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ON g-MELLIN TRANSFORM: CONSTRUCTION, CONVEXITY AND APPLICATIONS

PANKAJ JAIN, CHANDRANI BASU, VIVEK PANWAR

Abstract. In the framework of g-calculus, the Mellin transform has been defined and studied. For the new g-Mellin transform, the appropriate convolution is defined and its connection with the Hausdorff operator is pointed out. The notion of pseudo-logarithmic convexity (concavity) has been introduced and it is proved that the g-Mellin transform is pseudo-logarithmically convex (concave) for a suitable pseudo-exponential function. This leads to defining the g-gamma function. Finally, certain applications of g-Mellin transform are provided, namely, solving integral equations and a Titchmarsh type theorem.

1. Introduction

The theory of pseudo-calculus or more commonly known as g-calculus is well established now. The motivation of this notion arose as a result of the work of Sugeno [35] who developed the theory of fuzzy measures having monotonicity property instead of additivity. We refer to [8, 14, 15, 22, 24, 36, 37, 38] and references therein for more such work.

In a series of papers [27, 28, 29, 30, 31, 33, 34], Pap initiated the so called $\oplus$-decomposable measures and as a consequence the pseudo-algebraic operations, pseudo-derivative and pseudo-integral were defined. Since the pseudo-operations involve a generator function, usually denoted by g, the corresponding notions are also referred to as g-addition, g-multiplication, g-derivative, g-integral etc. The "g-theory" has been growing rapidly and getting much attention during the recent past. In [2, 16, 32, 34], the authors have considered famous classical inequalities like Young's, Hölder's , Minkowski's and Jensen's and in [23, 25], double g-integral has been considered.

The Mellin transform (see [9]) is an important integral transform given by

$$(Mf)(s) = \int_0^\infty x^{s-1} f(x)\, dx, \quad s \in \mathbb{C}. \tag{1.1}$$

The Mellin transform along with the related Fourier transform, Laplace transform etc. have wide range of applications not only in mathematical sciences but also in physical sciences and engineering. A lot of text is available on this subject.

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Among properties of Mellin transform, let us mention that it is a linear operator and possesses the scaling and shifting properties. The corresponding convolution is defined by

$$(f *_M h)(x) = \int_0^\infty f(\frac{x}{t})\frac{h(t)}{t}\,dt$$

so that the convolution equality

$$M(f *_M h) = (Mf) \cdot (Mh)$$

holds. From the Mellin transform, the function f can be recovered by using the inversion formula

$$(M^{-1}F)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)\,ds.$$

The classical theory of Mellin transform is very rich, a huge literature is available on this subject. Yet there has been a continued interest in the theory of Mellin transform. It has been studied (and going on) in different frameworks too, e.g., people have made a lot of contribution by studying it in quantum calculus, see [4, 5, 6, 11, 26], a very recent paper of the present authors [17] and references therein.

In [10], the authors have considered g-Mellin transform defined by

$$(M^{\oplus}f)(s) = \int_{\mathbb{R}^+}^{g} g^{-1}(x^{s-1}) \otimes f(x) \otimes dx, \quad s \in \mathbb{R} \tag{1.2}$$

In the present paper, we reinvestigate the g-Mellin transform (1.2), prove its various properties, provide an appropriate convolution and give the inversion formula. Most of the properties that we prove here were not considered in [10]. It is known that for $f(x) = e^{-x}$, the Mellin transform (1.1) is the Gamma function $\Gamma(s)$ and that it is logarithmically convex with respect to s. Here, we define the notion of pseudo-logarithmic convexity (concavity) and prove that the g-Mellin transform (1.2) is pseudo-logarithmically convex (concave) for f replaced by a suitable pseudo-exponential function. This motivates us to define the g-gamma function. As applications, we demonstrate the use of g-Mellin transform in solving integral equations and also a Titchmarsh type theorem is proved.

The paper is organized as follows. In Section 2, we collect basic notations, terminology and the structure of pseudo-operations and g-calculus which is required throughout the paper. The construction of g-Mellin transform is given in Section 3 where along with its several basic properties, the appropriate convolution is defined and its connection with the Hausdorff operator is pointed out. In Section 4, the notion of pseudo-logarithmic convexity (concavity) is given and proved its relation with the classical logarithmic convexity (concavity) which provides a nice tool to construct, in a natural way, the examples and counter examples of pseudo-logarithmically convex (concave) functions. Also, in this section, we prove that the g-Mellin transform (1.2) is pseudo-logarithmically convex (concave) for f replaced by a suitable pseudo-exponential function and as a result define the g-gamma function. Finally, in Section 5, some applications of g-Mellin transform are provided, namely, solving integral equations and a Titchmarsh type theorem.

2. Preliminaries

In this section, we collect some basic algebraic operations, elementary functions, derivative and integral in the framework of pseudo-algebra that is required throughout the paper.

The pseudo-addition is a function $\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is commutative, nondecreasing in each component, associative and has a zero element. We shall assume that $\oplus$ is strict pseudo-addition which means that $\oplus$ is strictly increasing in each component and continuous. Consequently, Aczel's Theorem [21] guarantees that there exists a monotone function g (called the generator for $\oplus$), $g : \mathbb{R} \to \mathbb{R}^+$ such that

$$x \oplus y = g^{-1}\left(g(x) + g(y)\right).$$

Similarly, we denote by $\otimes$, the pseudo-multiplication which is a function $\otimes : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which is commutative, nondecreasing in each component, associative and has a unit element. The operation $\otimes$ is defined, in terms of a monotone generator g, by

$$x \otimes y = g^{-1}\left(g(x) \cdot g(y)\right).$$

We shall denote by $\mathbb{R}_g$, the set $\mathbb{R}$ equipped with the pseudo-operations $\oplus$ and $\otimes$ with the corresponding generator g. The zero and the unit elements of $\mathbb{R}_g$ will be denoted, respectively, by 0_g and 1_g.

In [3], the authors considered that the generator function $g : \mathbb{R}_g \to \mathbb{R}$ is strictly monotone (either strictly increasing or strictly decreasing), onto, $g(0_g) = 0$, $g'(x) \neq 0$, for all x, $g \in C^2$ and $g^{-1} \in C^2$. Using this map, the following well defined operations have been defined:

$$x \ominus y = g^{-1}\left(g(x) - g(y)\right), \quad x \otimes^{-1} y = g^{-1}\left(\frac{g(x)}{g(y)}\right), \text{ provided } y \neq 0_g.$$

The order relation in $\mathbb{R}_g$, denoted by $\leq_g$, satisfies the following:

$$x \leq_g y \iff x \ominus y \leq 0_g.$$

If $x \leq_g y$, we can also write it as $y \geq_g x$. If $x \leq_g y$ and $x \neq y$, we shall write it as $x <_g y$, or equivalently, $y >_g x$. A number $x \in \mathbb{R}_g$ is said to be finite, written $x <_g \infty$, if there exists $b \in \mathbb{R}_g$ such that $x <_g b$.

