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FIXED POINT RESULTS FOR α-ψ-CONTRACTIVE MAPPINGS IN BIPOLAR METRIC SPACES

UTKU GÜRDAL [2], ALI MUTLU[1], AND KÜBRA ÖZKAN [1] *

Abstract. In this paper, we introduce the notion of α-ψ contractive type covariant and contravariant mappings in the bipolar metric spaces, which provides a framework to study distances between dissimilar objects. In addition this, we prove some fixed point theorems, which give existence and uniqueness of fixed point, for α-ψ contractive type covariant and contravariant mappings in complete bipolar metric spaces. On the other hand, we observe that some known fixed point theorems as Banach's and coupled are simple consequences of our obtained results. Once and for all, we state some examples to show usability of our main results.

1. Introduction

Recently, Samet et al. [24] introduced and studied new concepts, called contractive and α-admissible mapping. They proved various fixed point theorems for such mappings in complete metric spaces. Many researchers have focused to study such type mappings. They showed that fixed point results of such type mappings are applied to solve problems such as differential equations, nonlinear integral equations and the boundary value problem. Moreover, it was made many studies which extended the mentioned mappings to various metric spaces as cone, generalized, modular, e.g. [2, 3, 4, 6, 5, 7, 8, 9, 10, 11, 13, 14, 23, 24, 25].

In literature, there are many kind of metric spaces as partial, rectangular, cone, b-metric, etc. In 2016, Mutlu and Gürdal [15] added a new one to them and introduced bipolar metric spaces. Moreover, they proved some fixed point theorems as Banach's and Kannan's in such spaces. Afterward, Mutlu, Özkan and Gürdal extended the certain coupled fixed point theorems, previously introduced in metric spaces as standard, cone and modular, to bipolar metric spaces [5, 16, 17, 22]. And, they give some fixed theorems for multivalued mappings, and locally and weakly contractive mappings on this type metric spaces [19, 20]. Besides, Kishore et al. studied common fixed point theorems for Caristi type cyclic contraction in bipolar metric spaces [12].

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* Corresponding author.

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In this paper, we introduce the notion of α-ψ-contractive type mappings in the bipolar metric spaces. Also, we prove some fixed point theorems for such mappings in complete bipolar metric spaces. Moreover, we observe that some known fixed point theorems as Banach and Coupled are simple consequences of our obtained results. On the other hand, we state some examples to show usability of our main results.

2. Bipolar Metric Spaces

We begin by recalling some basic definitions and results will be used in sequel:

In this paper, $\mathbb{R}^+$ and $\mathbb{N}$ symbolise the set of all non-negative real numbers and the set of positive integers, respectively.

Definition. [15] Let $X, Y \neq \emptyset$ and $d : X \times Y \to \mathbb{R}^+$ be a function. d is called a bipolar metric on pair (X, Y), if the following properties are satisfied

(B0) if $d(x, y) = 0$, then $x = y$,

(B1) if $x = y$, then $d(x, y) = 0$,

(B2) if $x, y \in X \cap Y$, then $d(x, y) = d(y, x)$,

(B3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$,

for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$. Then the triple (X, Y, d) is called a bipolar metric space.

Definition. [15] Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets and given a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$.

(1) If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, then f is called a covariant map from (X_1, Y_1) to (X_2, Y_2) and denoted this with $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

(2) If $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, then f is called a contravariant map from (X_1, Y_1) to (X_2, Y_2) and denoted this $f : (X_1, Y_1) \nearrow\!\!\!\searrow (X_2, Y_2)$.

In particular, if d_1 and d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we also use the notations

$$f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2) \text{ and } f : (X_1, Y_1, d_1) \nearrow\!\!\!\searrow (X_2, Y_2, d_2).$$

Definition. [15] Let (X, Y, d) be a bipolar metric space.

(1) A point $u \in X \cup Y$ is called a left point if $u \in X$, a right point if $u \in Y$ and a central point if it is both left and right point.

(2) A sequence (x_n) on the set X is called a left sequence and a sequence (y_n) on Y is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

(3) A sequence (u_n) is called convergent to a point u, iff (u_n) is a left sequence, u is a right point and $\lim_{n\to\infty} d(u_n, u) = 0$; or (u_n) is a right sequence, u is a left point and $\lim_{n\to\infty} d(u, u_n) = 0$. A bisequence (x_n, y_n) on (X, Y, d) is a sequence on the set $X \times Y$. If the sequences (x_n) and (y_n) are convergent, then the bisequence (x_n, y_n) is called convergent, and if (x_n) and (y_n) converge to a common point, then (x_n, y_n) is called biconvergent.

