' = 0.$$

Therefore, $|u'|^p + |u|^q = 1$, hence it is reasonable to define u as a generalized sine function and u' as a generalized cosine function. Indeed, it is possible to show that u coincides with $\sin_{p,q}$ defined as above. The generalized trigonometric functions are often applied to the eigenvalue problem of p-Laplacian.

Now, to the authors' knowledge, no Redheffer-type inequalities have been obtained for the generalized trigonometric functions. Applying Theorem 1.1 to $\sin_{p,q} x$, we can prove the following inequalities.

Theorem 1.2. *Let* $2 \le p,\ q < \infty$. *Then,*

$$\frac{\pi_{p,q}^2 - x^2}{\pi_{p,q}^2 + x^2} \le \frac{\sin_{p,q} x}{x}, \quad x \neq 0. \tag{1.2}$$

In particular, for $2 \le p < \infty$,

$$\frac{\pi_p^2 - x^2}{\pi_p^2 + x^2} \le \frac{\sin_p x}{x}, \quad x \neq 0,$$

where $\sin_p x := \sin_{p,p} x$ *and* $\pi_p := \pi_{p,p}$.

Regarding the cosine function, Chen, Zhao and Qi [2] prove the Redheffer-type inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} < \cos\frac{x}{2}, \quad x \in (0, \pi). \tag{1.3}$$

Theorem 1.2 yields the following generalization of (1.3) and also offers an alternative proof of (1.3) than that given in [2].

Corollary 1.3. *Let* $2 \le q < \infty$. *Then,*

$$\frac{\pi_{q^*,q}^2 - x^2}{\pi_{q^*,q}^2 + x^2} < \cos_{q^*,q}^{q^*-1}\frac{x}{2}, \quad x \in (0, \pi_{q^*,q}),$$

where $q^* := q/(q-1)$.

2. Proofs of results

In this section, we give the proofs of Theorems 1.1, 1.2 and Corollary 1.3.

2.1. **Proof of Theorem 1.1.** We follow the idea of Sándor and Bhayo [8].

For $x = \pm a$, the equality of (1.1) clearly holds. Since the both sides of (1.1) are even functions by (S1), it is sufficient to prove the strict inequality of (1.1) for $x \in (0, a) \cup (a, \infty)$.

It is easy to show

$$\frac{a^2 - x^2}{a^2 + x^2} < \frac{S(x)}{x}, \quad x \in (a, \infty). \tag{2.1}$$

Indeed, there exists $t > 0$ such that $x = a + t$. Using the anti-periodicity of S in (S1), we know

$$\begin{aligned}\frac{S(x)}{x} - \frac{a^2 - x^2}{a^2 + x^2} &= -\frac{S(t)}{a+t} + \frac{2at + t^2}{2a^2 + 2at + t^2} \\ &= \frac{t}{a+t}\left(\frac{2a^2 + 3at + t^2}{2a^2 + 2at + t^2} - \frac{S(t)}{t}\right).\end{aligned}$$

Since $S(t)/t < 1$ for all $t \in (0, \infty)$, we have (2.1).

It remains to show that

$$\frac{a^2 - x^2}{a^2 + x^2} < \frac{S(x)}{x}, \quad x \in (0, a). \tag{2.2}$$

From (S2), the inequality above is equivalent to the following inequality.

$$a^2 < f(x), \quad x \in (0, a), \tag{2.3}$$

where

$$f(x) = \frac{x^2(x+S(x))}{x-S(x)}, \quad x \in (0,a). \tag{2.4}$$

Now, we will prove (2.3). Let $x \in (0,a)$. An easy calculation yields

$$-\frac{(x-S(x))^2}{2xS(x)} f'(x) = g(x), \tag{2.5}$$

where

$$g(x) = x + S(x) - \frac{x^2(1+S'(x))}{S(x)}. \tag{2.6}$$

Let b be any number in $(0,a)$. If $1+S'(b) \le 0$, then (2.6) yields $g(b) > 0$. We consider the case where $1+S'(b) > 0$. First, we suppose that $1+S'(x) > 0$ for $x \in [0,b]$. Then, for $x \in (0,b] \setminus P$,