In order to make $\mathbb{R}_g$ a linear space over the field $\mathbb{R}$, we define the pseudo-scalar product:

$$n \odot x = g^{-1}(ng(x)), \quad x \in \mathbb{R}_g,\ n \in \mathbb{R}.$$

It was pointed out in [3] that the operations $\odot$ and $\otimes$ are different. There was a need to define the scalar product $\odot$ since the compatibility condition $1 \odot x = x$ is not satisfied by $\otimes$.

Remark 2.1. $(\mathbb{R}_g, \oplus, \otimes, \leq_g)$ is an ordered and complete algebra.

The pseudo-derivative or more commonly called g-derivative of a suitable function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_g$ is defined by

$$D^g f(x) := \frac{d^g f(x)}{dx} = g^{-1}\left((g \circ f)'(x)\right).$$

while the pseudo-integral or the g-integral is defined by

$$\int_{[a,b]}^{g} f(t) \otimes dt = g^{-1}\left(\int_a^b (g \circ f)(x)\, dx\right).$$

Some of the properties of g-integral are mentioned below:

(a) $\int_{[a,b]}^{g} (f \oplus h) \otimes dt = \int_{[a,b]}^{g} f \otimes dt \oplus \int_{[a,b]}^{g} h \otimes dt$

(b) $\int_{[a,b]}^{g} (\lambda \otimes f) \otimes dt = \lambda \otimes \int_{[a,b]}^{g} f \otimes dt$

(c) $\int_{[a,b]}^{g} (\lambda \odot f) \otimes dt = \lambda \odot \int_{[a,b]}^{g} f \otimes dt$

(d) $f \leq_g h \Rightarrow \int_{[a,b]}^{g} f \otimes dt \leq_g \int_{[a,b]}^{g} h \otimes dt$

The following version of the fundamental theorem of calculus holds [29]: If $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_g$ is a continuous function, then for all $x \in [a,b]$

$$D^g \left(\int_{[a,x]}^{g} f(t) \otimes dt \right) = f(x). \tag{2.1}$$

Define the set

$$\mathbb{R}_g^+ = \{x \in \mathbb{R}_g : 0_g \leq_g x\}.$$

In view of the operation $\otimes$, for $x \in \mathbb{R}_g$ and $n \in \mathbb{N}$, we define

$$x^{(n)} := \underbrace{x \otimes x \otimes \ldots \otimes x}_{n-\text{times}} = g^{-1}(g^n(x)).$$

This notion of exponent can be extended for a general $p \in (0, \infty)$ which is defined (see [2], [15]) for all $x \in \mathbb{R}_g^+$ as

$$x^{(p)} = g^{-1}(g^p(x)).$$

It can further be generalized to cover negative powers as well: For $p \in (0, \infty)$, we define

$$x^{(-p)} = 1_g \otimes^{-1} x^{(p)}, \quad x \in \mathbb{R}_g^+.$$

For $p = 0$, we define $x^{(0)} = 1_g$. It is easy to check that for $p, q \in (0, \infty)$ and $x \in \mathbb{R}_g^+$, the following laws of exponent hold [16]:

(i) $x^{(p)} \otimes x^{(q)} = x^{(p+q)}$

(ii) $\left(x^{(p)}\right)^{(q)} = x^{(pq)}$

(iii) $(x \otimes y)^{(p)} = x^{(p)} \otimes y^{(p)}$

(iv) $(x \otimes^{-1} y)^{(p)} = x^{(p)} \otimes^{-1} y^{(p)}$

(v) $(\alpha \odot x)^{(p)} = \alpha^p \odot x^{(p)}$

The g-exponential function for $x \in \mathbb{R}_g$, as defined in [3], is given by

$$E^{(x)} = g^{-1}(e^{g(x)}),$$

where $e^{g(x)}$ is the standard exponential function. The g-logarithm function defined in [16] is given by

$$\operatorname{Ln} x = g^{-1} \left(\ln g(x) \right),$$

where $\ln g(x)$ is the standard logarithm function.

Remark 2.2. Unlike in the standard case, we may have that $E^{(x)} <_g 0_g$. In fact, if the generator g is monotonically decreasing, then for any $x \in \mathbb{R}_g$, $E^{(x)} \leq_g 0$, since g^{-1} is also monotonically decreasing. This suggests that in the pseudo case, logarithm can be defined for "negative" numbers, which in fact is true if, again, the generator g is decreasing.

The following proposition provides several properties of $E^{(x)}$ and $\operatorname{Ln} x$:

Proposition A. [16] *The following hold:*

(i) $E^{(x)} \otimes E^{(y)} = E^{(x \oplus y)}$
(ii) $E^{(\operatorname{Ln} x)} = x$
(iii) $\operatorname{Ln} E^{(x)} = x$
(iv) $\operatorname{Ln}(x \otimes y) = \operatorname{Ln} x \oplus \operatorname{Ln} y$

3. The g-Mellin transform

Let us write the operator $M^{\oplus}$ defined in (1.2) in a different way and call it M^g. Precisely, we define the following:

Definition 3.1. The g-Mellin transform of a function $f : \mathbb{R}^+ \to \mathbb{R}_g^+$ is defined by

$$(M^g f)(s) = \int_{\mathbb{R}^+}^{g} (g^{-1}(x))^{(s-1)} \otimes f(x) \otimes dx, \quad s \in \mathbb{R}$$

or, equivalently

$$(M^g f)(s) = g^{-1}\left(\int_0^\infty x^{s-1} g(f(x))\, dx\right), \quad s \in \mathbb{R}.$$

Note that the operator M^g is essentially the same as $M^{\oplus}$ defined in (1.2), their classical integral representations being the same.

