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A NOTE ON POLY-FUBINI POLYNOMIALS OF TWO VARIABLES

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Abstract. In this paper, we introduce and explore poly-Fubini polynomials of two variables and obtain relationships with the Stirling numbers of the second kind. Moreover, we also establish identities relating poly-Fubini polynomials to the poly-Bernoulli, poly-Euler, and poly-Genocchi polynomials.

1. Introduction

In recent years, extensive researches on various families of special polynomials such as the Bernoulli polynomials, Euler polynomials, Genocchi polynomials, Fubini polynomials, and also their generalizations and unifications (see, for instance the recent works in [2, 12, 13, 19, 25, 28, 29, 30]) have become popular due to the abundance of their applications in many branches of mathematics such as in p-adic analytic number theory, umbral calculus, special functions and mathematical analysis, numerical analysis, combinatorics and other related fields.

The classical Fubini polynomials or geometric polynomials $F_n(y)$ are defined in [31] by

$$F_n(y) = \sum_{k=0}^{n} S_2(n,k)k!y^k, \tag{1.1}$$

where $S_2(n,k)$ is the Stirling numbers of the second kind (see [7, 14] for detailed information). The polynomials $F_n(y)$ satisfy the generating function:

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \tag{1.2}$$

and the recurrence relation:

$$F_{n+1}(y) = y\frac{d}{dy}\left[F_n(y) + yF_n(y)\right] \quad (see\ [10]). \tag{1.3}$$

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Setting $y = 1$ in (1.1), we obtain the n^{th} Fubini number (sometimes called ordered Bell number) F_n, defined by

$$F_n(1) := F_n = \sum_{k=0}^{n} S_2(n,k)k!. \tag{1.4}$$

Combinatorially, the number F_n counts all the possible set partitions of an n-element set such that the order of the blocks matters.

As a generalization of the classical Fubini polynomials, the Fubini polynomials of two variables $F_n(x,y)$ were introduced by Kargin [18] given by

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} \quad \text{(see also [19, 21])}. \tag{1.5}$$

Setting $x = 0$, (1.5) reduces to the classical Fubini polynomials.

Some of the most recent works on new families of Fubini numbers and polynomials include the $q-$class of Fubini polynomials [11], higher-order central Fubini polynomials of two variables [22], bivariate Apostol-Fubini polynomials of higher order [3], $\omega-$torsion Fubini polynomials [23], Fubini-type numbers and polynomials via generating functions and p-adic integral approach [20], degenerate Fubini polynomials [21, 26], and degenerate central Fubini polynomials [27].

For an integer k, the polylogarithm function $\mathrm{Li}_k(x)$ is defined via the formal power series

$$\mathrm{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}, \quad x \in \mathbb{C}, |x| < 1. \tag{1.6}$$

If $k \le 0$, say $k = -s$, then it converges for $|x| < 1$ and is given by

$$\mathrm{Li}_{-s}(x) = \frac{\sum_{j=0}^{s} \left\langle {s \atop j} \right\rangle x^{s-j}}{(1-x)^{s+1}},$$

where $\left\langle {s \atop j} \right\rangle$ are the Eulerian numbers. The number $\left\langle {s \atop j} \right\rangle$ is the number of permutations of $\{1, 2, \cdots, s\}$ with j permutation ascents. Moreover,

$$\left\langle {s \atop j} \right\rangle = \sum_{l=0}^{j+1} \binom{s+1}{l}(j-l+1)^s.$$

In the case when $k = 1$ in (1.6),

$$\mathrm{Li}_1(x) = -\ln(1-x),$$

where $\ln(z)$ is the principal branch of the complex logarithm $\ln(z)$ with the imaginary part restricted by $-\pi < \mathrm{Im}(\ln(z)) \le \pi$.

In [6], Bayad and Hamahata introduced poly-Bernoulli polynomials $B_n^{(k)}(x)$ by means of the following exponential generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.7}$$

The numbers $B_n^{(k)} := B_n^{(k)}(0)$ are called the poly-Bernoulli numbers which were introduced by Kaneko [17] as generalizations of the classical Bernoulli numbers B_n.

