Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 11 Issue 4(2020), Pages 1-15.

AN ITERATIVE SCHEME FOR QUASI-NONEXPANSIVE MAPPING IN CAT(0) SPACES

PIM SANBOONSIRI

Abstract. In this work, we study a common fixed point theory on E be a nonempty closed and convex subset of a complete CAT(0) space K, and using $T, S : E \to E$ are quasi-nonexpansive mapping having demiclosed principle such that $F(T) \cap F(S) \neq \emptyset$, we introduce the modified algorithm in frame Noor iterative prove strong convergence and prove that the sequence $\{x_n\}$ $\triangle-$ converges to quasi-nonexpansive mappings in CAT(0) spaces which enough to approximate a common fixed point.

1. Introduction

A CAT(0) space plays a primary role in various mathematic areas (see [1], [2], [3]). Moreover, it is also beneficial to biology and computer science (see [4], [5]). A metric space K is a CAT(0) space, if it is geodesically connected and if every geodesic triangle in K is at least as thin as its comparison triangle in the Euclidean plane. The CAT(0) space is the well-known method that provides complete, simply connected Riemannian manifold, showing non-positive sectional curvature. The complex Hilbert ball with a hyperbolic metric is the CAT(0) space (see [6]). Other examples of the CAT(0) space include preHilbert spaces, R-trees (see[1], [7], [8], [9], [10]) and Euclidean buildings (see [11]).

A mapping $T : K \to K$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. The generalization of nonexpansive mappings which we are interested in are quasi-nonexpansive mappings, i.e., $d(Tx, y) \leq d(x, y)$ for all $x \in K$ and for all $y \in F(T)$.

Petryshyn and Williamson (see [12]), in 1973, proved a sufficient and necessary condition for Mann iterative sequences to converge to fixed points for quasi-nonexpansive mappings.

In 1997, Ghosh and Debnath (see [13]) extended the results of [12] and gave the sufficient and necessary condition for Ishikawa iterative sequences to converge to fixed points for quasi-nonexpansive mappings.

2000 *Mathematics Subject Classification.* 47H09, 47H10.
Key words and phrases. Noor iterative scheme; CAT(0) spces; Quasi-nonexpansive mapping.

Submitted June 13, 2020. Published October 9, 2020.
Communicated by Ravi Agarwal.

The Noor iteration (see[14]) is defined by $x_1 \in K$ and

$$\begin{cases} z_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ y_n = (1-\varepsilon_n)x_n + \varepsilon_n T z_n, \\ x_{n+1} = (1-\delta_n)x_n + \delta_n T y_n, \end{cases} \tag{1.1}$$

for all $n \geq 1$,where $\{\lambda_n\}, \{\varepsilon_n\}$ and$\{\delta_n\}$ are sequences in $[0,1]$. If we take λ_n =λ_n=0 for all n, (1.1) reduces to the Mann iteration (see [15]), and we take $\gamma_n = 0$ for all n, (1.1) reduces to the Ishikawa iteration (see [16]).

Phuengrattana and Suantai [17] give a necessary and sufficient condition for the convergence of the SP-iteration of continuous functions on an arbitrary interval. Also prove that the Mann, Ishikawa, Noor and SP-iterations are equivalent.

Kitkuan and Padcharoen [18] have modified SP-iteration in frame of a CAT(0) space as follows:

$$\begin{cases} z_n = (1-\lambda_n)x_n \oplus \lambda_n T x_n, \\ y_n = (1-\varepsilon_n)z_n \oplus \varepsilon_n T z_n, \\ x_{n+1} = (1-\delta_n)y_n \oplus \delta_n T y_n, \end{cases} \tag{1.2}$$

for all $n \geq 1$, where is a nonempty convex subset of a CAT(0) space, $x_1 \in K$, $\{\lambda_n\}, \{\varepsilon_n\}$ and$\{\delta_n\}$ are sequences in $[0,1]$.

Currently, Prommai et al. [28], introduce a common fixed point for firmly non-spreading mappings and quasi-nonexpansive mappings in CAT(0) spaces. Using the concept of Ishikawa iterative scheme, we define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\lambda_n)x_n \oplus \lambda_n S(Tx_n), \end{cases} \tag{1.3}$$

where E is a nonempty closed and convex subset of a complete CAT(0) space, S and T are mappings defined on E.

Motivation by above, we introduce a common fixed point for quasi-nonexpansive using the concept of Noor iterative scheme in frame of a CAT(0) space, we define the sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1-\lambda_n)x_n \oplus \lambda_n S(Tx_n), \\ y_n = (1-\varepsilon_n)x_n \oplus \varepsilon_n S(Tz_n), \\ x_{n+1} = (1-\delta_n)x_n \oplus \delta_n S(Ty_n). \end{cases} \tag{1.4}$$

The definitions and known results are recalled in the existing literature on this concept. K is a nonempty subset of a CAT(0) space K and $T : K \to K$ is a mapping. A point $u \in K$ is called a fixed point of T if $Tu = u$.

We known that the iterative scheme (1.2) is suitable for finding a fixed point in the framework of CAT(0) spaces and more relaxed step-sizes. On the other hand, the iterative schemes (1.3) and (1.4) are suitable for finding a common fixed point in the framework of CAT(0) spaces and also the iterative schemes (1.4) have more relaxed step-sizes.

Now, we recall some definitions. Let K be a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in K. For $x \in K$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point ([19], Proposition 7). Also, every CAT(0) space has the *Opial* property, i.e., if $\{x_n\}$ is a sequence in K and $\triangle - \lim_{n\to\infty} x_n = x$, then for each $y \neq x \in K$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

We now give the definition and collect some basic properties of $\triangle-$ convergence and recall the related concepts which will be used in our work.