Remark 3.2. (i) From the definition of M^g, it follows that

$$(M^g f)(s) = g^{-1}(M(g \circ f))(s) = (g^{-1} \circ M \circ g \circ f)(s),$$

i.e., the connection between the g-Mellin and the Mellin transform is given by

$$M^g = g^{-1} \circ M \circ g.$$

(ii) In view of the existence of the classical Mellin transform, it is clear that for the existence of the integral in the definition of M^g, we should have for $u > v$

$$(g \circ f)(x) = O_{0^+}(x^u) \qquad \text{and} \qquad (g \circ f)(x) = O_{+\infty}(x^v),$$

3.1. Properties of g-Mellin transform.

Various properties of M^g are proved through the following theorem:

Theorem 3.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}_g^+$ and $s \in \mathbb{R}$. The following hold:*

(a) $(M^g(\lambda \otimes f))(s) = \lambda \otimes (M^g f)(s), \quad \lambda \in \mathbb{R}_g$.
(b) $(M^g(f \oplus h))(s) = (M^g f)(s) \oplus (M^g h)(s)$
(c) $(M^g(f(ax)))(s) = g^{-1}(a^{-s}) \otimes (M^g f)(s)$
(d) $\left(M^g\left((g^{-1}(x))^{(\alpha)} \otimes f(x)\right)\right)(s) = (M^g f)(s + \alpha)$
(e) $(M^g(f(x^\alpha)))(s) = g^{-1}(1/\alpha) \otimes (M^g f)(s/\alpha)$
(f) $\left(M^g(D^g f)\right)(s) = (-1)(s-1) \odot (M^g f)(s-1)$. *More generally,*

$$\left(M^g(D^{(n)g} f)\right)(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \odot (M^g f)(s-n),$$

where $D^{(n)g}$ denotes the nth g-derivative of f.

(g) $\Big(M^g(x \odot D^g f)\Big)(s) = (-1)(s) \odot (M^g f)(s)$. *More generally,*

$$\Big(M^g(x^n \odot D^{(n)g} f)\Big)(s) = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \odot (M^g f)(s),$$

where $D^{(n)g}$ denotes the nth g-derivative of f.

(h) $\Big(M^g(x \odot D^g)^2 f\Big)(s) = (-1)^2 s^2 \odot (M^g f)(s)$. *More generally,*

$$\Big(M^g(x \odot D^g)^n f\Big)(s) = (-1)^n s^n \odot (M^g f)(s),$$

where $D^{(n)g}$ denotes the nth g-derivative of f.

(i) *Let f be continuous and F denote the definite g-integral of f, i.e., $F(x) = \int_{[0,x]}^g f(t) \otimes dt$. Then*

$$(M^g F)(s) = (-1/s) \odot (M^g f)(s+1).$$

(j) *Let f be continuous and $F^*(x) = \int_{[x,\infty)}^g f(t) \otimes dt$. Then*

$$(M^g F^*)(s) = (1/s) \odot (M^g f)(s+1).$$

Proof. (a) We have

$$\begin{aligned} g\left((M^g(\lambda \otimes f))(s)\right) &= \int_0^\infty x^{s-1} g(\lambda \otimes f(x))\, dx \\ &= \int_0^\infty x^{s-1} g(\lambda) g(f(x))\, dx \\ &= g(\lambda) \int_0^\infty x^{s-1} g(f(x))\, dx \\ &= g(\lambda) \cdot g(M^g f)(s) \\ &= g\left(\lambda \otimes (M^g f)(s)\right) \end{aligned}$$

and the assertion follows.

(b) This is straightforward by the definition of $\oplus$.

(c) By making variable substitution, we obtain

$$\begin{aligned} g\left((M^g(f(ax)))(s)\right) &= \int_0^\infty x^{s-1} g(f(ax))\, dx \\ &= a^{-s} \int_0^\infty y^{s-1} g(f(y))\, dy \\ &= g(g^{-1}(a^{-s})) \cdot g((M^g f)(s)) \end{aligned}$$

and we are done.

(d) This is easily obtained just by using the definitions of M^g and various operations involved.

(e) This can be obtained by taking $x^\alpha = y$.

(f) We have

$$\begin{aligned}
g\left((M^g(D^g f))(s)\right) &= \int_0^\infty x^{s-1} g(D^g f(x))\,dx \\
&= \int_0^\infty x^{s-1} \frac{d}{dx}(g(f(x))\,dx \\
&= \Big[x^{s-1} g(f(x))\Big]_0^\infty - \int_0^\infty (s-1)x^{s-2} g(f(x))dx \\
&= (-1)(s-1)\cdot g\left((M^g f)(s-1)\right) \\
&= g\left((-1)(s-1)\odot (M^g f)(s-1)\right).
\end{aligned}$$

Similarly, putting $D^g(f(x)) = h(x)$, we have

$$\begin{aligned}
g\Big((M^g(D^{(2)g} f))(s)\Big) &= g\left((M^g(D^g(D^g f)))(s)\right) \\
&= g\left((M^g(D^g h))(s)\right) \\
&= (-1)(s-1)\cdot g\left((M^g h)(s-1)\right) \\
&= (-1)(s-1)\cdot g\left((M^g(D^g f))(s-1)\right) \\
&= (-1)^2(s-1)(s-2)\cdot g\left((M^g f)(s-2)\right) \\
&= g\left((-1)^2(s-1)(s-2)\odot (M^g f)(s-2)\right) \\
&= g\left((-1)^2\frac{\Gamma(s)}{\Gamma(s-2)}\odot (M^g f)(s-2)\right).
\end{aligned}$$

By induction, it can be shown that

$$\Big(M^g(D^{(n)g} f)\Big)(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \odot (M^g f)(s-n).$$

(g) We have

$$\begin{aligned}
g\left((M^g(x\odot D^g f(x)))(s)\right) &= \int_0^\infty x^{s-1}\cdot x g(D^g f(x))\,dx \\
&= \int_0^\infty x^s \frac{d}{dx}(g(f(x))\,dx \\
&= \Big[x^s g(f(x))\Big]_0^\infty - \int_0^\infty s x^{s-1} g(f(x))dx \\
&= (-1)s\cdot g\left((M^g f)(s)\right) \\
&= g\left((-1)s\odot (M^g f)(s)\right),
\end{aligned}$$

Similarly, putting $D^g(f(x)) = h(x)$, we have

$$\begin{aligned}
g\left((M^g(x^2 \odot D^{(2)g} f(x)))(s)\right) &= g\left((M^g(x^2 \odot D^g(D^g f(x))))(s)\right) \\
&= g\left((M^g(x^2 \odot D^g h(x)))(s)\right) \\
&= \int_0^\infty x^{s+1} \frac{d}{dx}(g(h(x))\,dx \\
&= -\int_0^\infty (s+1)x^s (g(h(x))\,dx \\
&= -\int_0^\infty (s+1)x^s \frac{d}{dx}(g(f(x))\,dx \\
&= (-1)^2(s+1)s \cdot g\left((M^g f)(s)\right) \\
&= g\left((-1)^2(s+1)s \odot (M^g f)(s)\right) \\
&= g\left((-1)^2 \frac{\Gamma(s+2)}{\Gamma(s)} \odot (M^g f)(s)\right).
\end{aligned}$$

More generally, by induction, it can be shown that

$$\left(M^g(x^n \odot D^{(n)g} f)\right)(s) = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \odot (M^g f)(s).$$

