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AN EFFICIENT METHOD FOR SOLVING NEW GENERAL MIXED VARIATIONAL INEQUALITIES

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Abstract. The mixed variational inequality containing a nonlinear term ϕ is a generalization of classical variational inequality. The authors have introduced a new class of inequalities namely the class of general mixed variational inequalities. By using the resolvent operator concept, the equivalence between general mixed variational inequalities and resolvent operator techniques has been established. From this equivalence formulation, a new self-adaptive method for solving general mixed variational inequalities is suggested. Furthermore, the convergence of proposed method is studied. Results of this study can be analyzed as extension of the previous results for mixed variational inequalities.

1. Introduction

Variational principles played an important role in the advancements of the various branches of pure and applied sciences such as general theory of relativity, gauge field theory in modern particle physics and soliton theory. Variational principles analyzed as a new field of mathematical and engineering sciences from a few decades, which may be used to interpret basic principles of mathematical sciences in a more elegant way. The origin of variational principles was started from Newton, Fermat, Leibniz, Bernoulli and Lagrange, see [1, 2, 6, 10, 13, 15, 26]. These principles have been facilitated by theory of variational inequalities.

Variational inequalities achieved great importance in 1971, when Biocchi and Capelo [1] proved that fluid through a porous media may be studied in the framework of variational inequalities. Smith [25] treated the traffic assignment problem like an inequality problem. Dafermos [6], realized that the transportation problem formulated by Smith [25] is a variational inequality. Kikuchi and Oden [10] studied the contact problems in elasticity, via variational inequalities. These developments put a significant influence in the area of transport engineering [17], economics equilibrium and other disciplines of sciences.

The development of numerical methods in the theory of variational inequalities play a significant role in solving a given problem. By applying these methods, variational inequalities are converted into fixed point problems. This equivalent

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formulation is used frequently to suggest and analyze several iterative methods of fixed point theory for solving variational inequalities [11, 19, 20, 22, 24]. Furthermore, during study of existence of a solution of the variational inequality this technique is utilized in common.

Variational inequalities have been extended and modified in different directions see [16, 17, 18, 19, 20, 21, 22, 23]. A useful generalization of the variational inequality is known as the mixed variational inequality involving the nonlinear term, say ϕ . Due to presence of this nonlinear term ϕ in the mixed variational inequality, the projection technique and its variants can not be applied in the establishment of equivalence between mixed variational inequalities and a fixed point problem unless the nonlinear term is proper, convex and lower semi continuous function.

The important feature of resolvent technique is the resolvent step which involves the sub-differential of a proper, convex, and lower semi continuous function part only and other part facilitates the problem decomposition. This technique leads to develop efficient methods for solving the mixed variational inequalities using resolvent equations, see [3, 4, 9]. In the upcoming section some preliminary results are given.

2. Preliminary Results

Let H be a real Hilbert space, $\langle .,. \rangle$ and $\|.\|$ are notations of inner product and norm respectively. Let $T : H \to H$ be a nonlinear operator and $\phi : H \to R \cup \{+\infty\}$ be a function. Then there exists $u \in H$ such that

$$\langle Tu,\ v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall\ v \in H. \tag{2.1}$$

This is called a mixed variational inequality. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the framework of mixed variational inequality (2.1), see [4, 9].

If the function ϕ in (2.1) is proper, convex and semi-lower continuous, then it is equivalent to find $u \in H$ such that

$$0 \in Tu + \partial\phi(u), \tag{2.2}$$

where $\partial\phi(.)$ is the sub-differential of the function ϕ.

Problem (2.2) is known as the variational inclusion problem. It is also known as: the problem of finding zeros of $T + \partial\phi$. For numerical methods and other aspects of mixed variational inequality inclusion, see [4, 9, 12, 16]. It is noted that if ϕ is the indicator function of a close convex set K in H, that is

$$\phi(u) \equiv I_k(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.3}$$

then the mixed variational inequality (2.1) is equivalent to find $u \in K$ such that

$$\langle Tu,\ v-u\rangle \geq 0, \quad \forall\ v \in K, \tag{2.4}$$

the problem (2.4) is known as the variational inequality which was introduced and studied by Stampacchia [26]. It has been shown that a large class of un-related odd-order and nonsymetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the variational inequalities, see [3, 5, 7, 8, 17, 19].