$$\begin{aligned} S^2 g'(x) &= ((S')^2 - S''S + S')x^2 - 2S(1+S')x + S^2(1+S') \\ &= (1+S')(x-S)^2 + ((S')^2 - S''S - 1)x^2, \end{aligned} \tag{2.7}$$

where $S = S(x)$. From (S4), the right-hand side of (2.7) is positive in $(0,b] \setminus P$. Hence, $g'(x) > 0$ in $(0,b] \setminus P$. Since $g \in C((0,b])$ and $S'(0) = 1$, $g(x)$ is strictly increasing in $(0,b]$ and

$$g(b) > \lim_{x \to +0} g(x) = 0.$$

Next we suppose that there exists $c \in (0,b)$ such that $1+S'(x) > 0$ for $x \in (c,b]$ and $1+S'(c) = 0$. Then, in a similar way as above, we can see that $g(x)$ is strictly increasing in $(c,b]$ and

$$g(b) > \lim_{x \to c+0} g(x) = c + S(c) > 0.$$

In either case, we obtain $g(b) > 0$. Thus, using (2.5), we have $f'(x) < 0$, hence $f(x)$ is strictly decreasing in $(0,a)$. Since $f \in C((0,a])$,

$$f(x) > f(a) = a^2.$$

This is the desired conclusion (2.3). □

2.2. **Proof of Theorem 1.2.** Let $2 \le p,\ q < \infty$. Then, we can see that $S(x) = \sin_{p,q} x$ satisfies the conditions (S1)-(S4) with $a = \pi_{p,q}$ and $P = \{\pi_{p,q}/2\}$. Indeed, (S1) and (S2) are easily checked. For (S3), it is known that $\sin_{p,q} x \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus Z)$, where $Z = \{(2n+1)\pi_{p,q}/2 \mid n \in \mathbb{Z}\}$ ([5, Proposition 2.1]). Finally, (S4) is proved as follows. For $x \in [0, \pi_{p,q}/2) \cup (\pi_{p,q}/2, \pi_{p,q})$,

$$S''(x) = -\frac{q}{p} \sin_{p,q}^{q-1} x |\cos_{p,q} x|^{2-p} \tag{2.8}$$

and

$$\begin{aligned} S'(x)^2 - S''(x)S(x) - 1 &= \cos_{p,q}^2 x + \frac{q}{p} \sin_{p,q}^q x |\cos_{p,q} x|^{2-p} - 1 \\ &= \left(1 - \frac{q}{p}\right) \cos_{p,q}^2 x + \frac{q}{p} |\cos_{p,q} x|^{2-p} - 1. \end{aligned}$$

Since $p,\ q \ge 2$, it is easy to show that $h(t) = (1-q/p)t^2 + (q/p)t^{2-p} - 1$ is nonincreasing in $(0,1]$; hence $h(t) \ge h(1) = 0$ in $(0,1]$. Therefore, S satisfies (S4). Thus, we can apply Theorem 1.1 to $S(x) = \sin_{p,q} x$, and the proof is complete. □

2.3. **Proof of Corollary 1.3.** Letting $p = 2,\ 2 \leq q < \infty$ and $x = 2^{2/q-1}y$ in (1.2), we obtain

$$\frac{\pi_{2,q}^2 - (2^{2/q-1}y)^2}{\pi_{2,q}^2 + (2^{2/q-1}y)^2} \leq \frac{\sin_{2,q}(2^{2/q-1}y)}{(2^{2/q-1}y)}, \quad y \neq 0. \tag{2.9}$$

