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SOME FORMULAE OF GENOCCHI POLYNOMIALS OF HIGHER ORDER

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Abstract. Formulas of Genocchi polynomials of higher order are derived using Bernoulli, Euler and Genocchi polynomials of higher order as bases for the polynomial space. The main results are expressed in terms of the Bernoulli and Euler polynomials of higher order.

1. Introduction

Motivated by their importance and potential for applications in a variety of research fields, numerous polynomials and their extensions have recently been introduced and investigated. One of these polynomials is the Genocchi polynomials that have been extensively studied in many different context in such branches of mathematics, for instance, in elementary number theory, complex analytic number theory, calculus and many more. Some applications for Genocchi polynomials were derived in [4] to study a matrix formulation. In another paper [5] Araci dealt with the applications of umbral calculus on fermionic p-adic integral on $\mathbb{Z}_p$ and from these applications he derived new identities on Genocchi numbers and polynomials. On the other hand, the paper [7] presented new numerical method for solving fractional differential equations based on Genocchi polynomials operational matrix through collocation method and the properties of Genocchi polynomials were utilized to reduce the given problems to a system of algebraic equations.

Many studies in the literature provide relations of Genocchi numbers and polynomials to Bernoulli and Euler numbers and polynomials. Bernoulli, Euler and

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Genocchi polynomials are defined via exponential generating functions to be, respectively,

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \quad |t| < 2\pi \tag{1.1}$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad |t| < \pi \tag{1.2}$$

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, \quad |t| < \pi, \tag{1.3}$$

where, when $x = 0$, $B_n(0) = B_n$, $E_n(0) = E_n$ and $G_n(0) = G_n$, the Bernoulli, Euler and Genocchi numbers, respectively (see [1, 3, 4, 6, 8]). Bernoulli and Euler polynomials have been extensively studied by various researchers, one specific paper [8] dealt with some formulae of the product of two Bernoulli and Euler polynomials, which were then extended in [4] to Genocchi polynomials.

It is known that the Bernoulli and Euler Polynomials of order k are defined respectively by the generating functions

$$\sum_{n=0}^{\infty} B_n^k(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^k e^{xt}, \quad |t| < 2\pi, \tag{1.4}$$

$$\sum_{n=0}^{\infty} E_n^k(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^k e^{xt}, \quad |t| < \pi. \tag{1.5}$$

In the special case, $x = 0$, $B_n^k(0) = B_n^k$ and $E_n^k(0) = E_n^k$ are called the Bernoulli and Euler numbers of order k, respectively. From (1.4) and (1.5), we have $B_n^0(x) = x^n$ and $E_n^0(x) = x^n$. It is not difficult to show that (see [9, 10]), for Bernoulli polynomials,

$$\frac{d}{dx}B_n^k(x) = nB_{n-1}^k(x)$$

and

$$B_n^k(x+1) - B_n^k(x) = nB_{n-1}^{k-1}(x).$$

For Euler polynomials,

$$\frac{d}{dx}E_n^k(x) = nE_{n-1}^k(x)$$

and

$$E_n^k(x+1) + E_n^k(x) = 2E_{n-1}^{k-1}(x).$$

Let $\mathcal{P}_n = \{p(x) \in \mathbb{Q}[x] : \deg p(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Using the basis property of higher order Bernoulli polynomials

$$\left\{B_0^k(x), B_1^k(x), \ldots, B_n^k(x)\right\} \tag{1.6}$$

for the space $\mathcal{P}_n$, some interesting identities of higher-order Bernoulli polynomials were derived in [9]. Similarly, using the basis property of higher order Euler polynomials

$$\left\{E_0^k(x), E_1^k(x), \ldots, E_n^k(x)\right\} \tag{1.7}$$

for $\mathcal{P}_n$, some intersting identities for the higher order Euler polynomials were derived in [10].

The Genocchi polynomials of higher order are defined by the relation (see [3]),

$$\sum_{n=0}^{\infty} G_n^k(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}, \quad k \in \mathbb{Z}^+, \quad |t| < \pi. \tag{1.8}$$

Note that when $k = 1$, (1.8) reduces to (1.3), hence $G_n^1(x) = G_n(x)$.