2. Preliminaries and lemmas

Lemma 2.1. [20] *A sequence $\{x_n\}$ in a CAT(0) space K is convergent to $x \in K$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. For this case, $\triangle - \lim_{n\to\infty} u_n = x$ and x is given by the $\triangle-$ limit of $\{x_n\}$.*

The concept of $\triangle-$ convergence in a fundamental metric space was reported by Lim (see [21]). Kirk and Panyanak (see [22]) recently used the notion of $\triangle-$ convergence begin by Lim (see [21]) to prove on the CAT(0) space analogous of some Banach space results, which relate to weak convergence. Furthermore, Dhompongsa and Panyanak (see [23]) achieved $\triangle-$ convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

Lemma 2.2. [23] *Let K be a CAT(0) space. Then*

$$d((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d(x, z) + \alpha d(y, z)$$

for all $\alpha \in [0, 1]$ and $x, y, z \in K$.

Lemma 2.3. [23] *Let K be a CAT(0) space. Then*

$$d^2((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d^2(x, z) + \alpha d^2(y, z) - \alpha(1-\alpha)d^2(x, y)$$

for all $\alpha \in [0, 1]$ and $x, y, z \in K$.

Lemma 2.4. [26] *Every bounded sequence in a complete CAT(0) space always contains a $\triangle-$ convergent subsequence.*

Lemma 2.5. [27] *Let E is a closed and convex subset of a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in E. Then the asymptotic center of $\{x_n\}$ is in E.*

Lemma 2.6. [23] *Let K be a be a nonempty closed and convex subset of a CAT(0) space (K, d). Let $\{x_n\}$ be a bounded sequence in K with $A(\{x_n\}) = \{x\}$, $\{t_n\}$ be a subsequence of $\{x_n\}$ with $A(\{t_n\}) = \{t\}$. If $\lim_{n\to\infty} d(x_n, t)$ exists, then $x = t$.*

Lemma 2.7. [24] *Let K be a CAT(0) space. Let$\{x_n\}$ and $\{y_n\}$ be two bounded sequences in K with $\lim_{n\to\infty} d(y_n, x_n) = 0$. If $\triangle - \lim_{n\to\infty} x_n = x$, then $\triangle - \lim_{n\to\infty} y_n = x$.*

We defined $\omega_w(\{x_n\}) := \bigcup A(\{u_n\})$where the union is taken over any subsequence $\{u_n\}$ of $\{x_n\}$. In order to prove our main theorem, the following facts are needed.

Lemma 2.8. [24] *Let E be a nonempty closed and convex subset of a complete CAT(0) space K and $T : E \to K$ be a generalized hybrid mapping. If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, w)\}$ converges for all $w \in F(T)$, then $\omega_w(\{x_n\}) \subset F(T)$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.*

Lemma 2.9. [24] *The conclusion of Lemma 2.8 is still true if $T : E \to K$ is any one of nonexpansive mappings, firmly nonspreading mapping, nonspreading mapping, TJ-1 mapping, TJ-2 mapping, and hybrid mapping.*

3. Results

Theorem 3.1. *Let E be a nonempty closed and convex subset of a complete CAT(0) space K, and let $T : E \to K$ be a quasi-nonexpansive mapping having demiclosed principle such that $F(T) \neq \emptyset$. If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\{d(x_n, w)\}$ converges for all $w \in F(T)$, $\omega_w(\{x_n\}) \subset F(T)$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.*

Proof. By the assumption $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $t \in \omega_w(\{x_n\})$, then there exists a subsequence $\{t_n\}$ of $\{x_n\}$ such that $A(\{t_n\}) = \{t\}$. By Lemma 2.5 and Lemma 2.6 there exists a subsequence $\{w_n\}$ of $\{t_n\}$ such that $\triangle - \lim_{n\to\infty} t_n = w \in E$. Because $\lim_{n\to\infty} d(Tw_n, w_n) = 0$, we can conclude that $w \in F(T)$. From Lemma 2.6 and $\{d(x_n, w)\}$ converges for all $w \in F(T)$, we have $t = w \in F(T)$. This implies that $\omega_w(\{x_n\}) \subset F(T)$. Finally $\omega_w(\{x_n\})$ consists of exactly one point. Indeed, let $A(\{x_n\}) = \{x\}$ and $\{t_n\}$ be a subsequence of $A(\{t_n\}) = \{t\}$. Because $t \in \omega_w(\{x_n\}) \subset F(T)$, we have $t = w \in F(T)$ and hence $\{d(x_n, t)\}$ converges. We can apply Lemma 2.6 again to conclude that $x = t$. □

Theorem 3.2. *Let E be a nonempty closed and convex subset of a complete CAT(0) space K, and let $T, S : E \to E$ are quasi-nonexpansive mapping having demiclosed principle such that $F(T) \cap F(S) \neq \emptyset$. If $\{x_n\}$ be a sequence defined by (1.4) then $\lim_{n\to\infty} d(x_n, u)$ exists for all $u \in F(T) \cap F(S)$.*

Proof. Let $(\{x_n\})$ be a sequence defined by (1.4) and $u \in F(T) \cap F(S)$. Then $d(Tx, u) \leq d(x, u)$ and $d(Sy, u) \leq d(x, u)$ for all $x, y \in E$. By Lemma 2.3, we have

$$\begin{aligned} d^2(z_n, u) &= d^2((1-\lambda_n)x_n \oplus \lambda_n S(Tx_n), u) \\ &\leq (1-\lambda_n)d^2(x_n, u) + \lambda_n d^2(S(Tx_n), u) - \lambda_n(1-\lambda_n)d^2(x_n, S(Tx_n)) \\ &\leq (1-\lambda_n)d^2(x_n, u) + \lambda_n d^2(Tx_n, u) - \lambda_n(1-\lambda_n)d^2(x_n, S(Tx_n)) \\ &\leq (1-\lambda_n)d^2(x_n, u) + \lambda_n d^2(x_n, u) - \lambda_n(1-\lambda_n)d^2(x_n, S(Tx_n)) \\ &\leq d^2(x_n, u) - \lambda_n(1-\lambda_n)d^2(x_n, S(Tx_n)) \\ &\leq d^2(x_n, u), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d^2(y_n,u) &= d^2((1-\varepsilon_n)x_n \oplus \varepsilon_n S(Tz_n),u) \\ &\leq (1-\varepsilon_n)d^2(x_n,u) + \varepsilon_n d^2(S(Tz_n),u) - \varepsilon_n(1-\varepsilon_n)d^2(x_n,S(Tz_n)) \\ &\leq (1-\varepsilon_n)d^2(x_n,u) + \varepsilon_n d^2(Tz_n,u) - \varepsilon_n(1-\varepsilon_n)d^2(x_n,S(Tz_n)) \\ &\leq (1-\varepsilon_n)d^2(x_n,u) + \varepsilon_n d^2(z_n,u) - \varepsilon_n(1-\varepsilon_n)d^2(x_n,S(Tz_n)). \end{aligned} \tag{3.2}$$

