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INEQUALITIES FOR THE SIZE OF A BODY DERIVED FROM ITS SCATTERING AMPLITUDE

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Abstract. The scattering solution $u := u(x, \alpha)$ for a bounded obstacle D with a connected boundary S on which the Dirichlet boundary condition holds satisfies the equation $u(x, \alpha) = u_0(x, \alpha) - \int_S g(x, s)h(s, \alpha)ds$, $h(s, \alpha) := u_N(s, \alpha)$, where N is a unit normal to S pointing out of D, $g := \frac{e^{ik|x-y|}}{4\pi|x-y|}$, α is the unit vector in the direction of the incident plain wave $u_0 := e^{ik\alpha\cdot x}$. The main result of the paper is a formula for the size of D in terms of the analytic continuation of the scattering amplitude with respect to α.

1. Introduction and formulation of the results

In [1] a method for estimating the size of a scatterer in terms of the analytic continuation of the scattering amplitude $A(\beta, \alpha)$ of this scatterer with respect to β, α is given. In this paper we give an asymptotic formula for the analytical continuation of the scattering solution and derive from this formula an estimate for the size of the scatterer.

First, let us recall the equation for the scattering solution u. Let $D \in \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S, $D' := \mathbb{R}^3 \setminus D$. The scattering solution solves equation

$$(\nabla^2 + k^2)u = 0 \quad x \in D', \tag{1.1}$$

$$u|_S = 0, \tag{1.2}$$

$$u = u_0 + v, \quad u_0 := e^{ik\alpha\cdot x}, \tag{1.3}$$

where the scattered field v satisfies the radiation condition

$$v_r - ikv = o(\frac{1}{r}), \quad r \to \infty. \tag{1.4}$$

Here $k = const > 0$ is fixed, $r := |x|$, $v_r := \frac{\partial v}{\partial r}$. In [4], [3], [5], [6], the existence and uniqueness of the scattering solutions are proved. In [2] some symmetry properties for the solutions to the Helmholtz equation are established.

It follows from (1.4) that

$$v = \frac{e^{ikr}}{r}A(\beta, \alpha) + o(\frac{1}{r}), \quad r \to \infty, \quad \beta := \frac{x}{r}. \tag{1.5}$$

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The function $A(\beta, \alpha)$ is called the scattering amplitude. The dependence of A on k is not shown since $k > 0$ is a fixed constant.

Let M be a complex analytic variety in $\mathbb{C}^3$ consisting of complex vectors $z \in \mathbb{C}^3$ satisfying the equation

$$z \cdot z = 1, \quad z \cdot z := \sum_{j=1}^{3} z_j^2. \tag{1.6}$$

Let S^2 be a unit sphere in $\mathbb{R}^3$. The following result is proved in [4], p. 62.

Lemma 1. *If D is bounded then the scattering amplitude admits an analytic continuation from S^2 to the variety M.*

Proof. From formula (1.10), see below, it follows that the scattering amplitude $A(\beta, \alpha)$ is an entire function of β. It is analytic with respect to β in M. The S^2 is a subset of M. From the known formula (see [4], p.53):

$$A(\beta, \alpha) = A(-\alpha, -\beta), \tag{1.7}$$

which is valid for β, α on $S^2 \subset M$ it follows that $A(\beta, \alpha)$ is analytic also with respect to α and formula (1.7) holds on M by analytic continuation. If $A(\beta, \alpha)$ is known on an open subset of S^2 it is continued analytically on the whole M since M is an analytic variety. Lemma 1 is proved. □

Applying the Green's formula to u one derives an integral equation

$$u(x, \alpha) = u_0(x, \alpha) - \int_S g(x, s) h(s) ds := u_0 - Th, \quad h := h(s, \alpha) := u_N, \tag{1.8}$$

where u_N is the normal derivative of u on S from D', N is the unit normal to S directed out of D. From (1.8) by taking the normal derivative and using the known formula

$$(Th)_N = (Ah - h)/2$$

(see [4],p.16) for the normal derivative from D' of the single layer potential Th one derives:

$$h = -Ah + 2u_{0N}, \quad Ah := \int_S \frac{\partial g(s, t)}{\partial N_s} h(t) dt, \quad h(t) := h(t, \alpha) := h. \tag{1.9}$$

One can prove that the system of equations (1.8)–(1.9) for u and $h := u_N$ has a solution and this solution is unique (see [4], [5]): the existence and uniqueness of the scattering solution u is proved in [4] and h is uniquely determined by u as its normal derivative on S from outside.