(h) We have

$$\begin{aligned}
(x \odot D^g)^2 f(x) &= (x \odot D^g)(x \odot D^g f(x)) \\
&= (x \odot D^g f(x)) \oplus (x^2 \odot D^{(2)g} f(x)).
\end{aligned}$$

Hence

$$\begin{aligned}
g\left(\left(M^g(x \odot D^g)^2 f(x)\right)(s)\right) &= g\left(\left(M^g(x \odot D^g f(x))\right)(s) \oplus \left(M^g(x^2 \odot D^{(2)g} f(x))\right)(s)\right) \\
&= g\left(\left(M^g(x \odot D^g f(x))\right)(s)\right) + g\left(\left(M^g(x^2 \odot D^{(2)g} f(x))\right)(s)\right) \\
&= (-s) \cdot g\left((M^g f)(s)\right) + (s+1)s \cdot g\left((M^g f)(s)\right) \\
&= s^2 \cdot g\left((M^g f)(s)\right) \\
&= g\left((-1)^2 s^2 \odot (M^g f)(s)\right).
\end{aligned}$$

More generally, it follows by induction that

$$\left(M^g(x \odot D^g)^n f\right)(s) = (-1)^n s^n \odot (M^g f)(s).$$

(i) Let $F(x) = \int_{[0,x]}^g f(t) \otimes dt$. Then, by fundamental theorem of g-calculus (2.1), $D^g F(x) = f(x)$. Consequently, we apply part (f) above for f and s replaced, respectively, by F and $s+1$, and obtain

$$g\left((M^g(f(x)))(s+1)\right) = (-1)s \cdot g\left(M^g\left(\int_{[0,x]}^g f(t) \otimes dt\right)(s)\right)$$

so that

$$g\left(M^g\left(\int_{[0,x]}^g f(t)\otimes dt\right)(s)\right) = \frac{(-1)}{s}\cdot g\left((M^g(f(x)))(s+1)\right)$$
$$= g\left(\frac{(-1)}{s}\odot (M^g(f(x)))(s+1)\right).$$

(j) As a consequence of the fundamental theorem of classical calculus, we know that for any function h defined on $(0,\infty)$

$$\frac{d}{dx}\left(\int_x^\infty h(t)\,dt\right) = -h(x)$$

using which we obtain

$$\begin{aligned} D^gF^*(x) &= D^g\left(g^{-1}\left(\int_x^\infty (g\circ f)(t)\,dt\right)\right)\\ &= g^{-1}\left(\frac{d}{dx}g\left(g^{-1}\left(\int_x^\infty (g\circ f)(t)\,dt\right)\right)\right)\\ &= g^{-1}\left(\frac{d}{dx}\left(\int_x^\infty (g\circ f)(t)\,dt\right)\right)\\ &= g^{-1}(-(g\circ f)(x))\\ &= (-1)\odot f(x). \end{aligned}$$

Now the assertion follows as in part (i) above.

□

3.2. **The inversion formula.** Here, we recall the g-Mellin inversion through that of classical Mellin transform M [10]:

An operator $(M^g)^{-1}$, which satisfies the equation

$$(M^g)^{-1}\circ(M^g f) = f,$$

is considered as the g-Mellin inverse transform. In view of Remark 3.2 and writing $M^g f = F$, we find that

$$\begin{aligned} f(x) &= (M^g)^{-1}(F)\\ &= (g^{-1}\circ M\circ g)^{-1}(F)\\ &= (g^{-1}\circ M^{-1}\circ g)(F)\\ &= g^{-1}\circ M^{-1}(g(F))(s). \end{aligned}$$

3.3. **Convolution.** Now, we define the appropriate convolution for g-Mellin transform.

Definition 3.4. The g-Mellin convolution product of two functions f and h is defined by

$$(f *_M h)(x) = \int_{\mathbb{R}^+}^g 1_g \otimes^{-1} g^{-1}(t)\otimes f(t)\otimes h\Big(\frac{x}{t}\Big)\otimes dt.$$

Remark 3.5. Let us rewrite the g-Mellin convolution as the following operator

$$(H^g_\phi h)(x) = \int^g_{\mathbb{R}^+} 1_g \otimes^{-1} g^{-1}(t) \otimes \phi(t) \otimes h\Big(\frac{x}{t}\Big) \otimes dt. \tag{3.1}$$

In the classical case the above operator becomes

$$(H_\phi h)(x) = \int_{\mathbb{R}^+} \frac{\phi(t)}{t} h\Big(\frac{x}{t}\Big)\, dt$$

which is the well known Hausdorff operator. The boundedness of H_ϕ between various function spaces has been obtained in [1, 7, 12, 13, 18, 19, 20]. Thus (3.1) represents the g-analogue of the Hausdorff operator.

In the following, we give an equivalent form of (3.1).

Proposition 3.6. *The operator H^g_ϕ is equivalent to*

$$(G^g_\psi h)(x) = (1_g \otimes^{-1} g^{-1}(x)) \otimes \int^g_{\mathbb{R}^+} \psi\Big(\frac{z}{x}\Big) \otimes h(z) \otimes dz.$$

Proof. In (3.1), make the following replacement

$$\phi(t) = (1_g \otimes^{-1} g^{-1} t) \otimes \psi\Big(\frac{1}{t}\Big)$$

so that H^g_ϕ becomes

$$(G^g_\psi h)(x) = \int^g_{\mathbb{R}^+} (1_g \otimes^{-1} g^{-1}(t)) \otimes (1_g \otimes^{-1} g^{-1}(t)) \otimes \psi\Big(\frac{1}{t}\Big) \otimes h\Big(\frac{x}{t}\Big) \otimes dt$$

which implies

$$\begin{aligned} g((G^g_\psi h)(x)) &= \int_0^\infty \frac{g(1_g)}{t}\frac{g(1_g)}{t} g\Big(\psi\Big(\frac{1}{t}\Big)\Big) g\Big(h\Big(\frac{x}{t}\Big)\Big) dt \\ &= \int_0^\infty \frac{z}{x} g(1_g) \frac{z}{x} g(1_g) g\Big(\psi\Big(\frac{z}{x}\Big)\Big) g(h(z)) \frac{x}{z^2}\, dz \\ &= \int_0^\infty \frac{g(1_g)}{x} g\Big(\psi\Big(\frac{z}{x}\Big)\Big) g(h(z))\, dz \\ &= \frac{g(1_g)}{x} \int_0^\infty g\Big(\psi\Big(\frac{z}{x}\Big)\Big) g(h(z))\, dz \\ &= g(1_g \otimes^{-1} g^{-1}(x)) \cdot g\Big(\int^g_{\mathbb{R}^+} \psi\Big(\frac{z}{x}\Big) \otimes h(z) \otimes dz\Big) \end{aligned}$$

Thus

$$(G^g_\psi h)(x) = (1_g \otimes^{-1} g^{-1}(x)) \otimes \int^g_{\mathbb{R}^+} \psi\Big(\frac{z}{x}\Big) \otimes h(z) \otimes dz.$$

and we are done. □

We prove below some of the properties of the convolution defined above:

Theorem 3.7. (i) $(f *_M h)(x) = (h *_M f)(x)$.
(ii) $((f *_M h) *_M k)(x) = (f *_M (h *_M k))(x)$.
(iii) $(M^g(f *_M h))(s) = (M^g f)(s) \otimes (M^g h)(s)$.