From (3.1), we get

$$\begin{aligned} d^2(y_n,u) &\leq (1-\varepsilon_n)d^2(x_n,u) + \varepsilon_n d^2(x_n,u) - \varepsilon_n(1-\varepsilon_n)d^2(x_n,S(Tz_n)) \\ &\leq d^2(x_n,u) - \varepsilon_n(1-\varepsilon_n)d^2(x_n,S(Tz_n)) \\ &\leq d^2(x_n,u), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d^2(x_{n+1},u) &= d^2((1-\delta_n)x_n \oplus \delta_n S(Ty_n),u) \\ &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(S(Ty_n),u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)) \\ &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(Ty_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)) \\ &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(y_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)). \end{aligned} \tag{3.4}$$

From (3.3), we get

$$\begin{aligned} d^2(x_{n+1},u) &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(x_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)). \\ &\leq d^2(x_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)) \\ &\leq d^2(x_n,u). \end{aligned} \tag{3.5}$$

Therefore, $\{d(x_n,u)\}$ is bounded and decreasing sequence which imply that $\lim_{n\to\infty} d(x_n,u)$ exists. □

Theorem 3.3. *Let E be a nonempty closed and convex subset of a complete CAT(0) space K, and let $T : E \to E$ be a firmly nonspreading mapping. Suppose that $S : E \to E$ is a quasi-nonexpansive mapping having demiclosed principle such that $F(T) \cap F(S) \neq \emptyset$ and let $\{x_n\}$ be defined as (1.4). If let $\{\delta_n\}$ is a sequences in $(0,1)$ such that $\liminf_{n\to\infty} \delta_n(1-\delta_n) > 0$, $\liminf_{n\to\infty} \varepsilon_n(1-\varepsilon_n) > 0$, and $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$, then $\liminf_{n\to\infty} d(y_n,Ty_n)) = \liminf_{n\to\infty} d(z_n,Tz_n)) = \liminf_{n\to\infty} d(x_n,Tx_n)) = 0$, and $\liminf_{n\to\infty} d(Ty_n,u)) = \liminf_{n\to\infty} d(Tz_n,u)) = \liminf_{n\to\infty} d(Tx_n,u)) = 0$ exists.*

Proof. Let $\{x_n\}$ be a sequence defined by (1.4) and $u \in F(T) \cap F(S)$. By Lemma 3.2 , we have $\lim_{n\to\infty} d(Tx_n,u))$ exists. Because $d(Tx_n,u) \leq d(x_n,u) \leq d(x_1,u)$, $\{x_n\}$ and $\{Tx_n\}$ are bounded. From (3.5) , we have

$$d^2(x_{n+1},u) \leq d^2(x_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n)).$$

Thus, we have

$$\delta_n(1-\delta_n)d^2(x_n,S(Ty_n)) \leq d^2(x_n,u) - d^2(x_{n+1},u).$$

Because $\liminf_{n\to\infty} \delta_n(1-\delta_n) > 0$, there exist $k > 0$ and $N \in \mathbb{N}$ such that $\delta_n(1-\delta_n) \geq k$ for all $n \geq N$, and hence

$$\begin{aligned}\limsup_{n\to\infty} kd^2(x_n, S(Ty_n)) &\leq \limsup_{n\to\infty} \delta_n(1-\delta_n)d^2(x_n, S(Ty_n))\\ &\leq \limsup_{n\to\infty}(d^2(x_n,u) - d^2(x_{n+1},u))\\ &= 0. \end{aligned} \tag{3.6}$$

Therefore, $0 \leq \liminf_{n\to\infty} d^2(x_n, S(Ty_n)) \leq \limsup_{n\to\infty} d^2(x_n, S(Ty_n)) \leq 0$, which implies that $\lim_{n\to\infty} d^2(x_n, S(Ty_n)) = 0$. Thus, we have

$$\lim_{n\to\infty} d(x_n, S(Ty_n)) = 0. \tag{3.7}$$

Furthermore, we have from (3.4) that

$$\begin{aligned} d^2(x_{n+1},u) &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(Ty_n,u) - \delta_n(1-\delta_n)d^2(x_n,S(Ty_n))\\ &\leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(Ty_n,u)\\ &= d^2(x_n,u) - \delta_n d^2(x_n,u) + \delta_n d^2(Ty_n,u). \end{aligned} \tag{3.8}$$

Therefore,

$$\delta_n[d^2(Ty_n,u) - d^2(x_n,u)] \leq d^2(x_n,u) - d^2(x_{n+1},u).$$

Because $\delta_n(1-\delta_n) < \delta_n$, $\lim_{n\to\infty} \delta_n > 0$. Using the same argument we can conclude that

$$\lim_{n\to\infty}[d^2(Ty_n,u) - d^2(x_n,u)] = 0.$$

Because $u \in F(T)$,

$$\begin{aligned} d^2(Ty_n,u) &= d^2(Ty_n,Tu)\\ &\leq \frac{1}{2}(d^2(y_n,Tu) + d^2(Ty_n,u) - d^2(y_n,Ty_n) - d^2(u,Tu))\\ &\leq d^2(y_n,Tu) - \frac{1}{2}d^2(y_n,Ty_n)\\ &= d^2(y_n,u) - \frac{1}{2}d^2(y_n,Ty_n). \end{aligned} \tag{3.9}$$