The results in [8] were extended to Genocchi polynomials in [4] while in [3] theorems on Genocchi polynomials of higher order arising from the Genocchi basis

$$\left\{G_k^k(x), G_{k+1}^k(x), \ldots, G_{n+k-1}^k(x), G_{n+k}^k(x)\right\}, \tag{1.9}$$

were proved.

In this paper, we use the method of D.S. Kim and T. Kim [9, 10] to establish new formulas of Genocchi polynomials of higher order parallel to those in [4]. Derived results are expressed in terms of Bernoulli, Euler and Genocchi polynomials of higher order.

2. Preliminary Results

In this section, some identities of Genocchi polynomials of higher order are obtained. These properties are parallel to those of Bernoulli and Euler polynomials of higher order.

When $x = 0$, (1.8) becomes

$$\sum_{n=0}^{\infty} G_n^k \frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k, \tag{2.1}$$

where G_n^k are called higher order Genocchi numbers. When $k = 1$, (2.1) reduces to (1.3).

Now, differentiating both sides of (1.8) gives

$$\sum_{n=0}^{\infty} \frac{d}{dx} G_n^k(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k te^{xt} = \sum_{n=0}^{\infty} G_n^k \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} nG_{n-1}^k(x)\frac{t^n}{n!}.$$

Hence,

$$dG_n^k(x) = nG_{n-1}^k(x)dx. \tag{2.2}$$

By integrating both sides of (2.2) we obtain

$$\int G_n^k(x)dx = \frac{G_{n+1}^k(x)}{n+1}.$$

Consequently,

$$\int_a^b G_n^k(x)dx = \frac{G_{n+1}^k(b) - G_{n+1}^k(a)}{n+1}. \tag{2.3}$$

The following lemma is a known result. It contains an expression of $G_n^k(x)$ as polynomial in x with Genocchi numbers of higher order as the coefficients.

Lemma 2.1. *The Genocchi polynomials of higher order satisfy the relation*

$$G_n^k(x) = \sum_{m=0}^{n} \binom{n}{m} G_n^k x^{n-m}. \tag{2.4}$$

Lemma 2.2. *The following relation holds:*

$$G_n^k(x) = (-1)^{n+k} G_n^k(k-x). \tag{2.5}$$

In particular, when $x = 1$,

$$G_n^k(1) = (-1)^{n+k} G_n^k(k-1). \tag{2.6}$$

Proof. By replacing x with $k-x$, equation (1.8) gives

$$\sum_{n=0}^{\infty} G_n^k(k-x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{kt}e^{-xt}.$$

Replacing t with $-t$ yields

$$\sum_{n=0}^{\infty}(-1)^n G_n^k(k-x)\frac{t^n}{n!} = (-1)^k\left(\frac{2t}{e^t+1}\right)^k e^{xt} = (-1)^k\sum_{n=0}^{\infty} G_n^k(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (2.5). $\square$

Let us introduce the polynomial

$$p(x) = \sum_{l=k}^{n+k} G_l^k(x)x^{n+k-l}, \quad n \in \mathbb{N}. \tag{2.7}$$

Taking the first derivative of (2.7),

$$p'(x) = \sum_{l=k}^{n+k}\{G_l^k(x)(n+k-l)x^{n+k-l-1} + lG_{l-1}^k(x)x^{n+k-l}\}.$$

Since $G_{k-1}^k(x) = 0$,

$$p'(x) = (n+k+1)\sum_{l=k+1}^{n+k} G_{l-1}^k(x)x^{n+k-l}.$$

For the second derivative, one can easily verify that

$$p''(x) = (n+k+1)(n+k)\sum_{l=k+2}^{n+k} G_{l-2}^k(x)x^{n+k-l}.$$

Continuing this process yields

$$p^{(j)}(x) = (n+k+1)(n+k)\dots(n+k+2-j)\sum_{l=k+j}^{n+k} G_{l-j}^k(x)x^{n+k-l}.$$

The following lemma states formally this result.

