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GENERALIZATIONS OF SOME CLASSICAL INTEGRAL INEQUALITIES CONTAINING EXTENDED MITTAG-LEFFLER FUNCTION IN THE KERNEL

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Abstract. Several classical integral inequalities have been proved in more general settings using fractional integral operator containing an extended Mittag-Leffler function in the kernel. A number of corollaries and consequences of the main results are also presented.

1. Introduction and preliminary results

In this paper we continue our research on classical integral inequalities, recently given in the paper [2], where we proved several fractional generalizations and extensions of known integral inequalities, obtained with an extended generalized Mittag-Leffler function and its fractional integral operator from [1].

Our motivation is the paper by W. Liu et al. [6] from which we point out, as an example, the following inequality:

Theorem 1.1. [6, Theorem 4] *Let f, g be positive continuous functions on $[a, b]$ such that f is decreasing and g is increasing. Then the following inequality*

$$\frac{\displaystyle\int_a^x f^{\beta}(t)\,dt}{\displaystyle\int_a^x f^{\gamma}(t)\,dt} \geq \frac{\displaystyle\int_a^x g^{\alpha}(t)\,f^{\beta}(t)\,dt}{\displaystyle\int_a^x g^{\alpha}(t)\,f^{\gamma}(t)\,dt} \tag{1.1}$$

holds for every $\alpha > 0$ and $\beta \geq \gamma > 0$. If f is increasing, then (1.1) *is reverse.*

We want to prove such inequalities in more general settings using fractional calculus, the theory of differential and integral operators of non-integer order (more about fractional calculus can be found in the monographs [5, 7]). This calculus has made a significant impact in numerical analysis by providing important applications

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in various fields of physics and engineering. Therefore, new forms of fractional integral operators are constantly being explored, but all of them, in a special case, are reduced to *the left-sided Riemann-Liouville fractional integral* $J_{a+}^{\sigma}f$ *of order* σ defined as in [5, 7] for $f \in L_1[a,b]$ by

$$J_{a+}^{\sigma}f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t)\, dt, \quad x \in (a,b]. \tag{1.2}$$

Of course, we have also the right-sided Riemann-Liouville fractional integral

$$J_{b-}^{\sigma}f(x) = \frac{1}{\Gamma(\sigma)} \int_x^b (t-x)^{\sigma-1} f(t)\, dt, \quad x \in [a,b).$$

For $\sigma = n \in \mathbb{N}$, these fractional integrals are actually n-folds integrals. Using this, Z. Dahmani made the following fractional extension and generalization of the inequality (1.1):

Theorem 1.2. [3, Theorem 3.6] *Let* $(f_i)_{i=1,2,\ldots,n}$ *and* g *be positive continuous functions on* $[a,b]$ *such that* $(f_i)_{i=1,2,\ldots,n}$ *are decreasing and* g *is increasing. Then the following inequality*

$$\frac{J_{a+}^{\sigma}\left[\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}(x)\right] J_{a+}^{\sigma}\left[g^{\alpha}(x)\prod_{i=1}^{n} f_i^{\gamma_i}(x)\right]}{J_{a+}^{\sigma}\left[g^{\alpha}(x)\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}(x)\right] J_{a+}^{\sigma}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(x)\right]} \geq 1 \tag{1.3}$$

holds for every $a < x \leq b$, $\sigma > 0$, $\alpha > 0$, $\beta \geq \gamma_s > 0$, *where* s *is a fixed integer in* $\{1,2,\ldots,n\}$.

The Mittag-Leffler function E_ρ is defined for $z \in \mathbb{C}$ and $\Re(\rho) > 0$ by the power series using the gamma function Γ

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}. \tag{1.4}$$

Since this function with its generalizations appear as a solution of fractional order differential or integral equations, in [1] we presented more extended and generalized version $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ as follows:

Definition 1.3. *Let* $\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \Re(\rho)$. *Then* $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ *is defined by*

$$E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq, c-\delta)}{\mathrm{B}(\delta, c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}. \tag{1.5}$$

Recall that $(c)_{nq}$ denotes the generalized Pochhammer symbol $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$ and for $\Re(x),\Re(y),\Re(p) > 0$, B_p is an extension of the beta function

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt.$$

As proved in [1], the series (1.5) is absolutely convergent for all values of z provided that $q < r + \Re(\rho)$. Moreover, if $q = r + \Re(\rho)$, then $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ converges for $|z| < \frac{r^r\, \Re(\rho)^{\Re(\rho)}}{q^q}$.