From (3.4), we have

$$d^2(x_{n+1},u) \leq (1-\delta_n)d^2(x_n,u) + \delta_n d^2(Ty_n,u).$$

From (3.9), we get

$$d^2(x_{n+1},u) \leq d^2(x_n,u) - \delta_n d^2(x_n,u) + \delta_n(d^2(y_n,u) - \frac{1}{2}d^2(y_n,Ty_n)). \tag{3.10}$$

From (3.4), we get

$$\begin{aligned} d^2(x_{n+1},u) &\leq d^2(x_n,u) - \delta_n d^2(x_n,u) + \delta_n(d^2(x_n,u) - \frac{1}{2}d^2(y_n,Ty_n))\\ &= d^2(x_n,u) - \frac{\delta_n}{2}d^2(y_n,Ty_n). \end{aligned} \tag{3.11}$$

Thus, we have

$$\delta_n d^2(y_n,Ty_n) \leq 2(d^2(x_n,u) - d^2(x_{n+1},u)).$$

Because $\delta_n(1-\delta_n) < \delta_n$, $\lim_{n\to\infty} \delta_n > 0$, we get

$$\lim_{n\to\infty} d^2(y_n,Ty_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0. \tag{3.12}$$

Because $\lim_{n\to\infty}(d^2(Ty_n, u) - d^2(y_n, u)) = 0$ and $\lim_{n\to\infty} d(y_n, u)$ exists, we may conclude that $\lim_{n\to\infty} d(Ty_n, u))$ exists.
Again by (3.10) and (3.2), we get

$$\begin{aligned}
&d^2(x_{n+1}, u)\\
&\le d^2(x_n, u) - \delta_n d^2(x_n, u) + \delta_n(d^2(y_n, u) - \frac{1}{2}d^2(y_n, Ty_n))\\
&\le d^2(x_n, u) - \delta_n d^2(x_n, u) + \delta_n[(1-\varepsilon_n)d^2(x_n, u) + \varepsilon_n d^2(Tz_n, u)] - \frac{1}{2}\delta_n d^2(y_n, Ty_n)\\
&= d^2(x_n, u) - \delta_n\varepsilon_n d^2(x_n, u) + \delta_n\varepsilon_n d^2(Tz_n, u) - \frac{1}{2}\delta_n d^2(y_n, Ty_n),
\end{aligned} \tag{3.13}$$

and $u \in F(T)$

$$\begin{aligned}
d^2(Tz_n, u) &= d^2(Tz_n, Tu)\\
&\le \frac{1}{2}(d^2(z_n, Tu) + d^2(Tz_n, u) - d^2(z_n, Tz_n) - d^2(u, Tu))\\
&\le d^2(z_n, Tu) - \frac{1}{2}d^2(z_n, Tz_n)\\
&= d^2(z_n, u) - \frac{1}{2}d^2(z_n, Tz_n).
\end{aligned} \tag{3.14}$$

Again by (3.13), (3.14) and (3.1), we have

$$\begin{aligned}
&d^2(x_{n+1}, u)\\
&\le d^2(x_n, u) - \delta_n\varepsilon_n d^2(x_n, u) + \delta_n\varepsilon_n[d^2(z_n, u) - \frac{1}{2}d^2(z_n, Tz_n)] - \frac{1}{2}\delta_n d^2(y_n, Ty_n)\\
&= d^2(x_n, u) - \delta_n\varepsilon_n d^2(x_n, u) + \delta_n\varepsilon_n d^2(z_n, u) - \frac{1}{2}\delta_n\varepsilon_n d^2(z_n, Tz_n) - \frac{1}{2}\delta_n d^2(y_n, Ty_n)\\
&\le d^2(x_n, u) - \delta_n\varepsilon_n d^2(x_n, u) + \delta_n\varepsilon_n d^2(x_n, u) - \frac{1}{2}\delta_n\varepsilon_n d^2(z_n, Tz_n) - \frac{1}{2}\delta_n d^2(y_n, Ty_n)\\
&\le d^2(x_n, u) - \frac{1}{2}\delta_n\varepsilon_n d^2(z_n, Tz_n) - \frac{1}{2}\delta_n\varepsilon_n d^2(y_n, Ty_n).
\end{aligned} \tag{3.15}$$

Thus, we have

$$\frac{1}{2}\delta_n\varepsilon_n[d^2(z_n, Tz_n) + d^2(y_n, Ty_n)] \le d^2(x_n, u) - d^2(x_{n+1}, u). \tag{3.16}$$

Because $\delta_n(1-\delta_n) < \delta_n$, $\lim_{n\to\infty}\delta_n > 0$, $\varepsilon_n(1-\varepsilon_n) < \varepsilon_n$, $\lim_{n\to\infty}\varepsilon_n > 0$ and $\lim_{n\to\infty} d^2(y_n, Ty_n) = 0$, we get

$$\lim_{n\to\infty} d^2(z_n, Tz_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0. \tag{3.17}$$