Lemma 2.3. *The jth derivative of the polynomial $p(x)$ in (2.7) is given by*

$$p^{(j)}(x) = \frac{(n+k+1)!}{(n+k+1-j)!}\sum_{l=k+j}^{n+k} G_{l-j}^k(x)x^{n+k-l}.$$

3. Main Results

Rewrite (1.8) into the following:

$$\sum_{n=k}^{\infty} G_n^k(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^k e^{xt}t^k = \sum_{n=0}^{\infty} E_n^k(x)\frac{t^{n+k}}{n!} = \sum_{n=k}^{\infty} \frac{n!E_{n-k}^k(x)}{(n-k)!}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$G_n^k(x) = \frac{n!}{(n-k)!}E_{n-k}^k(x). \tag{3.1}$$

Consequently, this gives

$$G_{n+k}^k(x) = \frac{(n+k)!}{((n+k)-k)!}E_{(n+k)-k}^k(x) = (n+k)_k E_n^k(x),$$

where $(n+k)_k = (n+k)(n+k-1)\dots(n+1)$ is called the falling factorial of $n+k$ of degree k. Note that, when $x=0$,

$$\frac{G_{n+k}^k(0)}{(n+k)_k} = E_n^k, \tag{3.2}$$

and when $x = k-1$,

$$\frac{G_{n+k}^k(k-1)}{(n+k)_k} = E_n^k(k-1). \tag{3.3}$$

Theorem 3.1. *The following identity holds:*

$$\sum_{l=k}^{n+k} G_l^k(x)x^{n+k-l} = \sum_{j=0}^{n} \frac{\binom{n+k+1}{j}}{n+k-j+2}\left(\sum_{l=k+j-1}^{n+k}(-1)^{k+l-j+1}G_{l-j+1}^k(k-1) - G_{n+k-j+1}^k\right)B_j^k(x),$$

where $B_j^k(x)$ denotes Bernoulli polynomial of order k.

Proof. On account of the properties of the higher order Bernoulli basis given in (1.6) for $\mathcal{P}_n$, then $p(x)$ in (2.7) can be written as

$$p(x) = \sum_{j=0}^{n} a_j B_j^k(x). \tag{3.4}$$

Taking the first and second derivatives of $p(x)$,

$$p'(x) = \sum_{j=1}^{n} a_j j B_{j-1}^k(x) = \sum_{j=1}^{n} a_j \frac{j!}{(j-1)!}B_{j-1}^k(x),$$
$$p''(x) = \sum_{j=2}^{n} a_j j(j-1) B_{j-2}^k(x) = \sum_{j=2}^{n} a_j \frac{j!}{(j-2)!}B_{j-2}^k(x).$$

Continuing in this manner yields

$$\begin{aligned} p^{(n-1)}(x) &= \sum_{j=n-1}^{n} a_j \frac{j!}{(j-n+1)!}B_{j-n+1}^k(x) \\ &= a_{n-1}(n-1)!B_0^k(x) + a_n n! B_1^k(x). \end{aligned}$$

Replacing n with j gives

$$p^{(j-1)}(x) = a_{j-1}(j-1)!B_0^k(x) + a_j j!B_1^k(x).$$

Hence, with $B_0^k(x) = 1$ and $B_1^k(x) = x - \frac{k}{2}$ (see [2]),

$$p^{(j-1)}(1) - p^{(j-1)}(0) = a_j j!\left(1 - \frac{k}{2}\right) - a_j j!\left(0 - \frac{k}{2}\right) = a_j j!.$$

Thus,

$$a_j = \frac{1}{j!}[p^{(j-1)}(1) - p^{(j-1)}(0)]. \tag{3.5}$$

Using Lemma 2.3 and (2.6),

$$a_j = \frac{\binom{n+k+1}{j}}{n+k-j+2}\left(\sum_{l=k+j-1}^{n+k}(-1)^{k+l-j+1}G_{l-j+1}^k(k-1) - G_{n+k-j+1}^k\right).$$

Thus, (3.4) becomes

$$p(x) = \sum_{j=0}^{n}\frac{\binom{n+k+1}{j}}{n+k-j+2}\left(\sum_{l=k+j-1}^{n+k}(-1)^{k+l-j+1}G_{l-j+1}^k(k-1) - G_{n+k-j+1}^k\right)B_j^k(x),$$

which is exactly the desired identity. □

Corollary 3.2. *The following equality holds:*

$$\sum_{l=k}^{n+k} G_l^k(x)x^{n+k-l} = \sum_{j=0}^{n}\frac{\binom{n+k+1}{j}}{n+k-j+2}\Big(\sum_{l=k+j-1}^{n+k}\big[(-1)^{k+l-j+1}(l-j+1)_k E_{l-j+1-k}^k(k-1) \\ -(n+k-j+1)_k E_{n-j+1}^k\big]\Big)B_j^k(x).$$

Proof. Applying (3.2) and (3.3) to Theorem 3.1 immediately proves the corollary. □

The identity in the next theorem is obtained using the set (1.7) of Euler polynomials of higher order as a basis for the space of polynomials $\mathcal{P}_n$.