Remark. *Notice,*

$$\begin{aligned} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;0) &= \sum_{n=0}^{\infty}\frac{(\delta)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{(\tau)_{nr}}, \\ E^{\delta,c,q,r}_{\rho,\sigma,\tau}(0;p) &= \frac{\mathrm{B}_p(\delta,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{1}{\Gamma(\sigma)}, \\ E^{\delta,c,q,r}_{\rho,\sigma,\tau}(0;0) &= \frac{1}{\Gamma(\sigma)}. \end{aligned}$$

We will use all these calculations in the proof of our results.

This extended Mittag-Leffler function is contained in the kernel of the following fractional integral operator (also defined in [1]):

Definition 1.4. *Let $w,\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\geq 0$, $r>0$ and $0<q\leq r+\Re(\rho)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then the generalized fractional integral operator $\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ is defined by*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)=\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)f(t)dt. \tag{1.6}$$

Considering Remark 1, if we set $p=0$ and $\sigma=\tau=r=q=\delta=1$ in Definition 1.3, then (1.5) reduces to the Mittag-Leffler function (1.4), and setting $p=\omega=0$ in Definition 1.4, (1.6) reduces to the Riemann-Liouville fractional integral $J^{\sigma}_{a^+}f$ of order σ, as defined in (1.2). More details on how to derive the known generalizations of the Mittag-Leffler function and its fractional integral operator from (1.5) and (1.6) can be seen in [1].

We give in [4] a further generalization of the fractional integral operator:

Definition 1.5. *Let $w,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\geq 0$, $\rho,r>0$ and $0<q\leq r+\rho$. Let $f\in L_1[a,b]$, $0<a<b<\infty$, be a positive function. Let $h:[a,b]\to\mathbb{R}$ be a differentiable function, strictly increasing. Also let $x\to\frac{\phi(x)}{x}$ be an increasing function on $[a,\infty)$ and $x\in[a,b]$. Then the generalized fractional integral operator ${}^{\phi}_{h}F^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ is defined by*

$$\begin{aligned} &\left({}^{\phi}_{h}F^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \\ &= \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};p)h'(t)f(t)dt. \end{aligned}$$

For the purpose of this paper, we will need a special case of the above operator, obtained for $\phi(x)=x^{\sigma}$, $\sigma>0$, in real domain. This operator we denote ${}_{h}\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ and define as follows:

Definition 1.6. *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f\in L_1[a,b]$ be a positive function and $h:[a,b]\to\mathbb{R}$ be a differentiable function, strictly increasing. Then for $x\in[a,b]$ the generalized fractional integral operator ${}_{h}\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ is defined by*

$$\begin{aligned} &\left({}_{h}\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \\ &= \int_a^x (h(x)-h(t))^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};p)h'(t)f(t)dt. \end{aligned} \tag{1.7}$$

The fractional integral operator (1.6) can be obtained from the above if the function h is the identity function. On the other hand, if we set $p = \omega = 0$ in this definition, then (1.7) reduces to *the left-sided Riemann-Liouville fractional integral of a function f with respect to another function h of order σ* ([5, 7]):

$$J^{\sigma}_{a+;h} f(x) \;=\; \frac{1}{\Gamma(\sigma)} \int_a^x (h(x) - h(t))^{\sigma-1} h'(t) f(t)\, dt, \quad x \in (a, b].$$

The aim of this paper is to present several classical integral inequalities for a generalized fractional integral operator ${}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f$ containing an extended Mittag-Leffler function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}$ in the kernel.

