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SOME NEW OPIAL-TYPE INEQUALITIES ON FRACTIONAL CALCULUS OPERATORS

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Abstract. In this article, we establish some new Opial-type inequalities on fractional calculus involving the generalized Riemann-Liouville fractional integral, the Riemann-Liouville k-fractional integral, the (k, r)-fractional integral of the Riemann-type and the k-Hilfer fractional derivative operator.

1. Introduction

Fractional calculus refers to the study of integral and derivative operators of fractional order. This subject is as permeative as calculus itself and have been of great importance in the last few decades. Fractional calculus has been applied in various areas of engineering, science, finance, applied mathematics, bio engineering etc. Mathematical inequalities are important to the study of mathematics as well as many related fields and their uses are broad in scope. Fractional integral inequalities are helpful in establishing the uniqueness of solutions for fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These recommendations have led various researchers in the field of integral inequalities to inquire into certain extensions by involving fractional calculus operators (see [15], [18], [19], [24]).

In [16] Z. Opial introduced one of the most fundamental integral inequality in 1960 involving a function and its derivative. In literature it is known as Opial's inequality and is stated as:

Theorem 1.1. *Let $f \in C^1[0, a]$ satisfies $f(0) = f(a) = 0$ and $f(x) > 0$ on $(0, a)$. Then*

$$\int_0^a |f(x)f'(x)|dx \le \frac{a}{4} \int_0^a |f'(x)|^2 dx.$$

The constant $\frac{a}{4}$ is the best choice.

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Opial's inequality has received considerable attention of many researchers and many papers have appeared which provides with the simple proofs. It has various generalizations, extensions and discrete analogues. The study of Opial-type inequalities have grown into substantial field with many important applications.

In recent years, Opials inequality has been further generalized to many different aspects for instance, to integral inequalities involving higher order derivatives of the given function and to integral inequalities involving many functions of multiple variables. Agarwal, Pang and Alzer [1, 2, 4] study the Opial-type inequalities involving ordinary derivatives and their applications in differential and difference equations. For more detail, we refer the reader [9, 10, 21, 22]. The purpose of the present work is to produce the Opial-type inequalities by involving fractional order integral and derivative operators. We also assume that all integrals under consideration exist and that they are finite.

2. Preliminaries and Basic results

We start with the generalized L_p space given in [13] and is defined as:

Definition 2.1. *A space $L_{p,r}[a,b]$ is defined as a space of continuous real valued function f on $[a,b]$, such that*

$$\left(\int_a^b |f(t)|^p t^r dt\right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$ and $r \geq 0$. Note that $L_{p,0}[a,b] = L_p[a,b]$.

The definition of incomplete beta function presented in [7, p. 910] defined as follows:

Definition 2.2. *The incomplete beta function $B_x(a,b)$ is defined by the formula*

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,\ 0 \leq x \leq 1,\ \Re(a) > 0,\ \Re(b) > 0.$$

Clearly for $x = 1$ it becomes classical beta function.

Next, we give the well known definition of Riemann-Liouville fractional integrals (see [12]).

Definition 2.3. *Let $[a,b]$ be a finite interval on $\mathbb{R}$. The left and right sided Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined as:*

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\ t > a$$

and

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}f(\tau)d\tau,\ t < b,$$

respectively. Here Γ represents usual Gamma function defined by

$$\Gamma(t) = \int_0^{\infty} \tau^{t-1}e^{-\tau}d\tau,\ \Re(t) > 0.$$

Katugampola in [11] introduce the following definition of generalized Riemann-Liouville fractional integral.

Definition 2.4. *Let $\alpha > 0$, $a \geq 0$ and $r \neq -1$, a real number and let $f \in L_{1,r}[a,b]$. Then the generalized Riemann-Liouville fractional integral $I_a^{\alpha,r} f$ is defined by*

$$I_a^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{r+1} - t^{r+1})^{\alpha-1} t^r f(t)dt, \ x \in (a,b). \tag{2.1}$$

We note that if $r \to -1^+$ the integral operator (2.1) reduces to the famous Hadamard fractional integral i.e.,

$$I_a^{\alpha,-1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

Diaz *et al.* in [6] originate the definition of gamma k-function which is defined as:

Definition 2.5. *The Γ_k function is the generalization of the classical Γ function and is defined as follows:*

$$\Gamma_k(t) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{t}{k}-1}}{(t)_{n,k}}, \ k > 0, \ \Re(t) > 0,$$

where $(t)_{n,k} = t(t+k)(t+2k)\ldots,(t+(n-1)k)$, $n \geq 1$, is called Pochhammer k symbol. The integral representation is given by

$$\Gamma_k(t) = \int_0^\infty x^{t-1} e^{\frac{-x^k}{k}} dx, \ \Re(t) > 0. \tag{2.2}$$

Specially for $k = 1$, $\Gamma_1(t) = \Gamma(t)$.