(4) (x_n, y_n) is a Cauchy bisequence, if $\lim_{n,m\to\infty} d(x_n, y_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

(5) A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition. [15] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces.

(1) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$, $d_2(f(x_0), f(y)) < \varepsilon$ as $y \in Y_1$.

(2) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$, $d_2(f(x), f(y_0)) < \varepsilon$ as $x \in X_1$.

(3) A map f is called continuous, if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.

(4) A contravariant map $f : (X_1, Y_1, d_1) \nearrow\!\!\!\searrow (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, d_2)$

From the definition we can deduce that a covariant or a contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if

$$(u_n) \to v \text{ on } (X_1, Y_1, d_1) \Rightarrow (f(u_n)) \to f(v) \text{ on } (X_2, Y_2, d_2).$$

Samet et al. [24] introduced the following concepts:

Definition. Let Ψ be a family of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions;

(1) ψ is nondecreasing,

(2) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

In literature, these functions are known as (c)-comparison functions or Bianchini-Grandolfi gauge functions. It is obvious that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$. For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds:

If ψ is nondecreasing, then for each $t > 0$,

$$\lim_{n\to\infty} \psi^n(t) = 0 \Rightarrow \psi(t) < t \Rightarrow \psi(0) = 0.$$

So, if $\psi \in \Psi$, then for each $t > 0$, $\psi(t) < t$ and $\psi(0) = 0$.

3. Main Results

Definition. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping. If there exist $\alpha : X \times Y \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x \in X \text{ and } y \in Y, \tag{1}$$

then T is called α-ψ-contractive covariant mapping.

Definition. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \nearrow\!\!\!\searrow (X, Y)$ be a contravariant mapping. If there exist $\alpha : X \times Y \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Ty, Tx) \leq \psi(d(x, y)) \text{ for all } x \in X \text{ and } y \in Y, \tag{2}$$

then T is called α-ψ-contractive contravariant mapping.

Remark 3.1. A mapping $T : (X, Y) \rightrightarrows (X, Y)$ satisfying Banach contraction is an α-ψ-contractive covariant mapping with $\alpha(x, y) = 1$ for all $x \in X,\ y \in Y$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. Similarly, a mapping $T : (X, Y) \nearrow\!\!\!\searrow (X, Y)$ satisfying Banach contraction is an α-ψ-contractive contravariant mapping with $\alpha(x, y) = 1$ for all $x \in X,\ y \in Y$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$.

Definition. Let $T : (X, Y) \rightrightarrows (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible (covariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$$

for all $x \in X$ and $y \in Y$.

Definition. Let $T : (X, Y) \nearrow\!\!\!\searrow (X, Y)$ and $\alpha : X \times Y \to [0, +\infty)$. Then T is called α-admissible (contravariant) if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Ty, Tx) \geq 1$$

for all $x \in X$ and $y \in Y$.

Example 3.2. Let $X = [0, +\infty)$ and $Y = (-\infty, 0]$. We define the covariant mapping $T : (X, Y) \rightrightarrows (X, Y)$ by $Tx = x$ and $\alpha : X \times Y \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 0, & x = y, \\ 2, & \text{otherwise} \end{cases}$$

for all $x \in X$ and $y \in Y$. Then T is α-admissible. Similarly, if we take contravariant mapping $T : (X, Y) \nearrow\!\!\!\searrow (X, Y)$ which is defined by $Tx = -x$ (or $Ty = -y$), it also satisfying the condition (2). Then T is α-admissible.

Theorem 3.3. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be an α-ψ-contractive covariant mapping. Suppose that the following conditions are satisfied;*

(i) *T is an α-admissible,*

(ii) *There exist $x_0 \in X, y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$,*

(iii) *T is continuous,*

then T has a fixed point.