Because $\lim_{n\to\infty}(d^2(Tz_n,u)-d^2(z_n,u))=0$ and $\lim_{n\to\infty} d(z_n,u)$ exists, we may conclude that $\lim_{n\to\infty} d(Tz_n,u))$ exists.
Again by (3.4), (3.2) and (3.1), we have

$$\begin{aligned} d^2(x_{n+1},u) &\le (1-\delta_n)d^2(x_n,u)+\delta_n d^2(y_n,u)\\ &\le (1-\delta_n)d^2(x_n,u)+\delta_n[(1-\varepsilon_n)d^2(x_n,u)+\varepsilon_n d^2(z_n,u)]\\ &= d^2(x_n,u)-\delta_n\varepsilon_n d^2(x_n,u)+\delta_n\varepsilon_n d^2(z_n,u)\\ &\le d^2(x_n,u)-\delta_n\varepsilon_n d^2(x_n,u)+\delta_n\varepsilon_n[(1-\lambda_n)d^2(x_n,u)+\lambda_n d^2(Tx_n,u)]\\ &= d^2(x_n,u)-\delta_n\varepsilon_n\lambda_n d^2(x_n,u)+\delta_n\varepsilon_n\lambda_n d^2(Tx_n,u), \end{aligned} \tag{3.18}$$

and $u\in F(T)$

$$\begin{aligned} d^2(Tx_n,u) &= d^2(Tx_n,Tu)\\ &\le \frac{1}{2}(d^2(x_n,Tu)+d^2(Tx_n,u)-d^2(x_n,Tx_n)-d^2(u,Tu))\\ &\le d^2(x_n,Tu)-\frac{1}{2}d^2(x_n,Tx_n)\\ &= d^2(x_n,u)-\frac{1}{2}d^2(x_n,Tx_n). \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we have

$$\begin{aligned} d^2(x_{n+1},u) &\le d^2(x_n,u)-\delta_n\varepsilon_n\lambda_n d^2(x_n,u)+\delta_n\varepsilon_n\lambda_n[d^2(x_n,u)-\frac{1}{2}d^2(x_n,Tx_n)]\\ &= d^2(x_n,u)-\delta_n\varepsilon_n\lambda_n\frac{1}{2}d^2(x_n,Tx_n). \end{aligned} \tag{3.20}$$

Thus, we have

$$\frac{1}{2}\delta_n\varepsilon_n\lambda_n d^2(x_n,Tx_n)\le d^2(x_n,u)-d^2(x_{n+1},u). \tag{3.21}$$

Because $\delta_n(1-\delta_n)<\delta_n$, $\lim_{n\to\infty}\delta_n>0$, $\varepsilon_n(1-\varepsilon_n)<\varepsilon_n$, $\lim_{n\to\infty}\varepsilon_n>0$, $\lambda_n(1-\lambda_n)<\lambda_n$, $\lim_{n\to\infty}\lambda_n>0$, we get

$$\lim_{n\to\infty} d^2(x_n,Tx_n)=0.$$

Thus, we have

$$\lim_{n\to\infty} d(x_n,Tx_n)=0. \tag{3.22}$$

Because $\lim_{n\to\infty}(d^2(Tx_n,u)-d^2(x_n,u))=0$ and $\lim_{n\to\infty} d(x_n,u)$ exists, we may conclude that $\lim_{n\to\infty} d(Tx_n,u))$ exists. □

Theorem 3.4. *Let E be a nonempty closed and convex subset of a complete CAT(0) space K, and let $T: E\to E$ be a firmly nonspreading mapping. Suppose that $S: E\to E$ is a quasi-nonexpansive mapping having demiclosed principle such that $F(T)\cap F(S)\neq\emptyset$ and let $\{x_n\}$ be a sequence defined by* (1.4)*. If $\{\delta_n\}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty}\delta_n(1-\delta_n)>0$, then $\triangle-\lim_{n\to\infty} x_n=u\in F(T)\cap F(S)$.*

Proof. Let $\{x_n\}$ be a sequence defined by (1.4) and $u\in F(T)\cap F(S)$. From Lemma 3.2, we have $\lim_{n\to\infty} d(x_n,u)$ exists. Then $\{x_n\}$ is bounded. From Lemma 3.3, we have $\lim_{n\to\infty} d(x_n,Tx_n)=0$ and $\lim_{n\to\infty} d(Tx_n,u)$ exists which implies that $\{Tx_n\}$ is also bounded. By (3.22) we have $\lim_{n\to\infty} d(x_n,Tx_n)=0$ and by (3.7) we have $\lim_{n\to\infty} d(x_n, S(Tx_n)) = 0$. Because $d(S(Tx_n), Tx_n) \leq d(S(Tx_n), x_n) + d(x_n, Tx_n)$, $\lim_{n\to\infty} d(S(Tx_n), Tx_n) = 0$. By Theorem 3.1 and Lemma 2.9, there exist $\bar{x}, \bar{y} \in E$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subset F(T)$ and $\omega_w(\{Tx_n\}) = \{\bar{y}\} \subset F(S)$. Thus, we have $\triangle - \lim_{n\to\infty} x_n = \bar{x}$ and $\triangle - \lim_{n\to\infty} Tx_n = \bar{y}$. Using Lemma 2.7, we have $\bar{x} = \bar{y}$. □

4. Numerical results

In this section, we show numerical experiment results for supporting main results.

Example 4.1. *Let $K = \mathbb{R}$ be a Euclidean metric space and $E = [1, 9]$. Let $S, T : \mathbb{R} \to \mathbb{R}$ be mappings defined by*

$$Tx = \sqrt{24 - 6x + x^2} \quad \text{and} \quad Sx = \sqrt[3]{48 + x^2}.$$

It is easy to check that T be a firmly nonspreading mapping and S be a quasi-nonexpansive mapping with $F(S) = F(T) = \{4\}$. We compare our iterative scheme (1.4) and iterative scheme (1.3) with difference step-sizes have numerical results in Table 1, the value of x_n in Figure 1, Figure 3, Figure 5, and Figure 7, the values error in Figure 2, Figure 4, Figure 6 and Figure 8.

Table 1. The numerical results of iterative schemes.