We emphasize that the right-sided versions of all inequalities in this paper can be established using

$$\begin{aligned}&\left({}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f\right)(x;p)\\ &= \int_x^b (h(t) - h(x))^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(t) - h(x))^{\rho}; p) h'(t) f(t) dt\end{aligned}$$

and proved analogously. Also, for the reader's convenience we will use a simplified notation

$$\begin{aligned}\boldsymbol{E}(z;p) &:= E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p),\\ (\boldsymbol{h\Upsilon} f)(x;p) &:= \left({}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p).\end{aligned}$$

2. Fractional integral inequalities containing the Mittag-Leffler function in the kernel

To generalize Theorem 1.1 and Theorem 1.2 using fractional calculus, we supplement methods from [3, 6] with the necessary steps.

Theorem 2.1. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in [a, b]$. Let $(f_i)_{i=1,2,\ldots,n}$ and g be positive continuous functions, such that $(f_i)_{i=1,2,\ldots,n}$ are decreasing and g is increasing with $(f_i)_{i=1,2,\ldots,n} \in L_\beta[a, b]$ and $g \in L_\alpha[a, b]$. Then for the fixed integer $s \in \{1, 2, \ldots, n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{h\Upsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{h\Upsilon}\left(\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)} \;\geq\; \frac{\left(\boldsymbol{h\Upsilon}\left(g^{\alpha}\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{h\Upsilon}\left(g^{\alpha}\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{2.1}$$

If $(f_i)_{i=1,2,\ldots,n}$ are increasing and g is decreasing, then the inequality (2.1) *also holds.*
If all functions are monotonic in the same sense, then the inequality (2.1) *is reverse.*

Proof. Let $u, v \in [a, x]$. Let $(f_i)_{i=1,2,\ldots,n}$ be decreasing and g increasing, all positive and continuous. Let $s \in \{1, 2, 3, ..., n\}$. Then

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0, \tag{2.2}$$

hence

$$\begin{aligned}&(g(u))^{\alpha}(f_s(v))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(u))^{\beta-\gamma_s}\\ &\geq\; (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(v))^{\beta-\gamma_s}.\end{aligned}$$

Multiplying both sides of the above inequality by

$$(h(x)-h(v))^{\sigma-1}\boldsymbol{E}\left(w(h(x)-h(v))^{\rho};p\right)\prod_{i=1}^{n}(f_i(v))^{\gamma_i}h'(v) \tag{2.3}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}(g(u))^{\alpha}&\left({}_{\boldsymbol{h}}\Upsilon\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)\\ &+(f_s(u))^{\beta-\gamma_s}\left({}_{\boldsymbol{h}}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)\\ \geq\ &(g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left({}_{\boldsymbol{h}}\Upsilon\left(\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)\\ &+\left({}_{\boldsymbol{h}}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p).\end{aligned}$$

Further multiplying by

$$(h(x)-h(u))^{\sigma-1}\boldsymbol{E}\left(w(h(x)-h(u))^{\rho};p\right)\prod_{i=1}^{n}(f_i(u))^{\gamma_i}h'(u) \tag{2.4}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$\begin{aligned}&\left({}_{\boldsymbol{h}}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)\left({}_{\boldsymbol{h}}\Upsilon\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)\\ \geq\ &\left({}_{\boldsymbol{h}}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)\left({}_{\boldsymbol{h}}\Upsilon\left(\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p).\end{aligned}$$

from which follows (2.1).
Analogously we can prove the case when $(f_i)_{i=1,2,\ldots,n}$ are increasing and g is decreasing, and obtain reversed inequality if all functions are monotonic in the same sense. □

Remark. *If the function h is the identity function, then we obtain an inequality for the generalized fractional operator $\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f$ defined by* (1.6)*:*

$$\frac{\left(\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)}{\left(\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}\left(\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)}\ \geq\ \frac{\left(\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}\left(g^{\alpha}\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)}{\left(\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}\left(g^{\alpha}\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)}.$$