It can be seen from the generating function (1.7) that, for any $n \geq 0$,

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

where $B_n(x)$ are the classical Bernoulli polynomials given by the generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

The poly-Bernoulli numbers $B_n^{(k)}$ satisfy the relation (see [5]):

$$B_n^{(-k)} = \sum_{m \geq 0} m! S_2(n+1, m+1) m! S_2(k+1, n+1),$$

In [15], Hamahata defined poly-Euler polynomials $E_n^{(k)}(x)$ via the generating function:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(1+e^t)} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}.$$

These polynomials satisfy the explicit formula:

$$E_n^{(k)}(x) = \frac{1}{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} E_n(x-j). \tag{1.8}$$

where $E_n(x) := E_n^{(1)}(x)$ are the classical Euler polynomials given by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

When $x = 0$, $E_n^{(k)} := E_n^{(k)}(0)$ are called the poly Euler numbers which reduce to classical Euler numbers when $k = 1$.

In [24], Kim *et al.* also introduced poly-Genocchi polynomials and numbers through the generating functions:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!},$$

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t + 1} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!}.$$

In the case when $k = 1$,

$$G_n^{(1)}(x) = G_n(x) \quad \text{and} \quad G_n^{(1)} = G_n,$$

where $G_n(x)$ and G_n are the classical Genocchi polynomials and numbers respectively.

Motivated by the above generalizations of special polynomials, we introduce a generalization of the Fubini polynomials of two variables using polylogarithm and establish relationships with the Stirling numbers of the second kind. Moreover, we will also obtain identitites involving poly-Bernoulli, poly-Euler, and poly-Genocchi polynomials.

2. The poly-Fubini polynomials of two variables

We now define a polylogarithm version of the Fubini polynomials of two variables.

Definition 2.1. For $k \in \mathbb{Z}$, we define the *poly-Fubini polynomials of two variables* $F_n^{(k)}(x,y)$ through the generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!}. \tag{2.1}$$

The polynomials $F_n^{(k)}(y) := F_n^{(k)}(0,y)$ are called *poly-Fubini polynomials*, and the numbers $F_n^k := F_n^{(k)}(1)$ are called *poly-Fubini numbers*.

Note that the polylogarithm function satisfies the derivative property:

$$\frac{d}{dt}\mathrm{Li}_k(1-e^{-t}) = \frac{1}{e^t-1}\mathrm{Li}_{k-1}(1-e^{-t}).$$

Thus, for $k \geq 1$, the generating function in (2.1) can be expressed as iterated integrals:

$$\sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{e^{xt}}{t(1-y(e^t-1))}\underbrace{\int_0^t \frac{1}{e^t-1}\int_0^t \frac{1}{e^t-1}\cdots\int_0^t \frac{t}{e^t-1}dtdt\cdots dt}_{(k-1)-times}.$$

In particular, when $k=2$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(2)}(x,y)\frac{t^n}{n!} &= \frac{e^{xt}}{t(1-y(e^t-1))}\int_0^t \frac{t}{e^t-1}dt \\ &= \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{B_n}{n+1}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{B_r}{r+1}F_{n-r}(x,y)\frac{t^n}{n!}.\end{aligned}$$

Hence, we have the following theorem.

Theorem 2.2. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(2)}(x,y) = \sum_{r=0}^{n}\binom{n}{r}\frac{B_r}{r+1}F_{n-r}(x,y). \tag{2.2}$$

The polynomials $F_n^{(k)}(x,y)$ satisfy the following addition formula.

Theorem 2.3. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(k)}(x+z,y) = \sum_{m=0}^{n}\binom{n}{m}F_m^{(k)}(x,y)z^{n-m} \tag{2.3}$$

$$= \sum_{m=0}^{n}\binom{n}{m}F_m^{(k)}(z,y)x^{n-m}. \tag{2.4}$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x+z,y)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{xt}e^{zt} \\ &= \sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty} z^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}F_m^{(k)}(x,y)z^{n-m}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (2.3), and interchanging the roles of x and z in (2.3) gives (2.4). □

Taking $z=0$ in (2.4), we obtain a relationship between the poly-Fubini polynomials of two variables $F_n^{(k)}(x,y)$ and the univariate poly-Fubini polynomials $F_n^{(k)}(y)$ in the next corollary.