Step-sizes	Iterative scheme (1.3)				Iterative scheme (1.4)			
	It.	Time	x_n	$\|x_n - x_{n-1}\|$	It.	Time	x_n	$\|x_n - x_{n-1}\|$
$\lambda_n = 0.5$, $\varepsilon_n = 0.7$, $\delta_n = 0.9$	28	0.006375	4.00000007	0.00000006	10	0.005084	4.00000000	0.00000001
$\lambda_n = \frac{n}{2n+1}$, $\varepsilon_n = \frac{3n+3}{5n+5}$, $\delta_n = 0.9 - \frac{1}{(n+1)^2}$	30	0.009131	4.00000008	0.0000007	11	0.003692	4.00000001	0.00000005
$\lambda_n = \frac{n}{2n+1}$, $\varepsilon_n = 0.9 - \frac{1}{(n+1)^2}$, $\delta_n = \frac{3n+3}{5n+5}$	30	0.009107	4.00000008	0.0000007	21	0.008554	4.00000003	0.00000004
$\lambda_n = \frac{3n+3}{5n+5}$, $\varepsilon_n = \frac{n}{2n+1}$, $\delta_n = 0.9 - \frac{1}{(n+1)^2}$	22	0.006320	4.00000008	0.00000006	11	0.004272	4.00000003	0.00000005

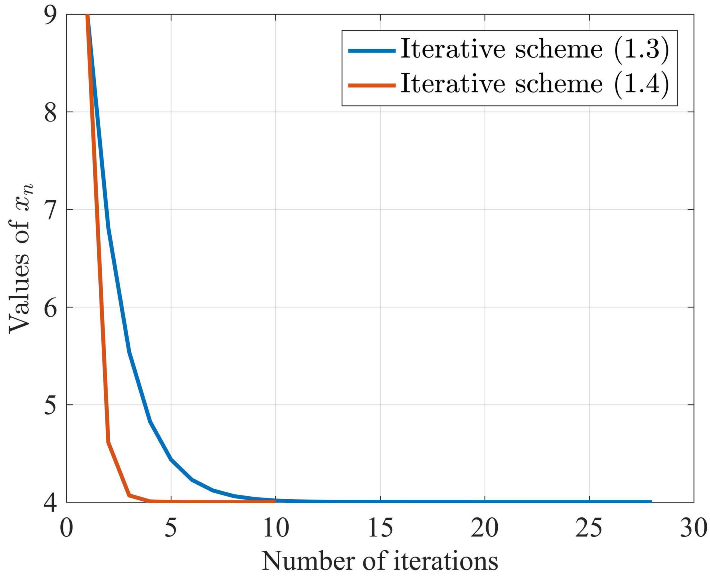

FIGURE 1. Graph of results values case $\lambda_n = 0.5$, $\varepsilon_n = 0.7$, $\delta_n = 0.9$.

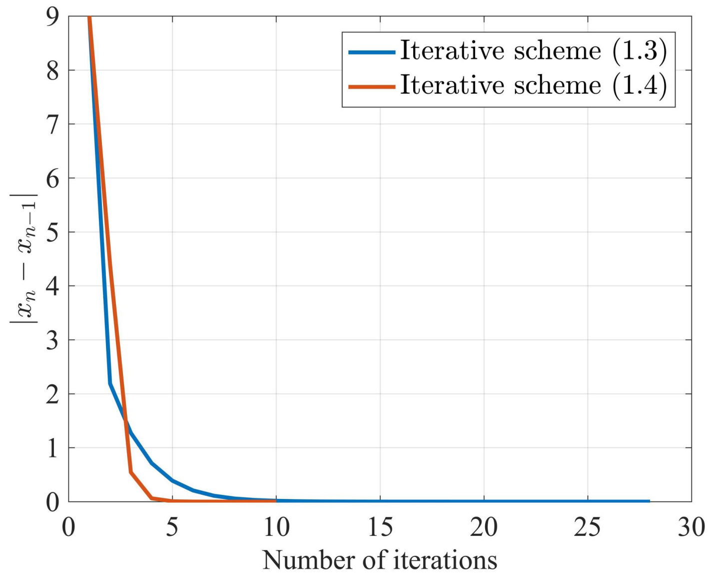

FIGURE 2. Graph of error values case $\lambda_n = 0.5$, $\varepsilon_n = 0.7$, $\delta_n = 0.9$.

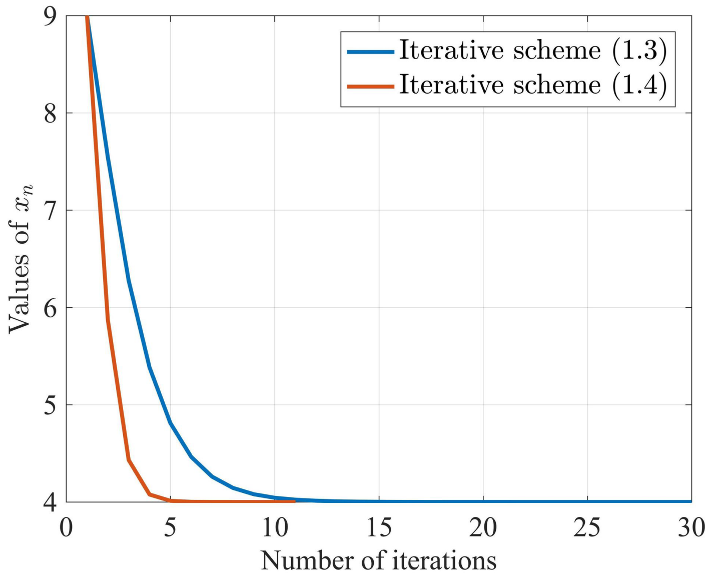

FIGURE 3. Graph of results values case $\lambda_n = \dfrac{n}{2n+1}$, $\varepsilon_n = \dfrac{3n+3}{5n+5}$, $\delta_n = 0.9 - \dfrac{1}{(n+1)^2}$.