This inequality is given in [2, Theorem 3.1]*. All other corresponding inequalities from* [2] *follow from this paper setting h to be the identity function.*

Also, as mentioned in the introductory section, if the h is the identity function and $p=w=0$, then we obtain the left-sided Riemann-Liouville fraction integral J^{σ}_{a+} of order σ (1.2)*, i.e. a special case of Mittag-Leffler function and its corresponding generalized fractional integral operator. Therefore, Theorem 2.1 generalizes Theorem 1.2.*

The conditions under which inequality (2.1) *and reverse inequality hold are complemented by the remaining cases of monotonicity of functions in the theorem.*

Next inequality follows by setting $g(x) = x - a$ and it is a generalization of [3, Theorem 3.1].

Corollary 2.2. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in (a, b]$. Let $(f_i)_{i=1,2,\ldots,n}$ be positive continuous decreasing functions with $(f_i)_{i=1,2,\ldots,n} \in L_\beta[a, b]$. Then for the fixed integer $s \in \{1, 2, \ldots, n\}$ the following inequality holds*

$$\frac{\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{2.5}$$

If $(f_i)_{i=1,2,\ldots,n}$ are increasing, then the inequality (2.5) *is reverse.*

For the next result we set $n = 1$ in Theorem 2.1.

Corollary 2.3. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a, b]$. Let f, g be positive continuous functions, monotonic in the opposite sense with $f \in L_\beta[a, b]$ and $g \in L_\alpha[a, b]$. Then the following inequality holds*

$$\frac{\left({}_{h}\Upsilon f^{\beta}\right)(x;p)}{\left({}_{h}\Upsilon f^{\gamma}\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon(g^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{h}\Upsilon(g^{\alpha} f^{\gamma})\right)(x;p)}. \tag{2.6}$$

If f and g are monotonic functions in the same sense, then the inequality (2.6) *is reverse.*

If additionally $g(x) = x - a$, then the following corollary holds.

Corollary 2.4. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in (a, b]$. Let $f \in L_\beta[a, b]$ be a positive continuous decreasing function. Then the following inequality holds*

$$\frac{\left({}_{h}\Upsilon f^{\beta}\right)(x;p)}{\left({}_{h}\Upsilon f^{\gamma}\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon((x-a)^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{h}\Upsilon\left((x-a)^{\alpha} f^{\gamma}\right)\right)(x;p)}. \tag{2.7}$$

If f is increasing, then the inequality (2.7) *is reverse.*

Remark. *If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then inequalities* (2.6) *and* (2.7) *imply Theorem 1.1 and* [6, Theorem 3]*, respectively.*

Further, from the condition (2.2) *with $n = 1$ (i.e. we have only one function f) and for $u, v \in [a, x]$ we obtain*

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0.$$

Now it is easy to see that although Theorem 1.1 is stated only for the case of decreasing function f and increasing function g, inequality (1.1) *remains valid even if f is increasing and g is decreasing function g, hence monotone functions in the opposite sense.*

Theorem 2.5. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \dots, n$ and let $x \in [a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ and g be positive continuous functions, $(f_i)_{i=1,2,\dots,n} \in L_{\alpha+\beta}[a,b]$ and $g \in L_\alpha[a,b]$. Let $s \in \{1, 2, \dots, n\}$ be fixed integer and for $u, v \in [a,x]$ let*

$$\left[(g(u))^\alpha (f_s(v))^\alpha - (g(v))^\alpha (f_s(u))^\alpha\right]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0. \tag{2.8}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{2.9}$$

If the condition (2.8) *is reverse, then the inequality* (2.9) *is reverse.*

Proof. From the (2.8) we obtain

$$\begin{aligned}
&(g(u))^\alpha (f_s(v))^{\alpha+\beta-\gamma_s} + (g(v))^\alpha (f_s(u))^{\alpha+\beta-\gamma_s} \\
&\geq (g(u))^\alpha (f_s(v))^\alpha (f_s(u))^{\beta-\gamma_s} + (g(v))^\alpha (f_s(u))^\alpha (f_s(v))^{\beta-\gamma_s}.
\end{aligned}$$