Theorem 3.3. *The following identity holds:*

$$\sum_{l=k}^{n+k} G_l^k(x)x^{n+k-l} = \sum_{j=0}^{n}\frac{1}{2}\binom{n+k+1}{j}\left(\sum_{l=k+j}^{n+k}(-1)^{k+l-j}G_{l-j}^k(k-1) + G_{n+k-j}^k\right)E_j^k(x),$$

where $E_j^k(x)$ denotes Euler polynomial of order k.

Proof. Express the polynomial $p(x)$ as a linear combination of Euler polynomials of higher order (1.7),

$$p(x) = \sum_{j=0}^{n} b_j E_j^k(x). \tag{3.6}$$

Taking the first and second derivatives,

$$p'(x) = \sum_{j=1}^{n} b_j j E_{j-1}^k(x) = \sum_{j=1}^{n} b_j \frac{j!}{(j-1)!} E_{j-1}^k(x)$$
$$p''(x) = \sum_{j=2}^{n} b_j j(j-1) E_{j-2}^k(x) = \sum_{j=2}^{n} b_j \frac{j!}{(j-2)!} E_{j-2}^k(x).$$

Continue in this manner to obtain,

$$p^{(n)}(x) = \sum_{j=n}^{n} b_j \frac{j!}{(j-n)!} E_{j-n}^k(x) = b_n n! E_0^k(x).$$

Replacing n with j gives

$$p^{(j)}(x) = b_j j! E_0^k(x).$$

Hence, with $E_0^k(x) = 1$,

$$p^{(j)}(1) + p^{(j)}(0) = b_j j![E_0^k(1) + E_0^k(0)] = b_j j![1+1] = 2b_j j!.$$

Thus,

$$b_j = \frac{1}{2j!}[p^{(j)}(1) + p^{(j)}(0)]. \tag{3.7}$$

Using Lemma 2.3 and (2.6),

$$b_j = \frac{1}{2}\binom{n+k+1}{j}\left(\sum_{l=k+j}^{n+k} (-1)^{k+l-j} G_{l-j}^k(k-1) + G_{n+k-j}^k\right).$$

Thus, (3.6) becomes

$$p(x) = \sum_{j=0}^{n} \frac{1}{2}\binom{n+k+1}{j}\left(\sum_{l=k+j}^{n+k} (-1)^{k+l-j} G_{l-j}^k(k-1) + G_{n+k-j}^k\right) E_j^k(x),$$

which is the desired identity. □

Using (3.2) and (3.3), the next corollary immediately follows from Theorem 3.3.

Corollary 3.4. *The following equality holds:*

$$\sum_{l=k}^{n+k} G_l^k(x) x^{n+k-l} = \sum_{j=0}^{n} \frac{1}{2}\binom{n+k+1}{j}\Big(\sum_{l=k+j-1}^{n+k} \big[(-1)^{k+l-j+1}(l-j)_k E_{l-j-k}^k(k-1)$$
$$+(n+k-j)_k E_{n-j}^k\big]\Big) E_j^k(x).$$

Let us now consider the polynomial,

$$p(x) = \sum_{l=k}^{n+k} \frac{1}{l!(n+k-l)!} G_l^k(x) x^{n+k-l}. \tag{3.8}$$

Taking the derivatives,

$$\begin{aligned} p'(x) &= \sum_{l=k}^{n+k} \frac{1}{l!(n+k-l)!}\left[G_l^k(x)(n+k-l)x^{n+k-l-1} + lG_{l-1}^k(x)x^{n+k-l}\right] \\ &= 2\sum_{l=k+1}^{n+k} \frac{1}{(l-1)!(n+k-l)!}G_{l-1}^k(x)x^{n+k-l}. \end{aligned}$$

$$p''(x) = 2^2\sum_{l=k+2}^{n+k} \frac{1}{(l-2)!(n+k-l)!}G_{l-2}^k(x)x^{n+k-l}.$$

By induction, the following lemma is proved.