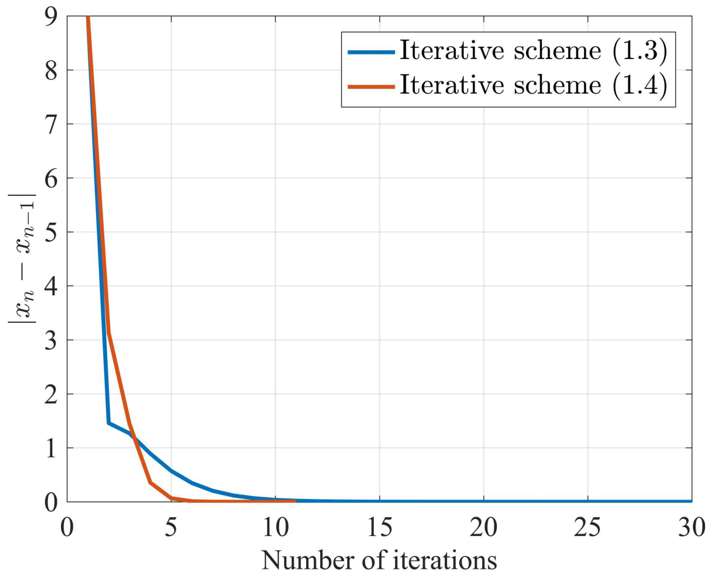

FIGURE 4. Graph of error values case $\lambda_n = \dfrac{n}{2n+1}$, $\varepsilon_n = \dfrac{3n+3}{5n+5}$, $\delta_n = 0.9 - \dfrac{1}{(n+1)^2}$.

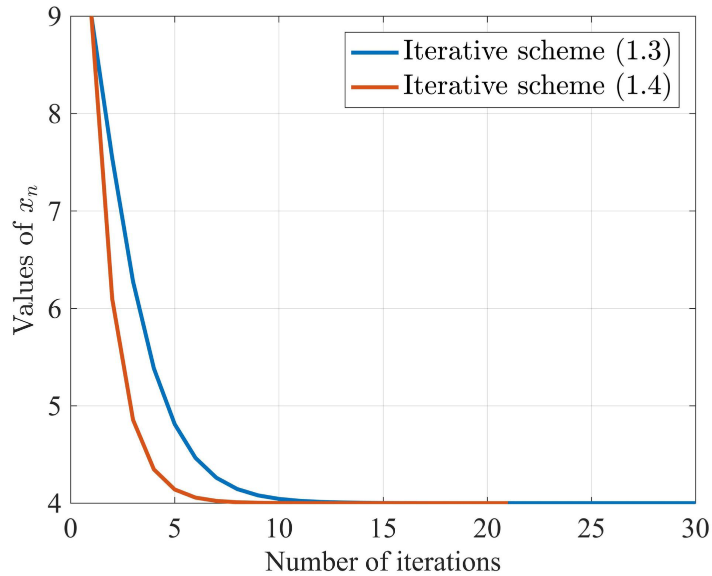

FIGURE 5. Graph of results values case $\lambda_n = \dfrac{n}{2n+1}$, $\varepsilon_n = 0.9 - \dfrac{1}{(n+1)^2}$, $\delta_n = \dfrac{3n+3}{5n+5}$.

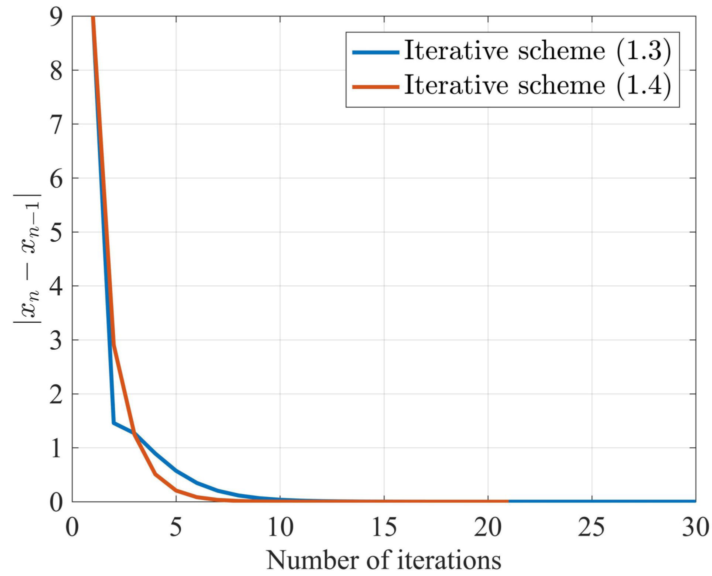

FIGURE 6. Graph of error values case $\lambda_n = \dfrac{n}{2n+1}$, $\varepsilon_n = 0.9 - \dfrac{1}{(n+1)^2}$, $\delta_n = \dfrac{3n+3}{5n+5}$.

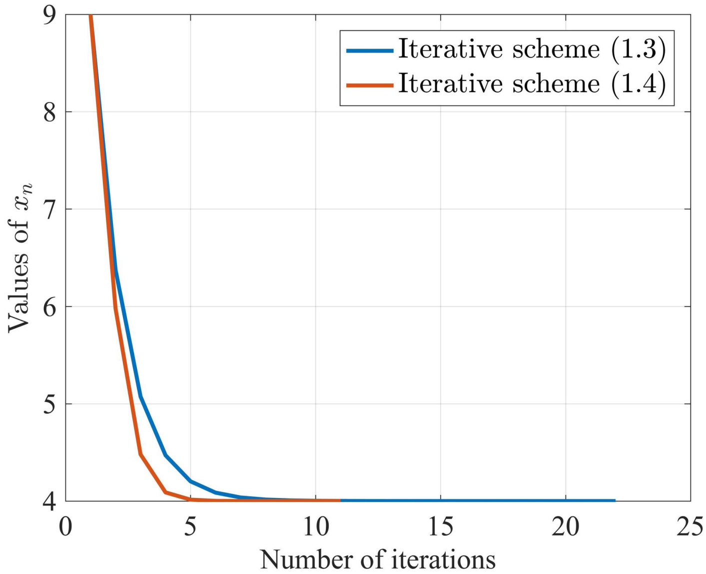

FIGURE 7. Graph of results values case $\lambda_n = \dfrac{3n+3}{5n+5}$, $\varepsilon_n = \dfrac{n}{2n+1}$, $\delta_n = 0.9 - \dfrac{1}{(n+1)^2}$.