Multiplying both sides of the above inequality by (2.3) and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}
&(g(u))^\alpha \left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p) \\
&\qquad + (f_s(u))^{\alpha+\beta-\gamma_s}\left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p) \\
\geq\ & (g(u))^\alpha (f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p) \\
&\qquad + (f_s(u))^\alpha \left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).
\end{aligned}$$

Further multiplying the above by (2.4) and then integrating on $[a,x]$ with respect to the variable u, we obtain

$$\begin{aligned}
&\left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)\left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p) \\
\geq\ & \left(\boldsymbol{h}\Upsilon\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left(\boldsymbol{h}\Upsilon\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p).
\end{aligned}$$

from which follows (2.9).
If the condition (2.8) is reverse, then the reverse inequality of (2.9) can be proved analogously. □

Remark. *If the function h is the identity function and $p = w = 0$, then Theorem 2.5 generalizes* [3, Theorem 3.10].

Corollary 2.6. *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1,2,\dots,n$ and let $x \in (a,b]$. Let $(f_i)_{i=1,2,\dots,n} \in L_{\alpha+\beta}[a,b]$ be positive continuous functions. Let $s \in \{1,2,\dots,n\}$ be fixed integer and for $u,v \in [a,x]$ let*

$$[(u-a)^{\alpha}(f_s(v))^{\alpha} - (u-v)^{\alpha}(f_s(u))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0. \tag{2.10}$$

Then the following inequality holds

$$\frac{\left({}_h\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left({}_h\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)} \geq \frac{\left({}_h\Upsilon\left((x-a)^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_h\Upsilon\left((x-a)^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{2.11}$$

If the condition (2.10) *is reverse, then the inequality* (2.11) *is reverse.*

If we set $n = 1$ in Theorem 2.5, then we obtain the following inequality.

Corollary 2.7. *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a,b]$. Let f,g be positive continuous functions, $f \in L_{\alpha+\beta}[a,b]$, $g \in L_{\alpha}[a,b]$, such that for $u,v \in [a,x]$*

$$[(g(u))^{\alpha}(f(v))^{\alpha} - (g(v))^{\alpha}(f(u))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0. \tag{2.12}$$

Then the following inequality holds

$$\frac{\left({}_h\Upsilon f^{\alpha+\beta}\right)(x;p)}{\left({}_h\Upsilon f^{\alpha+\gamma}\right)(x;p)} \geq \frac{\left({}_h\Upsilon(g^{\alpha}f^{\beta})\right)(x;p)}{\left({}_h\Upsilon(g^{\alpha}f^{\gamma})\right)(x;p)}. \tag{2.13}$$

If the condition (2.12) *is reverse, then the inequality* (2.13) *is reverse.*

Corollary 2.8. *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in (a,b]$. Let $f \in L_{\alpha+\beta}[a,b]$ be positive continuous function such that for $u,v \in [a,x]$*

$$[(u-a)^{\alpha}(f(v))^{\alpha} - (v-a)^{\alpha}(f(u))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0. \tag{2.14}$$

Then the following inequality holds

$$\frac{\left({}_h\Upsilon f^{\alpha+\beta}\right)(x;p)}{\left({}_h\Upsilon f^{\alpha+\gamma}\right)(x;p)} \geq \frac{\left({}_h\Upsilon((x-a)^{\alpha}f^{\beta})\right)(x;p)}{\left({}_h\Upsilon((x-a)^{\alpha}f^{\gamma})\right)(x;p)}. \tag{2.15}$$

If the condition (2.14) *is reverse, then the inequality* (2.15) *is reverse.*

Remark. *If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then inequalities* (2.13) *and* (2.15) *imply* [6, Theorem 6] *and* [6, Theorem 5]*, respectively.*