For solving the mixed variational inequality, resolvent operator technique is used. This technique is helpful to establish equivalence between the variational inequality and resolvent operator to establish and formulate a fixed point problem. We now recall some basic concepts and results.

Definition 1. [4] *If A is a maximal monotone operator on H, then for constant $\rho > 0$, the resolvent operator associated with A is defined by*

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall\ u \in H.$$

It is known that a monotone operator is maximal, if and only if, its resolvent operator is defined everywhere. In addition, the resolvent operator is single-valued and non-expansive, that is

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|, \quad \forall\ u, v \in H.$$

Remark. *It is well known that the sub-differential $\partial\phi$ of a proper, convex and lower semi-continuous function $\phi : H \to H \cup \{+\infty\}$ is a maximal monotone, so*

$$J_\phi(u) = (I + \phi\partial\phi)^{-1}(u), \quad \forall\ u \in H.$$

The resolvent operator J_ϕ has the following characterization.

Lemma 2.1. [4] *For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, \quad v - u\rangle + \rho\phi(v) - \rho\phi(u) \geq 0, \qquad \forall\ v \in H, \tag{2.5}$$

if and only if

$$u = J_\phi z, \tag{2.6}$$
$$J_\phi = (I + \phi\partial\phi)^{-1},$$

where J_ϕ is the resolvent operator.

In this research, the authors propose a new class of general mixed variational inequalities, which is extension of the class of mixed variational inequalities. A self-adaptive technique involving step size is introduced and followed to solve mixed variational inequalities. The convergence of this technique is studied under some particular conditions. A numerical example is also given for implementation of these results.

3. MAIN RESULTS

Firstly, a new class of general mixed variational inequalities is introduced and stated in Theorem 3.1. The relevant theory is established to formulate an iterative scheme for solving the class of these inequalities. The convergence of this iterative scheme is also studied.

Theorem 3.1. *Let $(H, \langle .,. \rangle, \|.\|)$ be a real Hilbert space, T, $g : H \to H$ be nonlinear operators and the function $\phi : H \to R \cup \{+\infty\}$ be a nonlinear term. Then there exists $u \in H$ such that the following inequality holds:*

$$\langle \rho Tu + u - g(u),\ v - u\rangle + \phi(v) - \phi(u) \geq 0, \qquad \forall\ v \in H, \tag{3.1}$$

where, $\rho > 0$ is a constant.

Remark. *(i) For $g \equiv I$ and $\rho = 1$, where I is the identity operator, problem (3.1) reduces to (2.1).*
(ii) For $g \equiv I$ and $\rho = 1$, where I is the identity operator and $\phi = 0$, the problem reduces to (2.4).

This new general mixed variational inequality can be used to establish the fixed point problem by using resolvent operator technique. Iterative schemes are established by using this fixed point formulation for solving the mixed variational inequalities.

By using Lemma 2.1, it can be shown that general mixed variational inequality (2.1) is equivalent to the fixed point problem.

Lemma 3.2. *The function $u \in H$ is a solution of the new general mixed variational inequality (2.1) if and only if, $u \in H$ satisfies*

$$u = J_\phi[g(u) - \rho Tu], \tag{3.2}$$

where $J_\phi = (I + \rho\partial\phi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