Proof. (i) We have

$$\begin{aligned} g\left((f *_M h)(x)\right) &= \int_0^\infty \frac{g(1_g)}{t} g(f(t)) g\Big(h\Big(\frac{x}{t}\Big)\Big) dt \\ &= \int_0^\infty \frac{g(1_g)}{x/y} g\Big(f\Big(\frac{x}{y}\Big)\Big) g(h(y)) \frac{x}{y^2} dy \\ &= \int_0^\infty \frac{g(1_g)}{y} g(h(y)) g\Big(f\Big(\frac{x}{y}\Big)\Big) dy \\ &= g\left((h *_M f)(x)\right) \end{aligned}$$

and the assertion follows.

(ii) We have

$$\begin{aligned} g\left(((f *_M h) *_M k)(x)\right) &= \int_0^\infty \frac{g(1_g)}{t} g((f *_M h)(t)) g\Big(k\Big(\frac{x}{t}\Big)\Big) dt \\ &= \int_0^\infty \frac{g(1_g)}{t} \left(\int_0^\infty \frac{g(1_g)}{z} g(f(z)) g\Big(h\Big(\frac{t}{z}\Big)\Big) dz \right) g\Big(k\Big(\frac{x}{t}\Big)\Big) dt \\ &= \int_0^\infty \frac{g(1_g)}{z} g(f(z)) \int_0^\infty \frac{g(1_g)}{t} g\Big(h\Big(\frac{t}{z}\Big)\Big) g\Big(k\Big(\frac{x}{t}\Big)\Big) dt\, dz \\ &= \int_0^\infty \frac{g(1_g)}{z} g(f(z)) \int_0^\infty \frac{g(1_g)}{uz} g(h(u)) g\Big(k\Big(\frac{x}{zu}\Big)\Big) z du\, dz \\ &= \int_0^\infty \frac{g(1_g)}{z} g(f(z)) g\Big((h *_M k)\Big(\frac{x}{z}\Big)\Big) dz \\ &= g\left((f *_M (h *_M k))(x)\right) \end{aligned}$$

and the assertion follows.

(iii) Here, we have

$$\begin{aligned} g\left((M^g(f *_M h))(s)\right) &= \int_0^\infty x^{s-1} g((f *_M h)(x)) dx \\ &= \int_0^\infty x^{s-1} \int_0^\infty \frac{g(1_g)}{t} g(f(t)) g\Big(h\Big(\frac{x}{t}\Big)\Big) dt\, dx \\ &= \int_0^\infty \frac{g(1_g)}{t} g(f(t)) \int_0^\infty x^{s-1} g\Big(h\Big(\frac{x}{t}\Big)\Big) dx\, dt \\ &= \int_0^\infty \frac{g(1_g)}{t} g(f(t)) \int_0^\infty (ty)^{s-1} g(h(y)) t\, dy\, dt \\ &= g(1_g) \int_0^\infty t^{s-1} g(f(t))\, dt \int_0^\infty y^{s-1} g(h(y))\, dy \\ &= g(1_g) \cdot g\left((M^g f)(s)\right) \cdot g\left((M^g h)(s)\right) \\ &= g\left(1_g \otimes (M^g f)(s) \otimes (M^g h)(s)\right) \\ &= g\left((M^g f)(s) \otimes (M^g h)(s)\right) \end{aligned}$$

and we are done. □

4. Convexity

The following was defined in [16]:

Definition 4.1. A function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_g$ is said to be pseudo-convex on $[a,b]$ if for all $x, y \in [a,b]$ and all $0 \leq \lambda \leq 1$

$$f(\lambda x + (1-\lambda)y) \leq_g \lambda \odot f(x) \oplus (1-\lambda) \odot f(y)$$

and is said to be pseudo-concave if the last inequality holds in the reverse direction.

We prove the following:

Theorem 4.2. *Let $h : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a given function and $g : \mathbb{R}_g \to \mathbb{R}$ be a generator. Write $f = g^{-1} \circ h$.*

(a) *If g is increasing then h is convex (concave) if and only if f is pseudo-convex (pseudo-concave).*
(b) *If g is decreasing then h is convex (concave) if and only if f is pseudo-concave (pseudo-convex).*

Proof. We shall only prove (a) and that too for convexity only. Other considerations can be proved similarly.

Since $g : \mathbb{R}_g \to \mathbb{R}$ is increasing, it follows that $g^{-1} : \mathbb{R} \to \mathbb{R}_g$ is so. Also note that $g \circ f = h$. Assume first that h is convex. We have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= g^{-1}\left(h(\lambda x + (1-\lambda)y)\right) \\ &\leq_g g^{-1}\left(\lambda h(x) + (1-\lambda)h(y)\right) \\ &= g^{-1}\left(\lambda g(f(x)) + (1-\lambda)g(f(y))\right) \\ &= g^{-1}\left(g(\lambda \odot f(x)) + g((1-\lambda) \odot f(y))\right) \\ &= \lambda \odot f(x) \oplus (1-\lambda) \odot f(y), \end{aligned}$$

i.e., f is pseudo-convex. Assume, now, that f is pseudo-convex. We have

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= g(f(\lambda x + (1-\lambda)y)) \\ &\leq g(\lambda \odot f(x) \oplus (1-\lambda) \odot f(y)). \end{aligned}$$

Now,

$$\lambda \odot f(x) = g^{-1}(\lambda g(f(x))) = g^{-1}(\lambda h(x))$$

and similarly

$$(1-\lambda) \odot f(y) = g^{-1}((1-\lambda)h(y))$$

so that

$$\lambda \odot f(x) \oplus (1-\lambda) \odot f(y) = g^{-1}(\lambda h(x) + (1-\lambda)h(y)).$$

Combining these estimates, we obtain

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y)$$

and we are done. □

Remark 4.3. Theorem 4.2 provides a natural way to construct examples and counter examples of pseudo-convex (pseudo-concave) functions. For example, since the function $h(x) = x^2$ is convex, for any increasing generator g, the function $f(x) = g^{-1}(x^2)$ is pseudo-convex. Similarly, since for $x > 0$, the function $h(x) = \ln x$ is not convex, the function $f(x) = g^{-1}(\ln x)$ is not pseudo-convex.