Proof. Let $x_0 \in X, y_0 \in Y$ such that $\alpha(x_0, Ty_0) \geq 1$. We define the bisequence (x_n, y_n) by $x_{n+1} = Tx_n$, $y_{n+1} = Ty_n$ for all $n \in \mathbb{N}$. Since T is α-admissible, using condition (ii), we obtain that

$$\begin{aligned}
&\alpha(x_0, y_0) \geq 1 \Rightarrow \alpha(Tx_0, Ty_0) \geq 1, \\
&\alpha(x_0, y_1) = \alpha(x_0, Ty_0) \geq 1 \Rightarrow \alpha(Tx_0, Ty_1) = \alpha(x_1, y_2) \geq 1, \\
&\alpha(x_1, y_1) = \alpha(Tx_0, Ty_0) \geq 1 \Rightarrow \alpha(Tx_1, Ty_1) = \alpha(x_2, y_2) \geq 1, \\
&\alpha(x_1, y_2) = \alpha(x_1, Ty_1) \geq 1 \Rightarrow \alpha(Tx_1, Ty_2) = \alpha(x_2, y_3) \geq 1, \\
&\alpha(x_2, y_2) = \alpha(Tx_1, Ty_1) \geq 1 \Rightarrow \alpha(Tx_2, Ty_2) = \alpha(x_3, y_3) \geq 1.
\end{aligned}$$

By repeating this process, we have

$$\alpha(x_n, y_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, y_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3}$$

Using equations (1) and (3), we find that for $x = x_{n-1},\ y = y_n$

$$d(x_n, y_{n+1}) = d(Tx_{n-1}, Ty_n) \leq \alpha(x_{n-1}, y_n) d(Tx_{n-1}, Ty_n) \leq \psi(d(x_{n-1}, y_n))$$

and for $x = x_n,\ y = y_n$

$$d(x_{n+1}, y_{n+1}) = d(Tx_n, Ty_n) \leq \alpha(x_n, y_n) d(Tx_n, Ty_n) \leq \psi(d(x_n, y_n))$$

as $n \geq 1$. By induction, we obtain

$$d(x_n, y_{n+1}) \leq \psi^n(d(x_0, y_1)) \text{ and } d(x_{n+1}, y_{n+1}) \leq \psi^{n+1}(d(x_0, y_0))$$

for all $n \in \mathbb{N}$. There exists $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(d(x_0, y_1)) < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}.$$

Now for $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, by applying the property $(B3)$, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_{n+2}) \\ &\quad + \cdots + d(x_{m-1}, y_{m-1}) + d(x_{m-1}, y_m) \\ &\leq \textstyle\sum_{k=n}^{m-1} d(x_k, y_{k+1}) + \sum_{k=n}^{m-2} d(x_{k+1}, y_{k+1}) \\ &\leq \textstyle\sum_{k=n}^{m-1} \psi^k(d(x_0, y_1)) + \sum_{k=n}^{m-2} \psi^{k+1}(d(x_0, y_0)) \\ &\leq \textstyle\sum_{n \geq n(\epsilon)} \psi^n(d(x_0, y_1)) + \sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_0)) \\ &< \epsilon \end{aligned}$$

for all $k \geq 1$. On the other hand, for $n > m > n(\epsilon)$ with, by using property $(B3)$, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_m, y_m) + d(x_m, y_{m+1}) + d(x_{m+1}, y_{m+1}) \\ &\quad + \cdots + d(x_n, y_{n+1}) + d(x_n, y_n) \\ &\leq \textstyle\sum_{k=m}^{n} d(x_k, y_k) + \sum_{k=m}^{n} d(x_k, y_{k+1}) \\ &\leq \textstyle\sum_{k=m}^{n} \psi^k(d(x_0, y_0)) + \sum_{k=m}^{n} \psi^{k+1}(d(x_0, y_1)) \\ &\leq \textstyle\sum_{n \geq n(\epsilon)} \psi^n(d(x_0, y_0)) + \sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_1)) \\ &< \epsilon \end{aligned}$$

for all $k \geq 1$. Therefore, we get the conclusion that (x_n, y_n) is a Cauchy sequence. Since (X, Y, d) is complete bipolar metric space, (x_n, y_n) biconverges. That is, there exists $u \in X \cap Y$ such that $(x_n) \to u$ and $(y_n) \to u$ as $n \to \infty$. Since the covariant map T is continuous, $y_n \to u$ implies that $y_{n+1} = Ty_n \to Tu$ and $x_n \to u$ implies that $x_{n+1} = Tx_n \to Tu$. By uniqueness of the limit, we get immediately that $Tu = u$. Then u is a fixed point of T. □

Theorem 3.4. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be an α-ψ-contractive contravariant mapping. Suppose that the following conditions are satisfied;*