We use this useful equivalent formulation to suggest and analyze a predictor-corrector method for solving the general mixed variational inequality (2.1). Let us define

$$w = J_\phi[u - \gamma Tu], \qquad \text{for } \gamma > 0, \tag{3.3}$$

$$u = J_\phi[g(w) - \rho Tw], \qquad \text{for } \rho > 0. \tag{3.4}$$

We now define the residue vector $R(u)$ as follows:

$$R(u) := u - J_\phi[g(u) - \rho Tu]. \tag{3.5}$$

From Lemma 3.2, it is clear that u is a solution of (3.1) if and only if $u \in H$ is the solution of the equation

$$R(u) := 0.$$

Related to the general mixed variational inequalities (3.1), we now consider the problem of solving the resolvent equations. Let $R_\phi = I - J_\phi$, where I is the identity operator and J_ϕ is the resolvent operator. For given operators $T,\ g : H \to H$, such that g^{-1} exists, consider the problem of finding $z \in H$ such that

$$\rho T J_\phi z + R_\phi z = 0, \tag{3.6}$$

which is known as the resolvent equation, introduced and studied by Noor [16]. It has been shown that the resolvent equations are flexible and provide a unified framework to develop several efficient and powerful numerical techniques.

Lemma 3.3. *By using Lemma 3.2, the general mixed variational inequality (3.1) has a solution $u \in H$ if and only if $z \in H$ satisfies the resolvent equation (3.6), provided*

$$u = J_\phi z \tag{3.7}$$

$$z = g(u) - \rho Tu, \tag{3.8}$$

where $\rho > 0$ is constant.

From Lemma 3.3, the new mixed variational inequality and the resolvent equation are equivalent. This alternate formulation is very important from numerical and approximation point of views and use to suggest and analyze a number of iterative algorithms for solving the mixed variational inequalities (3.1) and related optimization problems. This equivalent formulation has been used by Noor [16] to suggest and analyze several iterative methods for solving the variational inequalities and related optimization problems.

Using (3.5), (3.7) and (3.8) the resolvent equation (3.6) can be written in the form:

$$\begin{aligned} 0 &= u - J_\phi[g(u) - \rho Tu] - \rho Tu + \rho T J_\phi[g(u) - \rho Tu] \\ &= R(u) - \rho Tu + \rho T J_\phi[g(u) - \rho Tu]. \end{aligned} \tag{3.9}$$

We now define the relation

$$D(u) = R(u) - \rho Tu + \rho T J_\phi[g(u) - \rho Tu]. \tag{3.10}$$

It is clear that $u \in H$ is a solution of the mixed variational inequality if and only if $u \in H$ is a zero of the equation

$$D(u) = 0.$$

Using (3.3) and (3.10), we can obtain

$$w = J_\phi[u - \gamma D(u) - \gamma Tu]. \tag{3.11}$$

This technique will be used for establishing the iterative schemes for solving the mixed variational inequality problem (3.1).

The above modification in the result has motivated us to construct the following new self-adaptive iterative method for solving the general mixed variational inequality (3.1).

Algorithm 3.1.

Step 0: Given $\epsilon > 0,\ \gamma \in [1,\ 2),\ \mu \in (0,\ 1),\ \rho > 0,\ \delta_0,\ \delta \in (0,\ 1)$ and $u^0 \in H$, set $n = 0$.

Step 1: Set $\rho_n = \rho$. If $\|R(u_n)\| < \epsilon$, then stop; otherwise, find the smallest non-negative integer m_n, satisfying

$$\|\rho_n(T(u^n) - T(w^n))\| \leq \delta \left\|R(u^n)\right\|^2, \tag{3.12}$$

where

$$w^n = J_\phi[u^n - \gamma D(u^n) - \gamma T(u^n)], \tag{3.13}$$

Step 2: Compute

$$D(u^n) = R(u^n) - \rho T(u^n) + \rho T J_\phi[g(u^n) - \rho T(u^n)],$$

where

$$R(u^n) := u^n - J_\phi[g(u^n) - \rho T(u^n)].$$

Step 3: Get the next iterate

$$w^n = J_\phi[u^n - \gamma D(u^n) - \gamma T(u^n)], \tag{3.14}$$

$$u^{n+1} = J_\phi[g(w^n) - \rho T(w^n)], \tag{3.15}$$

then set $\rho = \frac{\rho_n}{\mu}$, else set $\rho = \rho_n$. Set $n = n + 1$, and go to Step 1.

This technique is closely related to projection residue and differs from the standard procedure.