Here we define the following:

Definition 4.4. A function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_g$ is said to be pseudo-logarithmically convex on $[a,b]$ if for all $x, y \in [a,b]$ and all $0 \le \lambda \le 1$

$$\operatorname{Ln}\left(f(\lambda x + (1-\lambda)y)\right) \le_g \lambda \odot \operatorname{Ln} f(x) \oplus (1-\lambda) \odot \operatorname{Ln} f(y)$$

or, equivalently

$$f(\lambda x + (1-\lambda)y) \le_g (f(x))^{(\lambda)} \otimes (f(y))^{(1-\lambda)}$$

and is said to be pseudo-logarithmically concave if the above inequalities hold in the reverse direction.

On the lines similar to Theorem 4.2, the following can be proved:

Theorem 4.5. *Let $h : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a given function and $g : \mathbb{R}_g \to \mathbb{R}$ be a generator. Write $f = g^{-1} \circ h$.*

(a) *If g is increasing then h is logarithmically convex (logarithmically concave) if and only if f is pseudo-logarithmically convex (pseudo-logarithmically concave).*
(b) *If g is decreasing then h is logarithmically convex (logarithmically concave) if and only if f is pseudo-logarithmically concave (pseudo-logarithmically convex).*

Remark 4.6. In view of Remark 4.3 and Theorem 4.5, it is easy to construct examples for pseudo-logarithmically convex functions. Since the function $h(x) = \exp(|x|^p)$, $p \ge 1$ is logarithmically convex, for any increasing generator g, the function $g^{-1}(h(x)) = g^{-1}(\exp(|x|^p))$ is pseudo-logarithmically convex.

We shall be using the following g-Young's inequalities [16]:

Theorem A. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

(a) *If the generator g is increasing then for all $a, b \in \mathbb{R}_g^+$, the following Young's type inequality holds:*

$$a \otimes b \le_g \left(a^{(p)} \otimes^{-1} g^{-1}(p)\right) \oplus \left(b^{(p')} \otimes^{-1} g^{-1}(p')\right). \tag{4.1}$$

(b) *If the generator g is decreasing then for all $a, b \in \mathbb{R}_g^+$, the inequality (4.1) holds in the reverse direction.*

We prove the following:

Theorem 4.7. *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_g$ be a function and g be the generator of the pseudo operations.*

(a) *If either g is increasing and f is pseudo-logarithmically convex or g is decreasing and f is pseudo-logarithmically concave then f is pseudo-convex.*
(b) *If either g is increasing and f is pseudo-logarithmically concave or g is decreasing and f is pseudo-logarithmically convex then f is pseudo-concave.*

Proof. We shall only consider the case when g is increasing and f is pseudo-logarithmically convex. The other cases are similar.

Taking

$$a = (f(x))^{(\lambda)}, \quad b = (f(y))^{(1-\lambda)}, \quad p = 1/\lambda$$

in (4.1), we get

$$\begin{aligned}(f(x))^{(\lambda)} \otimes (f(y))^{(1-\lambda)} &\leq_g \left(f(x) \otimes^{-1} g^{-1}(1/\lambda\right) \oplus \left(f(y) \otimes^{-1} g^{-1}(1/(\lambda-1))\right) \\ &= g^{-1}\left(\frac{g(f(x))}{1/\lambda} + \frac{g(f(y))}{1/(\lambda-1)}\right) \\ &= g^{-1}\left(\lambda g(f(x)) + (1-\lambda) g(f(y))\right) \\ &= g^{-1}\left(g(\lambda \odot f(x)) + g((1-\lambda) \odot (f(y))\right) \\ &= \lambda \odot f(x) \oplus (1-\lambda) \odot (f(y)\end{aligned}$$

and the pseudo-convexity of f follows. □

Remark 4.8. The converse of any of the statements in Theorem 4.7 is not true in general. From Remark 4.3, $f(x) = g^{-1}(x^2)$ is pseudo-convex. However, it is not pseudo-logarithmically convex since $2\ln|x|$ is not logarithmically convex.

Now, consider the g-Mellin transform $M_g f$ for the function $f(x) = E^{(-g^{-1}(x))}$ and consider it as a function of s, i.e.

$$\Gamma_g(s) := \int_{\mathbb{R}^+}^{g} (g^{-1}(x))^{(s-1)} \otimes E^{(-g^{-1}(x))} \otimes dx. \tag{4.2}$$

For the next result, we shall be using the following g-Hölder's inequality proved in [16] (see also [2])

Theorem B. *Let $1 < p < \infty$ and write* $[f]^g_{p,\mathbb{R}_g} := \left(\int_{\mathbb{R}_g}^{g} |f(x)|_g^{(p)} \otimes dx\right)^{(\frac{1}{p})}$.

(a) *Let the generator g be increasing and $f, h : \mathbb{R} \to \mathbb{R}_g$ be measurable functions such that $[f]^g_{p,\mathbb{R}_g} <_g \infty$ and $[h]^g_{p',\mathbb{R}_g} <_g \infty$. Then $[f \otimes h]^g_{1,\mathbb{R}_g} <_g \infty$ and the following Hölder's type inequality holds:*

$$[f \otimes h]^g_{1,\mathbb{R}_g} \leq_g [f]^g_{p,\mathbb{R}_g} \otimes [h]^g_{p',\mathbb{R}_g}. \tag{4.3}$$

(b) *If the generator g is decreasing then the inequality* (4.3) *holds in the reverse direction.*

We prove the following:

Theorem 4.9. *If the generator g is increasing then Γ_g is pseudo-logarithmically convex and if g is decreasing then it is pseudo-logarithmically concave.*