The definition of Riemann-Liouville k-fractional integral presented in [14] is as follows:

Definition 2.6. *Let $f \in L_1[a,b]$, then the Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha} f$ of order $\alpha > 0$ and $k > 0$, is given by*

$$I_{a,k}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ t \in (a,b), \tag{2.3}$$

where Γ_k is defined by (2.2). Moreover, if we choose $k = 1$ the integral operator (2.3) represents the left sided Riemann-Liouville fractional integral.

The following definition of (k,r)-Riemann-Liouville fractional integral is given by Sarikaya *et al.* in [23].

Definition 2.7. *If $f \in L_{1,r}[a,b]$ and $k > 0$, then the Riemann-Liouville (k,r)-fractional integral $I_{k,a}^{\alpha,r} f$ of order $\alpha > 0$, is defined as:*

$$I_{k,a}^{\alpha,r} f(t) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{r+1} - \tau^{r+1})^{\frac{\alpha}{k}-1} \tau^r f(\tau) d\tau, \ t \in [a,b], \ r \in \mathbb{R} \setminus \{-1\}. \tag{2.4}$$

The next definition is presented in [8].

Definition 2.8. *Let $f \in L_1[a,b]$, $k > 0$, $f * K_{(1-\nu)(1-\mu)} \in AC^1[a,b]$. The k-fractional derivative operator ${}^k D^{\mu,\nu}_{a+} f$ of order $0 < \mu < 1$ and type $0 < \nu \leq 1$ with respect to $x \in [a,b]$ is defined by*

$$\left({}^k D^{\mu,\nu}_{a+} f\right)(x) := I^{\nu(1-\mu)}_{a+,k} \frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+,k} f(x)\right), \tag{2.5}$$

whenever the right hand side exists. The derivative (2.5) is usually called k-Hilfer fractional derivative. The more general integral representation of equation (2.5) is define as:
Let $f \in L^1[a,b], f * K_{(1-\nu)(n-\mu)} \in AC^n[a,b], n-1 < \mu < n, 0 < \nu \leq 1, n \in \mathbb{N}$ then the following equation holds true:

$$\left({}^k D^{\mu,\nu}_{a+} f\right)(x) = \left(I^{\nu(n-\mu)}_{a+,k} \frac{d^n}{dx^n}\left(I^{(1-\nu)(n-\mu)}_{a+,k} f(x)\right)\right). \tag{2.6}$$

Applying the properties of Riemann-Liouvile fractional integral the relation (2.6) can be rewritten in the form:

$$\begin{aligned}\left({}^k D^{\mu,\nu}_{a+} f\right)(x) &= \left(I^{\nu(n-\mu)}_{a+,k}\left(\left(D^{n-(1-\nu)(n-\mu)}_{a+,k} f\right)(x)\right)\right)\\ &= \frac{1}{k\Gamma_k\left(\nu\left(n-\mu\right)\right)}\int_a^x (x-y)^{\frac{\nu(n-\mu)}{k}-1}\left(\left(D^{\mu+\nu(n-\mu)}_{a+,k} f\right)(y)\right)dy.\end{aligned}$$

From the derivative (2.6), we obtain different classical fractional derivatives by setting

(i) $k = 1$, *we get Hilfer fractional derivative presented in* [8].
(ii) $k = 1, \nu = 0$, $D^{\mu,0}_{a+} f = D^{\mu}_{a+} f$, *we arrive at Riemann- Liouvile fractional derivative of order μ given in* [20].
(iii) $k = 1, \nu = 1, n = 1$ *it is a Caputo fractional derivative* $D^{\mu,1}_{a+} f = {}^C D^{\mu}_{a+} f$ *of order μ provided in* [12].

We say that a function $g : [a,b] \to \mathbb{R}$ belongs to the class $U(f,h)$ if it admits the representation

$$|g(t)| \leq \int_a^t h(t,\tau)\,|f(\tau)|\,d\tau,$$

where f is a continuous function and k a non-negative kernel, $f(t) > 0$ implies $g(t) > 0$ for every $t \in [a,b]$.

The following result proved in [17, p. 237] and is stated as:

Theorem 2.9. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U(v,h)$ and $(\int_a^x (h(x,t))^p dt)^{\frac{1}{p}} \leq \hat{M}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the inequality*

$$\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q dx \leq \frac{q}{\hat{M}^q}\phi\big(\hat{M}\big(\int_a^b |v(x)|^q dx\big)^{\frac{1}{q}} dx\big). \tag{2.7}$$

The reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ is concave.

In the following result, we have extension of the above result (see [5]).