Lemma 3.5. *The jth derivative of polynomial $p(x)$ in* (3.8) *is given by*

$$p^{(j)}(x) = 2^j\sum_{l=k+j}^{n+k} \frac{1}{(l-j)!(n+k-l)!}G_{l-j}^k(x)x^{n+k-l}. \tag{3.9}$$

Theorem 3.6. *The following relation holds:*

$$\begin{aligned} \sum_{l=k}^{n+k} \frac{1}{l!(n+k-l)!}G_l^k(x)x^{n+k-l} = \sum_{j=k}^{n+k} \frac{2^{j-k-1}}{j!}\Bigg(&\sum_{l=j}^{n+k} \frac{(-1)^{l-j}G_{l+k-j}^k(k-1)}{(l+k-j)!(n+k-l)!} \\ &+ \frac{G_{n+2k-j}^k}{(n+2k-j)!}\Bigg)G_j^k(x). \end{aligned}$$

Proof. Using the set of Genocchi polynomials of higher order given in (1.9) as a basis for the space of polynomials $\mathcal{P}_n$,

$$p(x) = \sum_{l=k}^{n+k} c_l G_l^k(x). \tag{3.10}$$

The nth derivative is given by

$$p^{(n)}(x) = \sum_{l=k+n}^{n+k} c_l \frac{l!}{(l-n)!}G_{l-n}^k(x) = c_{n+k}\frac{(n+k)!}{k!}G_k^k(x).$$

From (3.1),

$$G_k^k(x) = k!E_0^k(x) \text{ and } G_{k+1}^k(x) = (k+1)!E_1^k(x).$$

Thus,

$$p^{(n)}(x) = c_{n+k}\frac{(n+k)!}{k!}k!E_0^k(x) = c_{n+k}(n+k)!E_0^k(x).$$

Replacing n with $j-k$ yields

$$p^{(j-k)}(x) = c_j j!E_0^k(x).$$

Then, we have

$$\begin{aligned} p^{(j-k)}(0) + p^{(j-k)}(1) &= c_j j![E_0^k(0) + E_0^k(1)] \\ &= c_j j![1+1] = 2c_j j! \end{aligned}$$

Thus,

$$c_j = \frac{1}{2j!}\left(2^{j-k}\sum_{l=j}^{n+k}\frac{1}{(l+k-j)!(n+k-l)!}(-1)^{l+2k-j}G^k_{l+k-j}(k-1)\right.$$
$$\left.+2^{j-k}\frac{1}{(n+2k-j)!}G^k_{n+2k-j}\right).$$

Substituting this to (3.10) yields

$$p(x) = \sum_{j=k}^{n+k}\frac{2^{j-k-1}}{j!}\left(\sum_{l=j}^{n+k}\frac{(-1)^{l-j}G^k_{l+k-j}(k-1)}{(l+k-j)!(n+k-l)!}+\frac{G^k_{n+2k-j}}{(n+2k-j)!}\right)G^k_j(x).$$

□

Using the set of Bernoulli polynomials of higher order given in (1.6) as a basis for the space of polynomials $\mathcal{P}_n$, the next theorem is established.

Theorem 3.7. *The following relation holds:*

$$\sum_{l=k}^{n+k}\frac{G^k_l(x)}{l!(n+k-l)!}x^{n+k-l} = \sum_{j=k}^{n+k}\frac{2^{j-1}}{j!}\left(\sum_{l=k+j-1}^{n+k}\frac{(-1)^{l-j+1+k}G^k_{l-j+1}(k-1)B^k_j(x)}{(l-j+1)!(n+k-l)!}\right.$$
$$\left.-\frac{G^k_{n+k-j+1}B^k_j(x)}{(n+k-j+1)!}\right).$$