Theorem 2.9. *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1,2,\dots,n$ and let $x \in [a,b]$. Let $f,g,(\varphi_i)_{i=1,2,\dots,n} \in L_1[a,b]$ be positive continuous functions such that f/φ_s and g are monotonic in the opposite sense, for $s \in \{1,2,\dots,n\}$. Then the following inequality holds*

$$\frac{\left({}_h\Upsilon\left(f\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p)}{\left({}_h\Upsilon\left(\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)} \geq \frac{\left({}_h\Upsilon\left(gf\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p)}{\left({}_h\Upsilon\left(g\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)}. \tag{2.16}$$

If f/φ_s and g are monotonic in the same sense for $s \in \{1,2,\dots,n\}$, the inequality (2.16) *is reverse.*

Proof. From hypotheses on functions, for $u, v \in [a, x]$ we have

$$[g(u) - g(v)] \left[\frac{f(v)}{\varphi_s(v)} - \frac{f(u)}{\varphi_s(u)} \right] \geq 0,$$

that is

$$g(u) \frac{f(v)}{\varphi_s(v)} + g(v) \frac{f(u)}{\varphi_s(u)} \geq g(u) \frac{f(u)}{\varphi_s(u)} + g(v) \frac{f(v)}{\varphi_s(v)}.$$

Multiplying both sides of the above inequality by

$$(h(x) - h(v))^{\sigma - 1} \boldsymbol{E} \left(w(h(x) - h(v))^{\rho}; p \right) \prod_{i=1}^{n} \varphi_i(v) h'(v)$$

and integrating on $[a, x]$ with respect to the variable v, we get

$$\begin{aligned} & g(u) \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(f \prod_{i \neq s}^{n} \varphi_i \right) \right) (x; p) + \frac{f(u)}{\varphi_s(u)} \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(g \prod_{i=1}^{n} \varphi_i \right) \right) (x; p) \\ \geq \quad & g(u) \frac{f(u)}{\varphi_s(u)} \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(\prod_{i=1}^{n} \varphi_i \right) \right) (x; p) + \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(g f \prod_{i \neq s}^{n} \varphi_i \right) \right) (x; p). \end{aligned}$$

Again, multiplying the above by

$$(h(x) - h(u))^{\sigma - 1} \boldsymbol{E} \left(w(h(x) - h(u))^{\rho}; p \right) \prod_{i=1}^{n} \varphi_i(u) h'(u)$$

and then integrating on $[a, x]$ with respect to the variable u, we arrive at

$$\begin{aligned} & \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(g \prod_{i=1}^{n} \varphi_i \right) \right) (x; p) \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(f \prod_{i \neq s}^{n} \varphi_i \right) \right) (x; p) \\ \geq \quad & \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(\prod_{i=1}^{n} \varphi_i \right) \right) (x; p) \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \left(g f \prod_{i \neq s}^{n} \varphi_i \right) \right) (x; p). \end{aligned}$$

from which follows (2.16).
If f/φ_s and g are monotonic in the same sense for $s \in \{1, 2, \dots, n\}$, then the reverse inequality of (2.16) can be proved analogously. □

Remark. *If the function h is the identity function and $p = w = 0$, then Theorem 2.9 generalizes* [3, Theorem 3.14].

If we set $n = 1$ in Theorem 2.9, then we obtain the following generalization of [6, Theorem 7].

Corollary 2.10. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $x \in [a, b]$. Let $f, g, \varphi \in L_1[a, b]$ be positive continuous functions such that f/φ and g are monotonic in the opposite sense. Then the following inequality holds*

$$\frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f \right) (x; p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} \varphi \right) (x; p)} \geq \frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} (f g) \right) (x; p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\varphi g) \right) (x; p)}. \tag{2.17}$$

If f/φ and g are monotonic in the same sense, then the inequality (2.17) *is reverse.*

Next is a counterpart of the previous result, where we assume $f(x) \leq \varphi(x)$. Hence, inequality (2.17) remains satisfied if g is replaced by $f^{\alpha-1}$.