Theorem 2.10. *Let the assumptions of Theorem 2.9 be fulfilled, then we have*

$$\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q dx \le \frac{q}{\hat{M}^q(b-a)}\int_a^b \phi\big((b-a)^{\frac{1}{q}}\hat{M}|v(x)|\big)dx.$$

The reverse of above inequality holds if $\phi(x^{\frac{1}{q}})$ is concave.

3. Opial-type inequalities for generalized Riemann-Liouville fractional integral operator

This section contains Opial-type inequalities for generalized Riemann-Liouvile fractional integral operator. Our first main result is presented in the following theorem.

Theorem 3.1. *Let $I_a^{\alpha,r}\mathfrak{v}_\mathfrak{i} \in U(\mathfrak{v}_\mathfrak{i}, h)$, $(i = 1, 2)$ and $\mathfrak{v}_2(t) > 0$ for every $t \in [a, b]$. Further let $\phi(u)$ be convex, non-negative and increasing for $u \ge 0$, $f(u)$ be convex function for $u \ge 0$ and $f(0) = 0$. If f is differentiable function and* $\max h(t,\tau) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1} - a^{r+1})^{\alpha-1}b^r$, *then the following inequality holds:*

$$\begin{aligned}
&\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1} - a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt \\
\le\ & f\left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1} - a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right). \qquad (3.1)
\end{aligned}$$

Proof. Since $f'(t)$ is a non-decreasing and from (2.1) for $I_a^{\alpha,r}\mathfrak{v}_1(t)$, we get

$$\begin{aligned}
&\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1} - a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt \\
\le\ & \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1} - a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) \\
&\times f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\frac{\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{r+1} - \tau^{r+1})^{\alpha-1}\tau^r|\mathfrak{v}_1(\tau)|d\tau}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right)\right)dt.
\end{aligned}$$

Now using Jensen's inequality, we get

$$\begin{aligned}
&\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
&\leq \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)\\
&\times f'\left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r\mathfrak{v}_2(\tau)\phi\left(\left|\frac{\mathfrak{v}_1(\tau)}{\mathfrak{v}_2(\tau)}\right|\right)d\tau\right)dt\\
&\leq \int_a^b \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)\\
&\times f'\left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (b^{r+1}-a^{r+1})^{\alpha-1}b^r\mathfrak{v}_2(\tau)\phi\left(\left|\frac{\mathfrak{v}_1(\tau)}{\mathfrak{v}_2(\tau)}\right|\right)d\tau\right)dt.
\end{aligned}$$

Applying Fundamental Theorem of calculus, we get inequality (3.1). □ □

Remark. *Particularly* $r \to -1^+$ *in Theorem 3.1, we get the results for the classical Hadamard fractional integral operator.*

Theorem 3.2. *Let the assumptions of Theorem 3.1 be true, then we have the inequality*

$$\begin{aligned}
&\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)\\
\times\quad & f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\quad & \frac{1}{b-a}\int_a^b f\left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r(b-a)\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right).
\end{aligned}$$

Proof. By rearranging the inequality (3.1), we get

$$\begin{aligned}
&\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\quad & f\left(\frac{(r+1)^{1-\alpha}}{(b-a)\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b (b-a)\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right).
\end{aligned}$$

Now by Jensen's inequality, we obtain

$$\begin{aligned} & \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_a^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_a^{\alpha,r}\mathfrak{v}_1(t)}{I_a^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt \\ \leq & \frac{1}{b-a}\int_a^b f\left(\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}(b^{r+1}-a^{r+1})^{\alpha-1}b^r(b-a)\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right), \end{aligned}$$

which completes the proof of the result. □ □

Remark. *In particular if we take $r=0$ in Theorem 3.2 we arrive at* [3, Corollary 3.2].

Remark. *If we take $r\to -1^+$ in Theorem 3.2, we get the results for the Hadamard fractional integral operator.*

Theorem 3.3. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$, the function $\phi(t^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let $I_a^{\alpha,r}\mathfrak{v}\in U(\mathfrak{v},h)$, $(\int_a^t h(t,\tau)^p d\tau)^{\frac{1}{p}} \leq \frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned} & \int_a^b |I_a^{\alpha,r}\mathfrak{v}(t)|^{1-q}\phi'(|I_a^{\alpha,r}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt \\ \leq & \frac{q}{\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q} \\ \times & \phi\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\ \times & \left.\left(\int_a^b|\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.2}$$