Proof. Write

$$p(x) = \sum_{l=k}^{n+k}a_lB^k_l(x). \tag{3.11}$$

Now, using (3.5) and Lemma 3.5,

$$a_j = \frac{2^{j-1}}{j!}\left(\sum_{l=k+j-1}^{n+k}\frac{(-1)^{l-j+1+k}G^k_{l-j+1}(k-1)}{(l-j+1)!(n+k-l)!}-\frac{G^k_{n+k-j+1}}{(n+k-j+1)!(n+k-(n+k))!}\right).$$

Hence, (3.11) can be written as

$$p(x) = \sum_{j=k}^{n+k}\frac{2^{j-1}}{j!}\left(\sum_{l=k+j-1}^{n+k}\frac{(-1)^{l-j+1+k}G^k_{l-j+1}(k-1)}{(l-j+1)!(n+k-l)!}-\frac{G^k_{n+k-j+1}}{(n+k-j+1)!}\right)B^k_j(x),$$

which is exactly the identity in the theorem. □

Corollary 3.8. *The following equality holds:*

$$\sum_{l=k}^{n+k}\frac{G^k_l(x)}{l!(n+k-l)!}x^{n+k-l} = \sum_{j=k}^{n+k}\frac{2^{j-1}}{j!}\Big(\sum_{l=k+j-1}^{n+k}\frac{(-1)^{l-j+1+k}(l-j+1)_kE^k_{l-j+1-k}(k-1)B^k_j(x)}{(l-j+1)!(n+k-l)!}$$
$$-\frac{(n+k-j+1)_kE^k_{n-j+1}B^k_j(x)}{(n+k-j+1)!}\Big).$$

Using the set of Euler polynomials of higher order given in (1.7) as a basis for the space of polynomials $\mathcal{P}_n$, the next theorem is proved.

Theorem 3.9. *The following relation holds:*

$$\sum_{l=k}^{n+k} \frac{G_l^k(x)}{l!(n+k-l)!} x^{n+k-l} = \sum_{j=k}^{n+k} \frac{2^{j-1}}{j!} \Bigg(\sum_{l=k+j}^{n+k} \frac{(-1)^{l-j+k} G_{l-j}^k(k-1)}{(l-j)!(n+k-l)!} - \frac{G_{n+k-j}^k}{(n+k-j)!} \Bigg) E_j^k(x).$$

Proof. Write

$$p(x) = \sum_{l=k}^{n+k} b_l E_l^k(x). \tag{3.12}$$

Now, using (3.7) and Lemma 3.5,

$$b_j = \frac{1}{2j!} \left(2^j \sum_{l=k+j}^{n+k} \frac{G_{l-j}^k(1)}{(l-j)!(n+k-l)!} + 2^j \frac{G_{n+k-j}^k}{(n+k-j)!} \right).$$

Hence, (3.12) can be written as

$$p(x) = \sum_{j=k}^{n+k} \frac{2^{j-1}}{j!} \left(\sum_{l=k+j}^{n+k} \frac{(-1)^{l-j+k} G_{l-j}^k(k-1)}{(l-j)!(n+k-l)!} - \frac{G_{n+k-j}^k}{(n+k-j)!} \right) E_j^k(x),$$

which is the identity in the theorem. □

An immediate consequence of Theorem 3.9 is

Corollary 3.10. *The following equality holds:*

$$\sum_{l=k}^{n+k} \frac{G_l^k(x)}{l!(n+k-l)!} x^{n+k-l} = \sum_{j=k}^{n+k} \frac{2^{j-1}}{j!} \Bigg(\sum_{l=k+j-1}^{n+k} \frac{(-1)^{l-j+1+k} (l-j)_k E_{l-j-k}^k(k-1) E_j^k(x)}{(l-j)!(n+k-l)!} - \frac{(n+k-j)_k E_{n-j}^k E_j^k(x)}{(n+k-j)!} \Bigg).$$

4. Conclusion and Recommendation

In this paper, the Bernoulli, Euler and Genocchi higher order bases were used to derive some identities of Genocchi polynomials of higher order. Also, the results were presented in terms of Bernoulli and Euler polynomials. For further studies, the following are recommended by the authors: (1) To obtain identities of Genocchi Polynomials of complex order and (2) To obtain similar results for Apostol-Genocchi polynomials and higher order Apostol-Genocchi Polynomials.

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