We consider the convergence criteria of Algorithm 3.1 under some known conditions and this is the main motivation of this paper.

Theorem 3.4. *Let the operators $T, g : H \to H$ be both strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \alpha > \beta\sqrt{k(2-k)}\ , \quad k < 1,$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}$$

then the approximate solution u^n obtained from Algorithm 3.1 converges to a solution satisfying the mixed variational inequality (3.1).

Proof: Since u^* is a solution of mixed variational inequality (3.1), it follows from Lemma 3.3 that

$$w^* = J_\phi[u^* - \rho T u^*], \quad u^* = J_\phi[g(w^*) - \rho T w^*]. \tag{3.16}$$

Applying Algorithm 3.1, from (3.14), (3.16), we know that J_ϕ is nonexpansive

$$\begin{aligned}\left\|u^{n+1} - u^*\right\| &= \left\|J_\phi[g(w^n) - \rho T w^n] - J_\phi[g(w^*) - \rho T w^*]\right\| \\ &\leq \left\|g(w^n) - \rho T w^n - g(w^*) + \rho T w^*\right\|.\end{aligned} \tag{3.17}$$

Consider

$$\begin{aligned}&\left\|g(w^n) - \rho T w^n - g(w^*) + \rho T(w^*)\right\| \\ &= \left\|-[(w^n - w^*) - (g(w^n) - g(w^*))] + w^n - w^* - \rho(T(w^n) - T(w^*))\right\| \\ &\leq \left\|w^n - w^* - (g(w^n) - g(w^*))\right\| + \left\|w^n - w^* - \rho(T(w^n) - T(w^*))\right\|.\end{aligned} \tag{3.18}$$

Using the strongly monotonicity of T with constant α and Lipschitz continuity with constant β, we have

$$\begin{aligned}&\left\|w^n - w^* - \rho(T(w^n) - T(w^*))\right\|^2 \\ &\leq \langle w^n - w^*,\ w^n - w^*\rangle - 2\rho\,\langle w^n - w^*,\ T(w^n) - T(w^*)\rangle \\ &+ \langle T(w^n) - T(w^*),\ T(w^n) - T(w^*)\rangle \\ &\leq \left\|w^n - w^*\right\|^2 - 2\rho\alpha\left\|w^n - w^*\right\|^2 + \rho^2\beta^2\left\|w^n - w^*\right\|^2,\end{aligned}$$

which is equivalent to

$$\left\|w^n - w^* - \rho(T(w^n) - T(w^*))\right\| \leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\left\|w^n - w^*\right\|. \tag{3.19}$$

In a similar way, using the strongly monotonicity of g with $\delta > 0$ and Lipschitz continuity of g with $\sigma > 0$, we have

$$\left\|w^n - w^* - (g(w^n) - g(w^*))\right\| \leq \sqrt{1 - 2\sigma + \delta^2}\left\|w^n - w^*\right\|. \tag{3.20}$$

From (3.18), (3.19) and (3.20), we get

$$\left\|u^{n+1} - u^*\right\| \leq (\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2})\left\|w^n - w^*\right\|. \tag{3.21}$$

Let $t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}$ and $\theta_1 = \sqrt{1 - 2\sigma + \delta^2}$, equation (3.21) becomes

$$\left\|u^{n+1} - u^*\right\| \leq (\ t(\rho) + \theta_1)\left\|w^n - w^*\right\|. \tag{3.22}$$

From (3.3) and (3.14), we obtain

$$\begin{aligned}\left\|w^n - w^*\right\| &= \left\|J_\phi[u^n - \gamma d(u^n) - \gamma T u^n] - J_\phi[u^* - \rho T u^*]\right\| \\ &\leq \left\|[u^n - \gamma d(u^n) - \gamma T u] - [u^* - \rho T u^*]\right\| \\ &\leq \left\|u^n - u^* - \gamma d(u^n)\right\| + \gamma\left\|T u^n - T u^*\right\|,\end{aligned} \tag{3.23}$$

where, we use the definition of $d(u^n)$.