Proof. Since we have

$$I_a^{\alpha,r}\mathfrak{v}(t)=\frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r\mathfrak{v}(\tau)d\tau,$$

with

$$h(t,\tau)=\begin{cases}\frac{(r+1)^{1-\alpha}(t^{r+1}-\tau^{r+1})^{\alpha-1}\tau^r}{\Gamma(\alpha)}, & a\leq\tau\leq t;\\ 0, & t<\tau\leq b,\end{cases} \tag{3.3}$$

we can have

$$\left(\int_a^t h(t,\tau)^p d\tau\right)^{\frac{1}{p}} \leq \frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}.$$

Now using the inequality (2.7), with

$$\hat{M} = \frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)} \left(b^{p(r+1)\alpha-p+1} B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1, \frac{pr+1}{r+1}) \right)^{\frac{1}{p}},$$

we get inequality (3.2). □ □

Theorem 3.4. *Let the assumptions of Theorem 3.3 be satisfied, then we have the following inequality*

$$\begin{aligned} &\int_a^b |I_a^{\alpha,r}\mathfrak{v}(t)|^{1-q}\phi'(|I_a^{\alpha,r}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt \\ \leq \quad & \frac{q}{\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q (b-a)} \\ \times \quad & \int_a^b \phi\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\ \times \quad & \left.(b-a)^{\frac{1}{q}}|\mathfrak{v}(t)|\right) dt. \end{aligned}$$

Proof. By rearranging the inequality (3.2), we get

$$\begin{aligned} &\int_a^b |I_a^{\alpha,r}v(t)|^{1-q}\phi'(|I_a^{\alpha,r}v(t)|)|v(t)|^q dt \\ \leq \quad & \frac{q}{\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q} \\ \times \quad & \phi\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\ \times \quad & \left.\left(\frac{1}{b-a}\int_a^b (b-a)|v(t)|^q dt\right)^{\frac{1}{q}}\right). \end{aligned}$$

Since $\phi(t^{\frac{1}{q}})$ is convex. By Jensen's inequality, we have

$$\begin{aligned} &\int_a^b |I_a^{\alpha,r}v(t)|^{1-q}\phi'(|I_a^{\alpha,r}v(t)|)|v(t)|^q dt \\ \leq \quad & \frac{q}{\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q (b-a)} \\ \times \quad & \int_a^b \phi\left(\frac{(1+r)^{1-\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\left(b^{p(r+1)\alpha-p+1}B_{1-(\frac{a}{b})^{r+1}}((\alpha-1)p+1,\frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\ \times \quad & \left.(b-a)^{\frac{1}{q}}|v(t)|\right) dt. \end{aligned}$$

□

Remark. *In particular if we choose $r = 0$ in Theorem 3.4, we turns up at* [5, Theorem 3.1].

Remark. *If we choose $r \to -1^+$ in Theorem 3.4, we get the results for the Hadamard fractional integral operator.*

4. Opial-type inequalities for The Riemann-Liouville k-fractional integral operator

In this section, we develop the results for the Riemann-Liouville k-fractional integral operator.

Theorem 4.1. *Let $I^{\alpha}_{k,a}\mathfrak{v}_i \in U(v_i, h)$, $(i = 1, 2)$ and $\mathfrak{v}_2(t) > 0$ for every $t \in [a, b]$. Further let $\phi(u)$ be a non negative, convex and increasing function for $u \geq 0$, $f(u)$ be convex and $f(0) = 0$. If f is differentiable function and* max $h(t,\tau) = \frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}$, *then the following inequality holds*

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I^{\alpha}_{k,a}\mathfrak{v}_2(t)\phi\left(\left|\frac{I^{\alpha}_{k,a}\mathfrak{v}_1(t)}{I^{\alpha}_{k,a}\mathfrak{v}_2(t)}\right|\right)\right)dt \\
\leq\ & f\left(\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right). \qquad (4.1)
\end{aligned}$$

Proof. Since $f'(t)$ is a non-decreasing and from (2.3) for $I^{\alpha}_{k,a}\mathfrak{v}_1(t)$, we get

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I^{\alpha}_{k,a}\mathfrak{v}_2(t)\phi\left(\left|\frac{I^{\alpha}_{k,a}\mathfrak{v}_1(t)}{I^{\alpha}_{k,a}\mathfrak{v}_2(t)}\right|\right)\right)dt \\
\leq\ & \frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) \\
\times\ & f'\left(I^{\alpha}_{k,a}\mathfrak{v}_2(t)\phi\left(\frac{\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}|\mathfrak{v}_1(\tau)|d\tau}{I^{\alpha}_{k,a}\mathfrak{v}_2(t)}\right)\right)dt.
\end{aligned}$$