The scattering solution and h do not depend on β. The scattering amplitude depends on α and β but not on x. The $h(s, \alpha)$ is bounded on $S \times S^2$.

Let $\{e_j\}$ be an orthonormal basis of $\mathbb{R}^3$. Define $\alpha := ae_1 + ibe_2$, where $a^2 - b^2 = 1$, $a, b \in \mathbb{R}$. Then $\alpha \in M$. One derives from (1.8) a representation formula for the scattering amplitude:

$$A(\beta, \alpha) = -\frac{1}{4\pi} \int_S e^{-ik\beta \cdot s} h(s, \alpha) ds. \tag{1.10}$$

By formula (1.7) one has:

$$A(\beta, \alpha) = A(-\alpha, -\beta) = -\frac{1}{4\pi} \int_S e^{ik\alpha \cdot s} h(s, -\beta) ds. \tag{1.11}$$

Lemma 2. One has:

$$|u_0(x, \alpha)| \gg |\int_S g(x, s) h(s, \alpha) ds|, \quad |bx_2| \to \infty, \quad bx_2 < 0. \tag{1.12}$$

Proof. From formula (1.8) it follows that inequality (1.12) holds because the integral in (1.8) decays as $|x| \to \infty$. One has $|u_0| = e^{-kbx_2}$, $k > 0$, so $|u_0| \to \infty$ if $bx_2 < 0$ and $|bx_2| \to \infty$. Lemma 2 is proved. □

Lemma 3. *One has:*

$$|u(x,\alpha)| \sim |u_0(x,\alpha)|, \quad |bx_2| \to \infty, \quad -bx_2 \to +\infty, \quad x \in D'. \tag{1.13}$$

Proof. Formula (1.13) follows from (1.8) and (1.12). Note that the integral in Lemma 2 tends to zero as $|x| \to \infty$. Lemma 3 is proved. □

Lemma 4. *One has*

$$|u(x,\alpha)| \sim e^{-kbx_2}, \quad -bx_2 \to +\infty, \quad x \in D'. \tag{1.14}$$

Proof. Formula (1.14) follows from (1.8), (1.13) and (1.12). Lemma 4 is proved. □

Lemma 5. *One has:*

$$|4\pi A(\beta,\alpha)| \leq e^{\max_{s\in S}(-kbs_2)} c, \tag{1.15}$$

where $c > 0$ is a constant independent of b.

Proof. Using formula (1.11) one gets

$$|4\pi A(\beta,\alpha)| \leq e^{\max_{s\in S}(-kbs_2)} \int_S |h(s,-\beta)| ds.$$

Denote

$$\int_S |h(s,-\beta)| ds := c.$$

Lemma 5 is proved. □

In Lemma 5 an estimate of the growth rate of the analytic continuation of the scattering amplitude is given in terms of the size of the body.

Theorem 1. *Let $\alpha = ae_1 + ibe_2$, where $a^2 - b^2 = 1$, so that $\alpha \in M$. Then*

$$\lim_{b\to-\infty} \frac{\ln|A(-\alpha,-\beta)|}{k|b|} \leq x_{2max}, \tag{1.16}$$

$$\lim_{b\to+\infty} \frac{\ln|A(-\alpha,-\beta)|}{-kb} \geq x_{2min}. \tag{1.17}$$

Proof. Formula (1.16) follows from Lemma 5. Indeed, if $b < 0$, then

$$\max_{s\in S}(-kbs_2) \leq k|b| |\max_{s\in S} s_2 := k|b|x_{2max}. \tag{1.18}$$

Formula (1.17) is obtained similarly. If $b > 0$, then changing the direction of x_2 axis one gets the estimate similar to (1.18):

$$\min_{s\in S} s_2 \leq \lim_{b\to+\infty} \frac{\ln|A(-\alpha,-\beta)|}{-kb}.$$

In this argument we assumed that the origin of the coordinate system is located inside D. If the plane orthogonal to e_2 intersects D, then formulas (1.16)–(1.17) are not changed. The case when this plane does not intersect D is left to the reader. Theorem 1 is proved. □

The quantity x_{max} is the maximal value of x_2 on S and x_{min} is the minimal value of x_2 on S. One can choose as e_2 any direction, so the $|x_{2max} - x_{2min}|$ is the breadth of S in the direction orthogonal to e_2.

The geometrical meaning of $|x_{2max} - x_{2min}|$ is the distance between two parallel planes orthogonal to e_2 and tangent to S. These planes are called the support planes. The direction $e_2 := \ell$ can be chosen arbitrarily.

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