Theorem 2.11. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha \geq 1$ and $x \in [a, b]$. Let $f, \varphi \in L_\alpha[a, b]$ be positive continuous functions such that f/φ and f are monotonic in the opposite sense, with $f(x) \leq \varphi(x)$ on $[a, b]$. Then the following inequality holds*

$$\frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} f\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} \varphi\right)(x;p)} \geq \frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^{\alpha}\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} \varphi^{\alpha}\right)(x;p)}. \tag{2.18}$$

If f/φ and f are monotonic functions in the same sense, then the inequality (2.18) *is reverse.*

Proof. Assume that f/φ is a decreasing function and f an increasing one. Then for $\alpha \geq 1$ function $f^{\alpha-1}$ is also increasing. By applying Corollary 2.10 we obtain

$$\frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} f\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} \varphi\right)(x;p)} \geq \frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (f^{\alpha})\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\varphi f^{\alpha-1})\right)(x;p)}.$$

This together with the assumption $f(x) \leq \varphi(x)$ lead to (2.18). Analogously we can prove the case when f/φ is increasing and f decreasing, and obtain reversed inequality if f/φ and f are monotonic in the same sense. □

For the last theorem we involve a convex function in the inequality.

Theorem 2.12. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function and $x \in [a, b]$. Let $f, g, \varphi \in L_1[a, b]$ be positive continuous functions such that f/φ is decreasing function and f, g are increasing, with $f(x) \leq \varphi(x)$ on $[a, b]$. Let ϕ be a convex function on $[0, \infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} f\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} \varphi\right)(x;p)} \geq \frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\phi(f)g)\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\phi(\varphi)g)\right)(x;p)}. \tag{2.19}$$

Proof. The function $\frac{\phi(x)}{x}$ is increasing since ϕ is a convex function on $[0, \infty]$ with $\phi(0) = 0$. From the assumption $f(x) \leq \varphi(x)$ with the positivity of f and φ, we get

$$\frac{\phi(f(x))}{f(x)} \leq \frac{\phi(\varphi(x))}{\varphi(x)}.$$

Further, since f, g and $\frac{\phi(x)}{x}$ are increasing then the following function

$$\frac{\phi(f(x))}{f(x)} g(x)$$

is also increasing. Hence

$$\begin{aligned}
\frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\phi(f)g)\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\phi(\varphi)g)\right)(x;p)} &= \frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\frac{\phi(f)}{f} fg)\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\frac{\phi(\varphi)}{\varphi} \varphi g)\right)(x;p)} \\
&\leq \frac{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\frac{\phi(f)}{f} fg)\right)(x;p)}{\left(_{\boldsymbol{h}}\boldsymbol{\Upsilon} (\frac{\phi(f)}{f} \varphi g)\right)(x;p)},
\end{aligned}$$

and by applying Corollary 2.10 for f, φ, $\frac{\phi(f)}{f}g$ we obtain

$$\frac{\left({}_{h}\Upsilon(\phi(f)g)\right)(x;p)}{\left({}_{h}\Upsilon(\phi(\varphi)g)\right)(x;p)} \le \frac{\left({}_{h}\Upsilon(f)\right)(x;p)}{\left({}_{h}\Upsilon(\varphi)\right)(x;p)}.$$

□

Corollary 2.13. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \ge 0$ and $0 < q \le r + \rho$. Let $h : [a, b] \to \mathbb{R}$ be a differentiable, strictly increasing function and $x \in [a, b]$. Let $f, \varphi \in L_1[a, b]$ be positive continuous functions such that f/φ is decreasing function and f is increasing, with $f(x) \le \varphi(x)$ on $[a, b]$. Let ϕ be a convex function on $[0, \infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{\left({}_{h}\Upsilon f\right)(x;p)}{\left({}_{h}\Upsilon \varphi\right)(x;p)} \ge \frac{\left({}_{h}\Upsilon(\phi(f))\right)(x;p)}{\left({}_{h}\Upsilon(\phi(\varphi))\right)(x;p)}. \tag{2.20}$$

Remark. *If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then Theorem 2.11, Theorem 2.12 and Corollary 2.13 generalize Theorem 8, Theorem 10 and Theorem 9 from* [6]*, respectively.*

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