Applying Jensen's inequality, we have

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I^{\alpha}_{k,a}\mathfrak{v}_2(t)\phi\left(\left|\frac{I^{\alpha}_{k,a}\mathfrak{v}_1(t)}{I^{\alpha}_{k,a}\mathfrak{v}_2(t)}\right|\right)\right)dt \\
\leq\ & \frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) \\
\times\ & f'\left(\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}\mathfrak{v}_2(\tau)\phi\left(\left|\frac{\mathfrak{v}_1(\tau)}{\mathfrak{v}_2(\tau)}\right|\right)d\tau\right)dt,
\end{aligned}$$

which implies

$$\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_{k,a}^{\alpha}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha}\mathfrak{v}_2(t)}\right|\right)\right) dt$$
$$\leq \quad f\left(\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) dt\right).$$

This completes the proof of the result. □ □

Theorem 4.2. *With the same assumptions as in Theorem 4.1 we have*

$$\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_{k,a}^{\alpha}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha}\mathfrak{v}_2(t)}\right|\right)\right) dt$$
$$\leq \quad \frac{1}{b-a}\int_a^b f\left(\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}}\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)\right) dt.$$

Proof. By rearranging the inequality (4.1), we get

$$\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_{k,a}^{\alpha}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha}\mathfrak{v}_2(t)}\right|\right)\right) dt$$
$$\leq \quad f\left(\frac{1}{(b-a)k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b (b-a)\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) dt\right).$$

Now by applying Jensen's inequality, we get

$$\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}-1}\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right) f'\left(I_{k,a}^{\alpha}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha}\mathfrak{v}_2(t)}\right|\right)\right) dt$$
$$\leq \quad \frac{1}{b-a}\int_a^b f\left(\frac{1}{k\Gamma_k(\alpha)}(b-a)^{\frac{\alpha}{k}}\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)\right) dt.$$

This completes the proof of the result. □ □

Remark. *In particular if we take $k = 1$ in Theorem 4.2 we arrive at* [3, Corollary 3.2].

Theorem 4.3. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$, the function $\phi(t^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $I_{k,a}^{\alpha}\mathfrak{v} \in U(\mathfrak{v}, h)$, $(\int_a^t h(t,\tau)^p d\tau)^{\frac{1}{p}} \leq$*

$\frac{(b-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality hold.*

$$\int_a^b |I^{\alpha}_{k,a}\mathfrak{v}(t)|^{1-q}\phi'(|I^{\alpha}_{k,a}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt$$

$$\leq \frac{q(k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q}))^q}{\left((b-a)^{\frac{\alpha}{k}-\frac{1}{q}}\right)^q}\phi\left(\frac{(b-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}\left(\int_a^b |\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right). \quad (4.2)$$

The reverse of above inequality holds if $\phi(t^{\frac{1}{q}})$ *is concave.*

Proof. Since

$$I^{\alpha}_{k,a}\mathfrak{v}(t) = \int_a^t \frac{1}{k\Gamma_k(\alpha)}(t-\tau)^{\frac{\alpha}{k}-1}\mathfrak{v}(\tau)d\tau,$$

we can have

$$\left(\int_a^t h(t,\tau)^p d\tau\right)^{\frac{1}{p}} \leq \frac{(b-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}.$$

Now using the inequality (2.7), with

$$\hat{M} = \frac{(b-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}},$$

we get inequality (4.2). □ □

Theorem 4.4. *Let the assumptions of Theorem 4.3 be satisfied, then we have the following inequality.*

$$\int_a^b |I^{\alpha}_{k,a}v(t)|^{1-q}\phi'(|I^{\alpha}_{k,a}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt$$

$$\leq \frac{q(k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q}))^q}{\left((b-a)^{\frac{\alpha}{k}}\right)^q}\int_a^b \phi\left(\frac{(b-a)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}|\mathfrak{v}(t)|\right)dt.$$

Proof. By rearranging the inequality (4.2), we get

$$\int_a^b |I^{\alpha}_{k,a}\mathfrak{v}(t)|^{1-q}\phi'(|I^{\alpha}_{k,a}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt$$

$$\leq \frac{q(k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q}))^q}{\left((b-a)^{\frac{\alpha}{k}-\frac{1}{q}}\right)^q}\phi\left(\frac{(b-a)^{\frac{\alpha}{k}-\frac{1}{q}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}\left(\frac{1}{b-a}\int_a^b (b-a)|\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right).$$

Applying Jensen's inequality, we get

$$\int_a^b |I_{k,a}^{\alpha}\mathfrak{v}(t)|^{1-q}\phi'(|I_{k,a}^{\alpha}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt$$

$$\leq \frac{q(k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q}))^q}{\left((b-a)^{\frac{\alpha}{k}}\right)^q}\int_a^b \phi\left(\frac{(b-a)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)p^{\frac{1}{p}}(\frac{\alpha}{k}-\frac{1}{q})^{\frac{1}{p}}}|\mathfrak{v}(t)|\right)dt.$$

□ □

Remark. *In particular if we choose $k=1$ in Theorem 4.4, we turns up at* [5, Theorem 3.1].