Corollary 2.4. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(k)}(x,y) = \sum_{m=0}^{n}\binom{n}{m}F_m^{(k)}(y)x^{n-m}. \tag{2.5}$$

The poly-Fubini polynomials of two variables satisfy the following derivative and integral properties:

Theorem 2.5. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$\frac{\partial}{\partial x}F_{n+1}^{(k)}(x,y) = (n+1)F_n^{(k)}(x,y) \tag{2.6}$$

$$\int F_n^{(k)}(x,y)dx = \frac{1}{n+1}F_{n+1}^{(k)}(x,y) + C, \tag{2.7}$$

for any constant C.

Proof. Using Corollary 2.4, we have

$$F_{n+1}^{(k)}(x,y) = \sum_{m=0}^{n}\binom{n+1}{m}F_m^{(k)}(y)x^{n+1-m}. \tag{2.8}$$

Differentiating both sides of (2.8) with respect to x gives

$$\begin{aligned}\frac{d}{dx}F_{n+1}^{(k)}(x,y) &= \sum_{m=0}^{n}\binom{n+1}{m}(n+1-m)F_m^{(k)}(y)x^{(n+1-m)-1} \\ &= (n+1)\sum_{m=0}^{n}\binom{n}{m}F_m^{(k)}(y)x^{n-m} \\ &= (n+1)F_n^{(k)}(x,y).\end{aligned}$$

Equation (2.7) follows directly from (2.6). □

The next theorem gives us an explicit formula of the poly-Fubini polynomials of two variables.

Theorem 2.6. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(k)}(x,y) = \frac{1}{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} F_{n+1}(x-j,y). \tag{2.9}$$

Proof. Note that the left-hand side of (2.1) can be expressed as

$$\begin{aligned}
\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} e^{xt} &= \sum_{m=0}^{\infty} \frac{(1-e^{-t})^{m+1}}{(m+1)^k} \cdot \frac{e^{xt}}{t(1-y(e^t-1))} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{e^{(x-j)t}}{t(1-y(e^t-1))} \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \sum_{n=-1}^{\infty} F_{n+1}(x-j,y) \frac{t^n}{(n+1)!}.
\end{aligned}$$

□

3. Relations with the Stirling numbers of the second kind

In this section, we obtain some interesting relationships of the poly-Fubini polynomials of two variables and the Stirling numbers of the second kind. It is well-known from [24] that

$$\frac{\mathrm{Li}_k(1-e^{-t})}{t} = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{m+j} m!}{(m+1)^{k-1}(j+1)} S_2(j+1,m+1) \frac{t^j}{j!} \tag{3.1}$$

Thus, using (3.1), we obtain an explicit formula of $F_n^{(k)}(x,y)$ involving the Stirling numbers of the second kind and the Fubini polynomials of two variables.

Theorem 3.1. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(k)}(x,y) = \sum_{r=0}^{n} \sum_{m=0}^{r} \frac{(-1)^{m+r} m!}{(m+1)^{k-1}(r+1)} \binom{n}{r} S_2(r+1,m+1) F_{n-r}(x,y).$$

Proof. Using (2.1) and (3.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(k)}(x,y) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^{m+n} m!}{(m+1)^{k-1}(n+1)} S_2(n+1,m+1) \frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} F_n(x,y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \sum_{m=0}^{r} \frac{(-1)^{m+r} m!}{(m+1)^{k-1}(r+1)} \binom{n}{r} S_2(r+1,m+1) F_{n-r}(x,y) \right) \frac{t^n}{n!}.
\end{aligned}$$

□

Note that for $y \neq -1$,

$$\begin{aligned}\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} &= e^{xt}((y+1)-ye^t)^{-1} \\ &= \frac{1}{y+1}\sum_{j=0}\left(\frac{y}{y+1}\right)^j e^{(x+j)t} \\ &= \frac{1}{y+1}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j (x+j)^n \frac{t^n}{n!}.\end{aligned}$$

Thus,

$$F_n(x,y) = \frac{1}{y+1}\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j (x+j)^n. \tag{3.2}$$

Consequently, replacing $F_{n-r}(x,y)$ in Theorem 3.1 using expression (3.2), we obtain an expression of $F_n^{(k)}(x,y)$ in terms of $S_2(n,m)$ only in the next corollary.