Proof. If $\lambda = 0$ or $\lambda = 1$, then the result is trivially true. So, assume that $0 < \lambda < 1$. For the sake of simplicity, let us write $y := -g^{-1}(x)$. By applying g-Hölder's inequality (4.3) with the exponents $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$, we obtain

$$\begin{aligned}&\Gamma_g(\lambda s_1 + (1-\lambda)s_2) = \\ &= \int_{\mathbb{R}}^{g} (g^{-1}(x))^{(\lambda s_1 + (1-\lambda)s_2 - 1)} \otimes E^{(\lambda \odot y \oplus (1-\lambda)\odot y)} \otimes dx \\ &= \int_{\mathbb{R}}^{g} \left((g^{-1}(x))^{(\lambda(s_1-1))} \otimes E^{(\lambda\odot y)}\right) \otimes \left((g^{-1}(x))^{((1-\lambda)(s_2-1))} \otimes E^{((1-\lambda)\odot y)}\right) \otimes dx \\ &\leq_g \left(\int_{\mathbb{R}}^{g} (g^{-1}(x))^{(s_1-1)} \otimes E^{(y)} \otimes dx\right)^{(\lambda)} \otimes \left(\int_{\mathbb{R}}^{g} (g^{-1}(x))^{(s_2-1)} \otimes E^{(y)} \otimes dx\right)^{(1-\lambda)} \\ &= (\Gamma_g(s_1))^{(\lambda)} \otimes (\Gamma_g(s_2))^{(1-\lambda)}\end{aligned}$$

and we are done. □

In the setting of the usual calculus, the function Γ_g defined in (4.2) is the gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$$

which is the Mellin transform of e^{-x} and is known to be logarithmically convex. Can the function Γ_g be called g-gamma or pseudo-gamma function? Let us analyse Γ_g further for more properties similar to the gamma function Γ.

By the definition of Γ_g, it is clear that

$$g(\Gamma_g(s)) = \int_0^\infty x^{s-1} e^{-x}\, dx = \Gamma(s)$$

so that

$$\Gamma_g(s) = g^{-1}(\Gamma(s))$$

which gives a connection between Γ_g and Γ. Consequently, using the property $\Gamma(s+1) = s\Gamma(s)$, we observe that

(i) $\Gamma_g(s+1) = g^{-1}(s\Gamma(s)) = g^{-1}(s) \otimes \Gamma_g(s)$
(ii) $\Gamma_g(1) = g^{-1}(\Gamma(1)) = g^{-1}(1)$
(iii) For any positive integer n,

$$\Gamma_g(n) = g^{-1}(n!)$$

Remark 4.10. In view of the above discussion, it is reasonable to call $\Gamma_g(s)$ as the pseudo-gamma or g-gamma function which is the g-Mellin transform of the function $f(x) = E^{(-g^{-1}(x))}$. We stop the discussion on Γ_g here since it is out of scope of the present paper. However, it is of interest to explore more on this topic, e.g., the g-beta function can be defined as

$$B_g(x,y) = (\Gamma_g(x) \otimes \Gamma_g(y)) \otimes^{-1} \Gamma_g(x+y)$$

and studied for various properties.

5. Applications

In this section we provide some applications of g-Mellin transform. First we give the following lemma:

Lemma 5.1. *Let K and h be a pair of functions defined on $\mathbb{R}^+$. If*

$$f(x) = \int_{\mathbb{R}^+}^{g} h(t) \otimes K(xt) \otimes dt \tag{5.1}$$

then

$$(M^g f)(s) = (M^g K)(s) \otimes (M^g h)(1-s).$$

Proof. By the definition of M_g applied on f given by (5.1), we get

$$\begin{aligned} g\Big((M^g f)(s)\Big) &= \int_0^\infty x^{s-1} g\big(f(x)\big)dx \\ &= \int_0^\infty x^{s-1}\left(\int_0^\infty g\big(h(t)\big)g\big(K(xt)\big)dx\right)dt \\ &= \int_0^\infty g\big(h(t)\big)\left(\int_0^\infty x^{s-1} g\big(K(xt)\big)dx\right)dt \\ &= \int_0^\infty g\big(h(t)\big)\left(t^{-s}\, g\big((M^g K)(s)\big)\right)dt \\ &= \int_0^\infty t^{-s}\, g\big(h(t)\big)dt \cdot g\big((M^g K)(s)\big) \\ &= g\left((M^g h)(1-s)\right) \cdot g\left((M^g K)(s)\right) \\ &= g\Big(\left((M^g h)(1-s)\right) \otimes \left((M^g K)(s)\right)\Big) \end{aligned}$$

and we are done. □

5.1. **Application to integral equations.**

Example 5.2. We solve the integral equation

$$h(x) = \int_{\mathbb{R}^+}^{g} f(t) \otimes K(xt) \otimes dt, \qquad x > 0. \tag{5.2}$$

In view of Lemma 5.1, we have

$$(M^g h)(s) = (M^g f)(1-s) \otimes (M^g K)(s)$$

which gives that

$$(M^g h)(1-s) = (M^g f)(s) \otimes (M^g K)(1-s)$$

i.e.,

$$\begin{aligned} g\big((M^g f)(s)\big) &= \frac{g\big((M^g h)(1-s)\big)}{g\big((M^g K)(1-s)\big)} \\ &= g((M^g h)(1-s)) \cdot g((M^g L)(s)) \\ &= g((M^g h)(1-s)) \otimes ((M^g L)(s)), \end{aligned}$$

provided

$$\frac{1}{g\big((M^g K)(1-s)\big)} = g((M^g L)(s)).$$

So,

$$(M^g f)(s) = (M^g h)(1-s) \otimes (M^g L)(s)$$

which on applying the inversion formula gives

$$\begin{aligned} f(x) &= \big(M^g\big)^{-1}\big((M^g h)(1-s) \otimes (M^g L)(s)\big) \\ &= \int_{\mathbb{R}^+}^{g} h(t) \otimes L(xt) \otimes dt, \end{aligned}$$

provided $L(x) = (M^g)^{-1}(M^g L(s))$ exists and (5.2) is formally solved.
In particular, if

$$g((M^g L)(s)) = g((M^g K)(s))$$

then the solution becomes

$$f(x) = \int_{\mathbb{R}^+}^{g} h(t) \otimes K(xt) \otimes dt,$$

provided

$$g\big((M^g K)(s) \otimes (M^g K)(1-s)\big) = 1.$$

5.2. **A Titchmarsh type theorem.**

Theorem 5.3. *Let K be a function defined over $\mathbb{R}^+$. If the integral equation*

$$f(x) = \int_{\mathbb{R}^+}^{g} K(ux) \otimes \int_{\mathbb{R}^+}^{g} K(uy) \otimes f(y) \otimes dy \otimes du \tag{5.3}$$

has a suitable solution f, then we have

$$g\big((M^g K)(s) \otimes (M^g K)(1-s)\big) = 1.$$

Proof. We can write (5.3) as a pair of reciprocal formulas

$$h(u) = \int_{\mathbb{R}^+}^{g} f(y) \otimes K(uy) \otimes dy$$

$$f(x) = \int_{\mathbb{R}^+}^{g} h(u) \otimes K(xu) \otimes du.$$

Applying Lemma 5.1 to both of these equations, we have

$$(M^g f)(s) = (M^g h)(1-s) \otimes (M^g K)(s)$$
$$(M^g h)(s) = (M^g f)(1-s) \otimes (M^g K)(s)$$

i.e.,

$$g((M^g f)(s)) = g((M^g h)(1-s)) \cdot g((M^g K)(s))$$
$$g((M^g h)(s)) = g((M^g f)(1-s)) \cdot g((M^g K)(s)).$$

Changing s into $1-s$ in one of these equations and taking product with the other, we get the desired result. □

References

[1] C. Abdelkefi, M. Rachdi, *Further results for the Dunkl transform and the generalized Cesaro operator*, 2012, Available from: https:/arxiv.org/abs/1208.5034.