5. Opial-type inequalities for $(k,r)-$ Riemann-Liouville fractional integral operator

This section consists of the results of Opial-type inequalities for (k,r) fractional integral operator of the Riemann-type.

Theorem 5.1. *Let $I_{k,a}^{\alpha,r}\mathfrak{v}_i \in U(\mathfrak{v}_i,h)$, $(i=1,2)$ and $\mathfrak{v}_2(t)>0$ for every $t\in[a,b]$. Further let $\phi(u)$ be non negative convex and increasing function for $u\geq 0$, $f(u)$ be convex function and $f(0)=0$. If f is differentiable function and* $\max h(t,\tau) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}b^r$, *then the following inequality*

$$\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)$$

$$\times\ f'\left(I_{k,a}^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha,r}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt$$

$$\leq f\left(\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right)$$

holds.

Proof. Similar to the proof of Theorem 3.1. □ □

Theorem 5.2. *With the same assumptions as in Theorem 5.1, we have*

$$\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}b^r\int_a^b \mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)f'\left(I_{k,a}^{\alpha,r}\mathfrak{v}_2(t)\phi\left(\left|\frac{I_{k,a}^{\alpha,r}\mathfrak{v}_1(t)}{I_{k,a}^{\alpha,r}\mathfrak{v}_2(t)}\right|\right)\right)dt$$

$$\leq \frac{1}{b-a}\int_a^b f\left(\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(b^{r+1}-a^{r+1})^{\frac{\alpha}{k}-1}b^r(b-a)\mathfrak{v}_2(t)\phi\left(\left|\frac{\mathfrak{v}_1(t)}{\mathfrak{v}_2(t)}\right|\right)dt\right).$$

Proof. Similar to the proof of Theorem 3.2. □ □

Remark. *In particular, if we take $k=1$ and $r=0$, in Theorem 5.2 we arrive at* [3, Corollary 3.2].

Remark. *If we take $r \to -1^{+}$ in Theorem 5.2, we get result for the k-Hadamard fractional integral operator.*

Remark. *If we take $k = 1$ and $r \to -1^{+}$ in Theorem 5.2, we get result for the Hadamard fractional integral operator.*

Theorem 5.3. *Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$, the function $\phi(t^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $I_{k,a}^{\alpha,r}\mathfrak{v} \in U(\mathfrak{v}, h)$, $(\int\limits_a^t h(t,\tau)^p d\tau)^{\frac{1}{p}} \leq \frac{(1+r)^{1-\frac{\alpha}{k}-\frac{1}{p}}}{k\Gamma_k(\alpha)} \left(b^{p(r+1)\frac{\alpha}{k}-p+1} B_{1-(\frac{a}{b})^{r+1}}((\frac{\alpha}{k}-1)p+1, \frac{pr+1}{r+1})\right)^{\frac{1}{p}}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality*

$$\begin{aligned}
&\int\limits_a^b |I_{k,a}^{\alpha,r}\mathfrak{v}(t)|^{1-q}\phi'(|I_{k,a}^{\alpha,r}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt \\
\leq\; & \frac{q}{\left(\frac{(1+r)^{1-\frac{\alpha}{k}-\frac{1}{p}}}{k\Gamma_k(\alpha)} \left(b^{p(r+1)\frac{\alpha}{k}-p+1} B_{1-(\frac{a}{b})^{r+1}}((\frac{\alpha}{k}-1)p+1, \frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q} \\
\times\; & \phi\left(\frac{(1+r)^{1-\frac{\alpha}{k}-\frac{1}{p}}}{k\Gamma_k(\alpha)} \left(b^{p(r+1)\frac{\alpha}{k}-p+1} B_{1-(\frac{a}{b})^{r+1}}((\frac{\alpha}{k}-1)p+1, \frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\
\times\; & \left.\left(\int\limits_a^b |\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right)
\end{aligned}$$

is true. The reverse of above inequality holds if $\phi(t^{\frac{1}{q}})$ is concave.