Since $\pi_{2,q} = 2^{2/q-1}\pi_{q^*,q}$ by [9, (1.10)], the left-hand side of (2.9) can be rewritten as

$$\frac{\pi_{q^*,q}^2 - y^2}{\pi_{q^*,q}^2 + y^2}.$$

On the other hand, we know the multiple-angle formula [9, Theorem 1.1]: for $x \in [0, \pi_{2,q}/(2^{2/q})] = [0, \pi_{q^*,q}/2]$, then

$$\sin_{2,q}(2^{2/q}x) = 2^{2/q}\sin_{q^*,q} x \cos_{q^*,q}^{q^*-1} x.$$

Thus, for $y \in (0, \pi_{q^*,q})$, the right-hand side of (2.9) is equal to

$$\frac{\sin_{q^*,q}(y/2)\cos_{q^*,q}^{q^*-1}(y/2)}{y/2}.$$

Since $\sin_{q^*,q}(y/2) < y/2$ in $(0, \pi_{q^*,q})$, it is strictly less than $\cos_{q^*,q}^{q^*-1}(y/2)$. This completes the proof. □

3. Remarks

3.1. **Estimate from above.** For (2.2) in the proof of Theorem 1.1, it is also possible to obtain an estimate of $S(x)/x$ from above if one supposes (S2)-(S4) and that the negative limit

$$d := \lim_{x \to +0} \frac{S''(x)}{x} \in (-\infty, 0)$$

exists. Indeed, in this case, we have seen that $f(x)$, defined as (2.4), is strictly decreasing in $(0, a]$, hence l'Hospital rule yields

$$a^2 = f(a) < \lim_{x \to +0} f(x) = -\frac{12}{d}$$

and

$$\frac{a^2 - x^2}{a^2 + x^2} < \frac{S(x)}{x} < \frac{12 + dx^2}{12 - dx^2}, \quad x \in (0, a).$$

Applying this inequality to $S(x) = \sin_{p,2} x$ and $a = \pi_{p,2}$, we see that $d = -2/p$ by (2.8) and for $p \in [2, \infty)$,

$$\frac{\pi_{p,2}^2 - x^2}{\pi_{p,2}^2 + x^2} < \frac{\sin_{p,2} x}{x} < \frac{6p - x^2}{6p + x^2}, \quad x \in (0, \pi_{p,2}).$$

In particular, the case of $p = 2$ is due to Sándor and Bhayo [8, Theorem 1].

3.2. **Powers in the L.H.S.** We mention the left-hand side of (1.2) in Theorem 1.2. Paredes and Uchiyama [6, Theorem 1.1] show that $\sin_{p,q} x$ has a convergent expansion near $x = 0$ as

$$\sin_{p,q} x = x - \frac{1}{p(q+1)}|x|^q x + \frac{1 - p + 3q - pq}{2p^2(q+1)(2q+1)}|x|^{2q} x + \cdots.$$

From the expression, we see that $\sin_{p,q} x/x$ can be expressed in terms of power series of $|x|^q$. In this sense, one may expect that if the left-hand side of (1.2) is replaced with

$$\frac{\pi_{p,q}^q - |x|^q}{\pi_{p,q}^q + |x|^q},$$

it will always hold for all $p,\ q \in (1,\infty)$. Actually, it certainly holds near $x = 0$, however it does not hold near $x = \pi_{p,q}$ (see Figure 1).

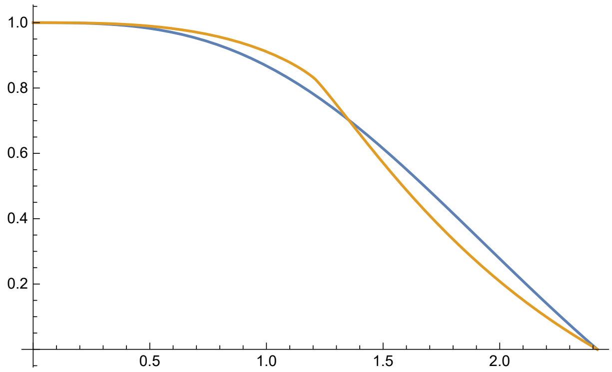

FIGURE 1. Graphs of $(\pi_{p,q}^q - |x|^q)/(\pi_{p,q}^q + |x|^q)$ and $\sin_{p,q} x/x$ in $(0, \pi_{p,q}]$ for $(p, q) = (3, 3)$.

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