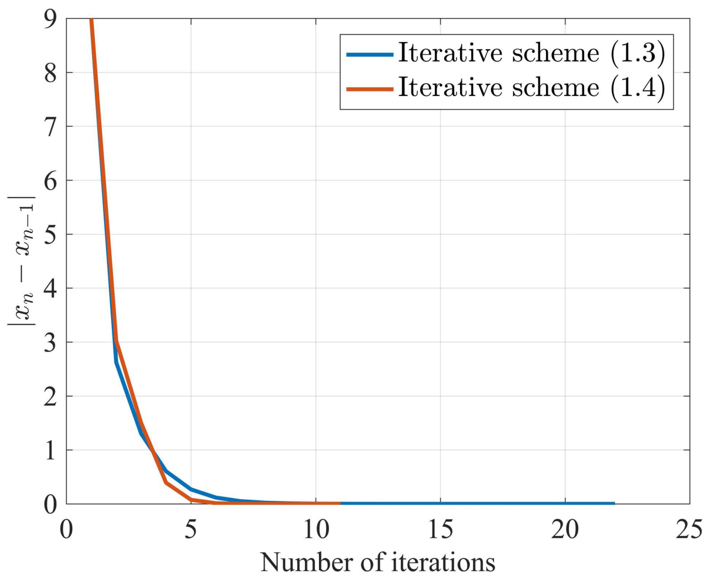

FIGURE 8. Graph of error values case $\lambda_n = \dfrac{3n+3}{5n+5}$, $\varepsilon_n = \dfrac{n}{2n+1}$, $\delta_n = 0.9 - \dfrac{1}{(n+1)^2}$.

5. Conclusions

We study fixed point theory on E be a nonempty closed and convex subset of a complete CAT(0) space K, and using T be a firmly nonspreading mapping and S be a quasi-nonexpansive mapping having demiclosed principle such that $F(T) \cap F(S) \neq \emptyset$ which enough to approximation a common fixed point.

Acknowledgments. The author would like to thank the support of Rambhai Barni Rajabhat University.

References

[1] M. R. Bridson, A. Haefliger, *A Metric Spaces of Non-Positive Curvature,* Springe, (1999).

[2] D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry.* In: Graduate Studies in Math., Am. Math. Soc.,Providence, RI, **33** (2001).

[3] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, In: Progress in Mathematics, Birkhuser, Boston, **152** (1999).

[4] I. Bartolini, P. Ciaccia, M. Patella, *String matching with metric trees using an approximate distance,* In: SPIRE Lecture Notes in Computer Science, Springer, Berlin, **2476** (1999), 271–283.

[5] C. Semple, *Phylogenetics*, Oxford Lecture Series in Mathematics and Its Application. Oxford University Press, Oxford, (2003).

[6] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings,* Dekker, New York (1984).

[7] G. Marino, B. Scardamaglia, E. Karapinar, *Strong convergence theorem for strict pseudo-contractions in Hilbert spaces,* J. Inequal. Appl., **134** (2016), 2016.

[8] E. Karapinar, H. Salahifard, S. M. Vaezpour, *Demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces,* Journal of Applied Mathematics, **2014** (2014), pp. 10.

[9] A. Padcharoen, *Convergence theorems for total asymptotically quasi-nonexpansive nonself mappings in uniformly convex metric spaces,* Adv. Fixed Point Theory, **5** (2014), no.1, 45–55.

[10] N. Akkasriworn, D. Kitkuan, A. Padcharoen, *Convergence theorems for generalized I-asymptotically nonexpansive mappings in a Hadamard spaces,* Commun. Korean Math. Soc., **31** (2016), no.3, 483–493.

[11] K. S. Brown, *Buildings*, Springer, New York (1989).

[12] W.V. Petryshyn, T. E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings,* J. Math. Anal. Appl, **43** (1973), 459–497.

[13] M. K. Gosh, L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings,* J. Math. Anal. Appl. **207** (1997), 96103.

[14] B. Xu, M. A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl. **267** (2002), 444–453.

[15] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.

[16] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147–150.

[17] W. Phuengrattana, S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval,* J. Comput. Appl. Math., **235** (2011), no. 9, 3006–3014.

[18] D. Kitkuan, A. Padcharoen, *Strong convergence of a modified SP-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces,* J. Nonlinear Sci. Appl. **9** (2016), 2126–2135.

[19] S. Dhompongsa, K. A. Kirk, B. Sims, *Fixed points of uniformly Lipschitzian mappings,* NonlinearAnal.,Theory Methods Appl. **65** (2006), 762–772.

[20] W. A. Kirk, *Geodesic geometry and fixed point theory*

[21] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Am. Math. Soc. **60** (1976), 179–182.

[22] W. A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal., Theory Methods Appl. **68** (2008) 3689–3696.

[23] S. Dhompongsa, B. Panyanak, *On △ - convergence theorems in CAT(0) spaces,* Comput. Math. Appl. **56** (2008), 2572-2579.

[24] L. J. Lin, C. S. Chuang, Z. T. Yu, *Fixed Point Theorems and Convergent Theorems for Generalized Hybrid Mapping on CAT(0) Spaces*, Fixed Point Theory Appl. **49** (2011).

[25] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces,* Bulletin of the American Mathematical Society **74** (1968), 660–665.

[26] W. A. Kirk, B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal., Theory Methods Appl. **68** (2008), 3689–3696.

[27] S. Dhompongsa, W. A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces,* J. Nonlinear and Convex Anal. **8** (2007), 35–45.

[28] T. Prommai, A. Kaewkhao, W. Inthakon, *Common Fixed Point Theorems for Firmly Nonspreading Mappings and Quasi-Nonexpansive Mappings in CAT(0) Spaces,* Special Issue (2020): Annual Meeting in Mathematics 2019, 293–301.

Pim Sanboonsiri

Department of Mathematics, Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand

E-mail address: pim.m@rbru.ac.th