Proof. Similar to the proof of Theorem 3.3. □ □

Theorem 5.4. *Let the assumptions of Theorem 5.3 be satisfied, then we have the following inequality.*

$$\begin{aligned}
&\int\limits_a^b |I_{k,a}^{\alpha,r}\mathfrak{v}(t)|^{1-q}\phi'(|I_{k,a}^{\alpha,r}\mathfrak{v}(t)|)|\mathfrak{v}(t)|^q dt \\
\leq\; & \frac{q}{\left(\frac{(1+r)^{1-\frac{\alpha}{k}-\frac{1}{p}}}{k\Gamma_k(\alpha)} \left(b^{p(r+1)\frac{\alpha}{k}-p+1} B_{1-(\frac{a}{b})^{r+1}}((\frac{\alpha}{k}-1)p+1, \frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right)^q (b-a)} \\
\times\; & \int\limits_a^b \phi\left(\frac{(1+r)^{1-\frac{\alpha}{k}-\frac{1}{p}}}{k\Gamma_k(\alpha)} \left(b^{p(r+1)\frac{\alpha}{k}-p+1} B_{1-(\frac{a}{b})^{r+1}}((\frac{\alpha}{k}-1)p+1, \frac{pr+1}{r+1})\right)^{\frac{1}{p}}\right. \\
\times\; & \left.(b-a)^{\frac{1}{q}}|\mathfrak{v}(t)|\right) dt.
\end{aligned}$$

Proof. Similar to the proof of Theorem 3.4. □ □

Remark. *In particular, if we choose $k = 1$ and $r = 0$ in Theorem 5.4, we turns up at* [5, Theorem 3.1].

Remark. *If we take $r \to -1^{+}$ in Theorem 5.4, we get result for the k-Hadamard fractional integral operator.*

Remark. *If we take $k = 0$ and $r \to -1^{+}$ in Theorem 5.4, we get result for the Hadamard fractional integral operator.*

6. Opial-type inequalities for $k-$Hilfer fractional derivative

In this section, we construct the Opial-type inequalities for $k-$Hilfer fractional derivative operator presented in (2.5). The first result of this section is as follows.

Theorem 6.1. *Let ${}^{k}D^{\nu,\mu}_{a+}\mathfrak{v}_i \in U(\mathfrak{v}_i, h)\,(i = 1,2)$ and ${}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t) > 0$ for every $t \in [a,b]$. Further let $\phi(u)$ be non negative convex and increasing function, for $u \geq 0$, $f(u)$ be convex with $f(0) = 0$. If f is differentiable function and $\max h(t,\tau) = \frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}$, then the following inequality holds.*

$$\begin{aligned}
& \frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)}\right|\right)\\
\times\quad & f'\left({}^{k}D^{\nu,\mu}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\nu,\mu}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\nu,\mu}_{a+}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\quad & f\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\right.\\
\times\quad & \left.\int_a^b {}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)}\right|\right)dt\right). \qquad (6.1)
\end{aligned}$$

Proof. Since $f'(t)$ is a non-decreasing and from (2.5), we have

$$\begin{aligned}
& \frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)}\right|\right)\\
\times\quad & f'\left({}^{k}D^{\mu,\nu}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\mu,\nu}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\mu,\nu}_{a+}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\quad & \frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_1(t)}{{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_2(t)}\right|\right)\\
\times\quad & f'\left({}^{k}D^{\mu,\nu}_{a+}\mathfrak{v}_2(t)\phi\left(\frac{\frac{1}{k\Gamma_k(\nu(n-\mu))}\int_a^t (t-\tau)^{\frac{\nu(n-\mu)}{k}-1}|{}^{k}D^{\mu+\nu(n-\mu)}_{a+}\mathfrak{v}_1(\tau)|d\tau}{D^{\mu,\nu}_{a+}\mathfrak{v}_2(t)}\right)\right)dt.
\end{aligned}$$

By applying Jensen's inequality, we get

$$\begin{aligned}
&\frac{(b-a)^{\frac{\nu(n-\mu)}{k}-1}}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)\\
\times\ & f'\left({}^kD_{a+}^{\mu,\nu}\mathfrak{v}_2(t)\ \phi\left(\left|\frac{{}^kD_{a+}^{\mu,\nu}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu,\nu}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\ & \int_a^b \frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)\\
\times\ & f'\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}\int_a^t (b-a)^{\frac{\nu(n-\mu)}{k}-1}{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(\tau)\right.\\
\times\ & \left.\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(\tau)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(\tau)}\right|\right)d\tau\right)dt.
\end{aligned}$$

By using the fundamental theorem of calculus, we get inequality (6.1). □ □

Theorem 6.2. *Let the assumptions of Theorem 6.1 be fulfilled, then we have the inequality*

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)\\
\leq\ & \frac{1}{b-a}\int_a^b f\left(\frac{1}{k\Gamma_k(\nu(n-\mu))(b-a)^{\frac{\nu(n-\mu)}{k}-1}}(b-a)\right.\\
\times\ & \left.{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)dt\right).
\end{aligned}$$

Proof. By rearranging the inequality (6.1), we get

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)\\
\times\ & f'\left({}^kD_{a+}^{\nu,\mu}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\nu,\mu}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\nu,\mu}\mathfrak{v}_2(t)}\right|\right)\right)dt\\
\leq\ & f\left(\frac{1}{(b-a)k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\right.\\
\times\ & \left.\int_a^b (b-a)^k D_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)dt\right).
\end{aligned}$$