Corollary 3.2. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$F_n^{(k)}(x,y) = \sum_{r=0}^{n}\left\{\sum_{m=0}^{r}\frac{(-1)^{m+r}m!}{(m+1)^{k-1}(r+1)}\binom{n}{r}S_2(r+1,m+1)\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j (x+j)^{n-r}\right\}.$$

Theorem 3.3. *For $k \in \mathbb{Z}$ and $n \geq 0$,*

$$(y+1)F_n^{(k)}(x,y) - yF_n^{(k)}(x+1,y) = \sum_{r=0}^{n}\sum_{m=0}^{r}\frac{(-1)^{m+r}m!}{m^{k-1}(r+1)}\binom{n}{r}S_2(r+1,m+1)x^{n-r}. \tag{3.3}$$

Proof. Note that

$$\frac{(y+1)\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{xt} - \frac{y\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{(x+1)t} = \frac{\mathrm{Li}_k(1-e^{-t})}{t}e^{xt}.$$

Using (3.1), we have

$$\sum_{n=0}^{\infty}\left\{(y+1)F_n^{(k)}(x,y) - yF_n^{(k)}(x+1,y)\right\}\frac{t^n}{n!} \tag{3.4}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(-1)^{m+n}m!}{m^{k-1}(n+1)}S_2(n+1,m+1)\right)\frac{t^n}{n!}\cdot\sum_{n=0}^{\infty}x^n\frac{t^n}{n!} \tag{3.5}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\sum_{m=0}^{r}\frac{(-1)^{m+r}m!}{m^{k-1}(r+1)}\binom{n}{r}S_2(r+1,m+1)x^{n-r}\right\}\frac{t^n}{n!}. \tag{3.6}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof. ∎

Theorem 3.4. *For $k \in \mathbb{Z}$ and $n \geq 0$, the poly-Fubini polynomials of two variables $F_n^{(k)}(x;y)$ satisfy the following relations:*

$$F_n^{(k)}(x,y) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} S_2(l,m) F_{n-l}^{(k)}(-m,y) x^{(m)}, \tag{3.7}$$

$$F_n^{(k)}(x,y) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} S_2(l,m) \binom{n}{m} F_{n-l}^{(k)}(y)(x)_m, \tag{3.8}$$

$$F_n^{(k)}(x,y) = \sum_{m=0}^{\infty} \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{l}} S_2(l+s,s) F_{n-m-l}^{(k)}(y) B_m^{(s)}(x), \tag{3.9}$$

where $B_n^{(s)}(x)$ are the higher-order Bernoulli polynomials (see [1]) defined by

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x) \frac{t^n}{n!}.$$

Here, $(x)_m$ and $x^{(m)}$ are the falling and rising factorials respectively, defined as

$$(x)_m = x(x-1)\cdots(x-m+1) \text{ and } x^{(m)} = x(x+1)\cdots(x+m-1) \text{ for } m \geq 1, \text{ and } (x)_0 := x^{(0)} := 1.$$

Proof. For relation (3.7), we note that (2.1) can be written as

$$\sum_{n=0}^{\infty} F_n^{(k)}(x,y) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} (1-(1-e^{-t}))^{-x}.$$

Applying Newton's binomial theorem, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(k)}(x,y) \frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} \sum_{m=0}^{\infty} \binom{x+m-1}{m} (1-e^{-t})^m. \\
&= \sum_{m=0}^{\infty} x^{(m)} \frac{(e^t-1)^m}{m!} \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} e^{-mt} \\
&= \sum_{m=0}^{\infty} x^{(m)} \left(\sum_{n=0}^{\infty} S_2(n,m) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} F_n^{(k)}(-m,y) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \sum_{l=m}^{n} S_2(l,m) \binom{n}{l} F_{n-l}^{(k)}(-m,y) x^{(m)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives relation (3.7).