[2] H. Agahi, Y. Ouyang, R. Mesiar, E. Pap, Endre and M. Štrboja, *Hölder and Minkowski type inequalities for pseudo-integral*, Appl. Math. Comput., 217 (2011), 8630–8639.

[3] A. Boccuto and D. Candeloro, *Differential calculus in Riesz spaces and applications to g-calculus*, Mediterr. J. Math., 8 (2011), 315–329.

[4] K. Brahim, R. Ouanes, *Some application of the q-Mellin transform*, Tamsui Oxford J. Math. Sci. 26 (2010), 335–343.

[5] K. Brahim, L. Riahi, *Two dimensional Mellin transform in quantum calculus*, Acta Math. Sci., 38 (2018), 546–560.

[6] K. Brahim, B.Nefzi, A. Bsaissa, *The symmetric Mellin transform in quantum calculus*, Le Matematiche, 70 (2015), 255–270.

[7] J. Chen, D. Fan, J. Li, *Hausdorff operators on function spaces*, Chin. Ann. Math. Ser. B, 33 (2013), 537–556.

[8] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1953/54), 131–295.

[9] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications* (2nd Ed.), Chapman & Hall / CRC, 2007.

[10] N. Duraković, T. Grbić, S. Rapajić, S. Medić and S. Buhmiler, *g-Mellin Transform*, IEEE 16th International Symposium on Intelligent Systems and Informatics (SISY), Subotica, (2018), 000075–000080.

[11] A. Fitouhi, N. Bettaibi, K. Brahim, *The Mellin transform in quantum calculus*, Constr. Approx., 23 (2006), 305–323.
[12] R. Bandaliyev, P. Gorka, *Hausdorff operator in Lebesgue spaces*, Mathematical Inequalities and Applications, 22 (2019), 657–676.
[13] R.R. Goldberg, *Certain operators and Fourier transforms on* L^2, Proc. Amer. Math. Soc., 10 (1959), 385–390.
[14] T. Grbić, I. Štajner-Papuga and M. Štrboja, *An approach to pseudo-integration of set-valued functions*, Inform. Sci., 181 (2011), 2278–2292.
[15] H. Ichihashi, H. Tanaka and K. Asai, *Fuzzy integrals based on pseudo-additions and multiplications*, J. Math. Anal. Appl., 130 (1988), 354–364.
[16] P. Jain, *Classical inequalities for pseudo integrals*, https://arxiv.org/abs/2006.06927.
[17] P. Jain, C. Basu and V. Panwar, *Finite Mellin Transform for* (p,q) *and Symmetric Calculus*, J. Pseudo-Differ. Operators Appl., 11 (2020),1595–1620.
[18] P. Jain, S. Jain, V. D. Stepanov, *LCT based integral transforms and Hausdorff operators*, Eurasian Math. J., 11 (2020), 57–71.
[19] E. Liflyand, *Hausdorff operators on Hardy spaces*, Eurasian Math. J., 4 (2013), 101–141.
[20] E. Liflyand, A. Miyachi, *Boundedness of the Hausdorff operators in* H_p *spaces,* $0 < p < 1$, Studia Math., 194 (2009), 279–292.
[21] C.H. Ling, *Representation of associative functions*, Publ. Math. Debrecen, 12 (1965), 189–212.
[22] A. Markoví, *A note on g-derivative and g-integral*, Real functions '94 (Liptovsk Jn, 1994), Tatra Mt. Math. Publ. 8 (1996), 71–76.
[23] A. Markoví and B. Riečan, *On the double g-integral*, Novi Sad J. Math. 26 (1996), 161–171.
[24] V.P. Maslov, *Asymptotic methods for solving pseudo-differential equations* (in Russian) "Nauka", Moscow, 1987.
[25] R. Mesiar, *Pseudo-linear integrals and derivatives based on a generator g*, Real functions '94 (Liptovsk Jn, 1994), Tatra Mt. Math. Publ. 8 (1996), 67–70.
[26] B. Nefzi, K. Brahim, A. Fitouhi, *On the finite Mellin transform in quantum calculus and Application*, Acta Math. Sci., 38 (2018), 1393–1410.
[27] E. Pap, *An integral generated by a decomposable measure*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 20 (1990), 135–144.
[28] E. Pap, *Decomposable measures and applications on nonlinear partial differential equations* Measure theory (Oberwolfach, 1990). Rend. Circ. Mat. Palermo (2) Suppl. No. 28 (1992), 387403.
[29] E. Pap, *g-calculus*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 23 (1993), 145–156.
[30] E. Pap, *The Lebesgue decomposition of the null-additive fuzzy measures*, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 24 (1994), 129–137.
[31] E. Pap, *Extension of* $\oplus$*-decomposable measures*, Atti Sem. Mat. Fis. Univ. Modena, 41 (1993), 109–119.
[32] E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Inform. Sci., 180 (2010) 543–548.
[33] E. Pap, M. Štrboja and I. Rudas, *Pseudo-*L^p *space and convergence*, Fuzzy Sets and Systems, 238 (2014), 113–128.
[34] H. Román-Flores, A. Flores-Franulič and Y. Chalco-Cano, *A Jensen type inequality for fuzzy integrals*, Inform. Sci., 177 (2007), 3192–3201.
[35] M. Sugeno, *Theory of fuzzy integrals and its application*, Doctoral dissertation, Tokyo Institute of Technology, 1974.
[36] H.-H. Tsai and I.-Y.. Lu, *The evaluation of service quality using generalized Choquet integral*, Inform. Sc., 176 (2006), 640–663.
[37] S. Weber, $\perp$*-decomposable measures and integrals for Archimedean t-conorms* $\perp$, J. Math. Anal. Appl., 101 (1984), 114–138.
[38] S. Weber, *Measures of fuzzy sets and measures of fuzziness*, Fuzzy Sets and Systems 13 (1984), 247–271.

Pankaj Jain
Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi-110021, India,
E-mail address: pankaj.jain@sau.ac.in, pankajkrjain@hotmail.com

Chandrani Basu
Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi-110021, India,
E-mail address: chandrani.basu@gmail.com

Vivek Panwar
Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi-110021, India,
E-mail address: vivek.pan1992@gmail.com