By applying Jensen's inequality, we have

$$\begin{aligned}&\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}\int_a^b {}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)\\ \leq\ &\frac{1}{b-a}\int_a^b f\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-1}(b-a)\right.\\ \times\ &\left.{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)\phi\left(\left|\frac{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_1(t)}{{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}_2(t)}\right|\right)dt\right).\end{aligned}$$

This completes the proof of the result. □ □

Theorem 6.3. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$, the function $\phi(t^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let ${}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)\in U(v,h)$, $(\int_a^t h(t,\tau)^p d\tau)^{\frac{1}{p}} \leq \frac{(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds.*

$$\begin{aligned}&\int_a^b |{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|^{1-q}\phi'(|{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|)|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt\\ \leq\ &\frac{q\left(p(\frac{\nu(n-\mu)}{k}-\frac{1}{q})\right)^{\frac{q}{p}}}{\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}\right)^q}\\ \times\ &\phi\left(\frac{(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}\left(\int_a^b |{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right).\end{aligned} \tag{6.2}$$

The reverse of the above inequality holds if $\phi(t^{\frac{1}{q}})$ is concave.

Proof. Since we have that

$${}^kD_{a+}^{\mu,\nu}\mathfrak{v}(t)=\frac{1}{k\Gamma_k(\nu(n-\mu))}\int_a^t (t-\tau)^{\frac{\nu(n-\mu)}{k}-1}\,{}^kD_{a+}^{\nu+\mu(n-\mu)}\mathfrak{v}(\tau)d\tau,$$

and

$$\left(\int_a^t h(t,\tau)^p d\tau\right)^{\frac{1}{p}} \leq \frac{(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}.$$

Now using the inequality (2.7), with

$$\hat{M}=\frac{(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}},$$

we get inequality (6.2). □ □

Theorem 6.4. *Let the assumptions of Theorem 6.3 be satisfied, then we have the following inequality*

$$\begin{aligned}
&\int_a^b |{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|^{1-q}\phi'(|{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|)|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt \\
\leq\ & \frac{q\left(p(\frac{\nu(n-\mu)}{k}-\frac{1}{q})\right)^{\frac{q}{p}}}{\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}}\right)^q} \\
\times\ & \int_a^b \phi\left(\frac{(b-a)^{\frac{\nu(n-\mu)}{k}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|\right)dt.
\end{aligned}$$

Proof. By rearranging the inequality (6.2), we get

$$\begin{aligned}
&\int_a^b |{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|^{1-q}\phi'(|{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|)|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt \\
\leq\ & \frac{q\left(p(\frac{\nu(n-\mu)}{k}-\frac{1}{q})\right)^{\frac{q}{p}}}{\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}\right)^q}\phi\left(\frac{(b-a)^{\frac{\nu(n-\mu)}{k}-\frac{1}{q}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}\right. \\
\times\ & \left.\left(\frac{1}{b-a}\int_a^b (b-a)|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Since $\phi(t^{\frac{1}{q}})$ is convex, therefore by Jensen's inequality, we get

$$\begin{aligned}
&\int_a^b |{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|^{1-q}\phi'(|{}^kD_{a+}^{\nu.\mu}\mathfrak{v}(t)|)|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|^q dt \\
\leq\ & \frac{q\left(p(\frac{\nu(n-\mu)}{k}-\frac{1}{q})\right)^{\frac{q}{p}}}{\left(\frac{1}{k\Gamma_k(\nu(n-\mu))}(b-a)^{\frac{\nu(n-\mu)}{k}}\right)^q} \\
\times & \int_a^b \phi\left(\frac{(b-a)^{\frac{\nu(n-\mu)}{k}}}{k\Gamma_k(\nu(n-\mu))p^{\frac{1}{p}}(\frac{\nu(n-\mu)}{k}-\frac{1}{q})^{\frac{1}{p}}}|{}^kD_{a+}^{\mu+\nu(n-\mu)}\mathfrak{v}(t)|\right)dt.
\end{aligned}$$

□ □

Remark. *In particular, if we take $k=1$ in Theorems [6.1, 6.2, 6.3, 6.4], we get the results for Hilfer fractional derivative.*

Remark. *In particular, we take for $k=1,\nu=0$, in Theorems [6.1, 6.2, 6.3, 6.4], we get the results for Riemann-Liouville fractional derivative.*

Remark. *In particular, if we take $k=1,\nu=1$ in Theorems [6.1, 6.2, 6.3, 6.4], we get the results for Caputo fractional derivative.*

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