For relation (3.8), we can express (2.1) as

$$\sum_{n=0}^{\infty} F_n^{(k)}(x,y) \frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e-t)}{t(1-y(e^t-1))} ((e^t-1)+1)^x.$$

Again, using Binomial theorem, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!} &= \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}\sum_{m=0}^{\infty}\binom{x}{m}(e^t-1)^m \\ &= \sum_{m=0}^{\infty}(x)_m\frac{(e^t-1)^m}{m!}\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} \\ &= \sum_{m=0}^{\infty}(x)_m\left(\sum_{n=0}^{\infty}S_2(n,m)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n^{(k)}(y)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=m}^{n}S_2(l,m)\binom{n}{l}F_{n-l}^{(k)}(y)(x)_m\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients completes the proof of (3.8).

For relation (3.9), we express (2.1) as

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(k)}(x,y)\frac{t^n}{n!} &= \frac{(e^t-1)^s}{s!}\cdot\frac{t^s e^{xt}}{(e^t-1)^s}\cdot\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}\cdot\frac{s!}{t^s}. \\ &= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(x)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}F_n^{(k)}(y)\frac{t^n}{n!}\right)\frac{s!}{t^s} \\ &= \left(\sum_{n=0}^{\infty}S_2(n+s,s)\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}B_m^{(s)}(x)\frac{t^m}{m!}F_{n-m}^{(k)}(y)\frac{t^{n-m}}{(n-m)!}\right)\frac{s!}{t^s} \\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=m}^{\infty}\sum_{l=0}^{n-m}S_2(l+s,s)\frac{t^{l+s}}{(l+s)!}B_m^{(s)}(x)F_{n-m-l}^{(k)}(y)\frac{t^{n-m-l}}{(n-m-l)!}\frac{t^m}{m!}\frac{s!}{t^s}\right\} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{s!}}S_2(l+s,s)F_{n-m-l}^{(k)}(y)B_m^{(s)}(x)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients completes the proof of (3.9).

Theorem 3.5. *For $k\in\mathbb{Z}$ and nonnegative integers m and n,*

$$\sum_{r=0}^{n}\binom{n}{r}m!F_r^{(k)}(-m,y)S_2(n-r,m)=\sum_{r=0}^{n}\binom{n}{r}(-1)^{m+n-r}m!F_r^{(k)}(y)S_2(n-r,m).$$

Proof. Consider the expression

$$A(t)=\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}(1-e^{-t})^m. \tag{3.10}$$

Expanding $A(t)$ into series, we obtain

$$\begin{aligned}A(t) &= \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{-tm}m!\frac{(e^t-1)^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}F_r^{(k)}(-m,y)m!S_2(n-r,m)\frac{t^n}{n!}.\end{aligned} \tag{3.11}$$

Now, $A(t)$ can be also expressed into

$$A(t) = \frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))} m!(-1)^m \frac{(e^{-t}-1)^m}{m!}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} F_r^{(k)}(y)(-1)^{m+n-r} m! S_2(n-r,m)\frac{t^n}{n!}. \tag{3.12}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.11) and (3.12), we get the desired identity. □

4. Relationships with other poly-type polynomials

In this section, we establish identities for $F_n^{(k)}(x,y)$ involving poly-Bernoulli, poly-Euler, and poly-Genocchi polynomials. Now, we express $F_n^{(k)}(x,y)$ in terms of the poly-Bernoulli polynomials.

Theorem 4.1. *For $k \in \mathbb{Z}$ and $n \geq 1$,*

$$F_{n-1}^{(k)}(x,y) = \frac{1}{n(y+1)}\sum_{s=0}^{n}\binom{n}{s}\left[B_{n-s}^{(k)}(x) - B_{n-s}^{(k)}(x-1)\right] c_s(y), \quad y \neq -1 \tag{4.1}$$

where $c_s(y) := \sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j j^s.$

Proof. It follows from (2.1) that

$$\sum_{n=1}^{\infty} nF_{n-1}^{(k)}(x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-y(e^t-1)}e^{xt}. \tag{4.2}$$