It follows that

$$\begin{aligned}\|u^n - u^* - \gamma d(u^n)\|^2 &\le \|u^n - u^*\|^2 - 2\gamma \langle u^n - u^*,\ d(u^n)\ \rangle \\ &\quad + \gamma^2 \|\ d(u^n)\|^2 \\ &\le \|u^n - u^*\|^2 . \end{aligned}$$

$$\|u^n - u^* - \gamma d(u^n)\| \le \|u^n - u^*\| . \tag{3.24}$$

Similarly,

$$\gamma \|T(u^n)] - T(u^*)]\| \le \gamma\beta \|u^n - u^*\| . \tag{3.25}$$

From (3.23), (3.24) and (3.25), we have

$$\|w^n - w^*\| \le (1 + \gamma\beta) \|u^n - u^*\| . \tag{3.26}$$

From (3.22) and (3.26), we obtain

$$\left\|u^{n+1} - u^*\right\| \le (\ t(\rho) + \theta_1)(1 + \gamma\beta) \|u^n - u^*\| . \tag{3.27}$$

Let $(\ t(\rho) + \theta_1)(1 + \gamma\beta) = \theta$, where $0 < \theta < 1$. Inequality (3.27) becomes

$$\left\|u^{n+1} - u^*\right\| \le \theta \|u^n - u^*\| . \tag{3.28}$$

Repeating this process from (3.28), we have

$$\left\|u^{n+1} - u^*\right\| \le \theta^n \left\|u^0 - u^*\right\| . \tag{3.29}$$

since $0 < \theta < 1$, $\sum_{n=0}^{\infty} \theta^n = \infty$, then for arbitrarily chosen initial points $u^0_,$, and u^{n+1} obtained from algorithm 3.1 converge strongly to u^* respectively.

4. **Numerical Results**

In this section, we present numerical result, to illustrate the implementation and efficiency of the proposed method. We take a numerical example to solve variational inequality. Here, we know that if ϕ is an indicator, then the general mixed variational inequality reduces to general variational inequality and have taken the following example to represent numerical results.

Example 4.1. This example is a general variational inequality with $g(u) = Au + q$ and $Tu = u$, where

$$A = \begin{bmatrix} 4 & -2 & 0 & \cdots & 0 & 0 \\ 1 & 4 & -2 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & -2 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

For this problem, we take the domain set

$K = \{u \in R^n / \ 0 \le u_i \le 1,\ \text{for}\ i = 1, 2, ...n\}$. Table 1 provides the results for Algorithm 3.1 with starting point $u^0 = -A^{-1}q$ for order $n = 100$ of the matrix. In all tests, we take, $\mu, \delta \in (0, 1)$, $\rho > 0$ and $\gamma \in [1, 2]$. The computation stops as soon as $\|R(u^n)\| \le 10^{-7}$. All codes are written in Matlab.

Table 1 provides the results for the new proposed methods (Algorithm 3.1). From Table 1, we see that the number of iterations vary with the change of parameters for solving general variational inequalities. By changing the parameters appropriately, the number of iterations can be reduced.

Table 1. (Numerical Results for Algorithm 3.1)

Parameters	$\rho = 2$, $\delta = 0.05$, $\mu = 0.7$	$\rho = 3$, $\delta = 0.04$, $\mu = 0.7$	$\rho = 4$, $\delta = 0.06$, $\mu = 0.6$
Iterations	8	10	7

It can be observed that in the new self-adaptive method the number of iterations vary with the change of parameters and by changing the parameters appropriately, the number of iterations can be reduced.

5. **Conclusion**

In this paper, we have introduced a new general mixed variational inequality. A new self-adaptive method has been suggested and analyzed for solving the general mixed variational inequalities. This technique has been used to modify the resolvent equations and solution. This technique is closely related to projection residue and differs from the standard procedure. Example has been given to illustrate the efficiency of the new method. We have observed that in the new method, the number of iterations varies with the change of parameters ρ, μ and δ. The convergence analysis is provided under mild conditions.

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