Expanding the right-hand side of (4.2), we get

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-y(e^t-1)}e^{xt} = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{xt}(1-y(e^t-1))^{-1}$$
$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt} - \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{(x-1)t}\right)\frac{1}{y+1}\left(1-\frac{y}{y+1}e^t\right)^{-1}$$
$$= \frac{1}{y+1}\sum_{n=0}^{\infty}[B_n^{(k)}(x) - B_n^{(k)}(x-1)]\frac{t^n}{n!}\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j e^{tj}, \quad \left|\frac{y}{y+1}e^t\right| < 1$$
$$= \frac{1}{y+1}\sum_{n=0}^{\infty}[B_n^{(k)}(x) - B_n^{(k)}(x-1)]\frac{t^n}{n!}\sum_{j=0}^{\infty}\cdot\sum_{n=0}^{\infty}\left(\frac{y}{y+1}\right)^j j^n\frac{t^n}{n!}$$
$$= \frac{1}{y+1}\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\left[B_{n-s}^{(k)}(x) - B_{n-s}^{(k)}(x-1)\right]\sum_{j=0}^{\infty}\left(\frac{y}{y+1}\right)^j j^s\right)\frac{t^n}{n!}.$$

□

Theorem 4.2. *For $k \in \mathbb{Z}$ and $n \geq 1$,*

$$n\left[(y+1)F_{n-1}^{(k)}(x,y) - yF_{n-1}^{(k)}(x+1,y)\right] = B_n^{(k)}(x) - B_n^{(k)}(x-1), \tag{4.3}$$
$$2\left[(y+1)F_{n-1}^{(k)}(x,y) - yF_{n-1}^{(k)}(x+1,y)\right] = E_n^{(k)}(x) + E_n^{(k)}(x+1), \tag{4.4}$$
$$2n\left[(y+1)F_{n-1}^{(k)}(x,y) - yF_{n-1}^{(k)}(x+1,y)\right] = G_n^{(k)}(x) + G_n^{(k)}(x+1). \tag{4.5}$$

Proof. For relation (4.3), we use the equality

$$\frac{\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}(1-y(e^t-1))e^{xt}t = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{xt}. \tag{4.6}$$

Expressing the left-hand side of (4.6) in terms of the poly-Fubini polynomials of two variables, we get

$$\frac{(y+1)\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{xt}t - \frac{y\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}e^{(x+1)t}t$$
$$= \sum_{n=1}^{\infty}\left[(y+1)nF_{n-1}^{(k)}(x,y) - \sum_{n=1}^{\infty} ynF_{n-1}^{k}(x+1,y)\right]\frac{t^n}{n!},$$

and expressing the right-hand side of (4.6) in terms of poly-Bernoulli polynomials, we obtain

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt} - \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{(x-1)t}$$
$$= \sum_{n=0}^{\infty}\left[B_n^{(k)}(x) - B_n^{(k)}(x-1)\right]\frac{t^n}{n!}.$$

Similarly, for relations (4.4) and (4.5), use the equations

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}(1-y(e^t-1))e^{xt}t = \frac{2\mathrm{Li}_k(1-e^{-t})}{t(e^t+1)}(e^t+1)e^{xt}t \text{ and}$$
$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(1-y(e^t-1))}(1-y(e^t-1))e^{xt}t = \frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}(e^t+1)e^{xt}$$

respectively to obtain the desired identities. □

Conclusion

Introducing the poly-Fubini polynomials of two variables as a generalization of the Fubini polynomials of two variables gives standard identities and formulas parallel to that of the poly-Bernoulli, poly-Euler, and poly-Genocchi polynomials. In this paper, we have developed and established relationships of poly-Fubini polynomials of two variables with the Stirling numbers of the second kind and with the other special poly-type polynomials. We note that a generalization which we may call generalized poly-Fubini polynomials of parameters a, b, c can be defined akin to the generalized poly-Bernoulli polynomials and poly-Euler polynomials of parameters a, b, c in [4, 8, 16] via the generating function:

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{t(1-y(b^t-a^{-t}))}c^{xt} = \sum_{n=0}^{\infty} F_n^{(k)}(x,y;a,b,c)\frac{t^n}{n!}, \quad a,b,c>0.$$

It is interesting to establish properties of these generalized poly-Fubini polynomials and to correlate these to the other generalized poly-type polynomials of parameters a, b, c.

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