(i) *T is an α-admissible,*

(ii) *There exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *T is continuous,*

then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define the bisequence (x_n, y_n) by $y_n = Tx_n$ and $x_{n+1} = Ty_n$ for all $n \in \mathbb{N}$. Since T is α-admissible, we obtain that

$$\begin{aligned} &\alpha(x_0, y_0) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Ty_0, Tx_0) = \alpha(x_1, y_0) \geq 1 \\ &\alpha(x_1, y_0) \geq 1 \Rightarrow \alpha(Ty_0, Tx_1) = \alpha(x_1, y_1) \geq 1 \\ &\alpha(x_1, y_1) \geq 1 \Rightarrow \alpha(Ty_1, Tx_1) = \alpha(x_2, y_1) \geq 1 \\ &\alpha(x_2, y_1) \geq 1 \Rightarrow \alpha(Ty_1, Tx_2) = \alpha(x_2, y_2) \geq 1 \end{aligned}$$

By repeating this process, we have

$$\alpha(x_n, y_n) \geq 1 \text{ and } \alpha(x_{n+1}, y_n) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{4}$$

Using inequalities (2) and (4), we find that for $x = x_n,\ y = y_{n-1}$

$$d(x_n, y_n) = d(Ty_{n-1}, Tx_n) \leq \alpha(x_n, y_{n-1}) d(Ty_{n-1}, Tx_n) \leq \psi(d(x_n, y_{n-1}))$$

and for $x = x_{n+1},\ y = y_n$

$$d(x_{n+1}, y_n) = d(Ty_n, Tx_n) \leq \alpha(x_n, y_n) d(Ty_n, Tx_n) \leq \psi(d(x_n, y_n))$$

as $n \geq 1$. By induction, we obtain

$$d(x_n, y_n) \leq \psi^n(d(x_1, y_0)) \text{ and } d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0))$$

for all $n \in \mathbb{N}$. There exists $\epsilon > 0$ and $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(d(x_1, y_0)) < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}.$$

Now for $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, by applying the property $(B3)$, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &\leq \textstyle\sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k) \\ &\leq \textstyle\sum_{k=n}^{m} \psi^k(d(x_1, y_0)) + \sum_{k=n}^{m-1} \psi^k(d(x_0, y_0)) \\ &\leq \textstyle\sum_{n \geq n(\epsilon)} \psi^n(d(x_1, y_0)) + \sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_0)) \\ &< \epsilon \end{aligned}$$

for all $k \geq 1$.

On the other hand, for $n > m > n(\epsilon)$ with, by using property $(B3)$, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n-1}) + d(x_{n-1}, y_{n-1}) + d(x_{n-1}, y_{n-2}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &\leq \textstyle\sum_{k=m}^{n-1} d(x_k, y_k) + \sum_{k=m}^{n} d(x_k, y_{k-1}) \\ &\leq \textstyle\sum_{k=m}^{n-1} \psi^k(d(x_1, y_0)) + \sum_{k=m}^{n} \psi^k(d(x_0, y_0)) \\ &\leq \textstyle\sum_{n \geq n(\epsilon)} \psi^{n+1}(d(x_0, y_0)) + \sum_{n \geq n(\epsilon)} \psi^n(d(x_1, y_0)) \\ &< \epsilon \end{aligned}$$

for all $k \geq 1$. Therefore, we get the conclusion that (x_n, y_n) is a Cauchy sequence. Since (X, Y, d) is complete bipolar metric space, (x_n, y_n) biconverges. That is, there exists $u \in X \cap Y$ such that $(x_n) \to u$ and $(y_n) \to u$ as $n \to \infty$. Since the contravariant map T is continuous, $x_n \to u$ implies that $y_n = Tx_n \to Tu$ and combining this with $y_n \to u$ gives $Tu = u$. Then u is a fixed point of T. □

In the next theorem, we omit the continuity hypothesis of T.

Theorem 3.5. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be an α-ψ-contractive covariant mapping. Suppose that the following conditions are satisfied;*

(i) *T is an α-admissible,*

(ii) *There exist $x_0 \in X, y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$,*

(iii) *If (x_n, y_n) is a bisequence such that $\alpha(x_n, y_n) \geq 1$ for all n and $x_n \to a$, $y_n \to a$, $a \in X \cap Y$ as $n \to \infty$, then $\alpha(a, y_n) \geq 1$ for all n.*

Then T has a fixed point.

Proof. Following the proof of Theorem 3.3, we obtain the sequence (x_n, y_n) defined by $x_{n+1} = Tx_n$, $y_{n+1} = Ty_n$ for all $n \geq 0$, which is Cauchy bisequence in the complete metric space (X, Y, d) and converges to some $u \in X \cap Y$ such that $x_n \to u$, $y_n \to u$ as $n \to \infty$. From (3) and condition (iii), we get $\alpha(u, y_n) \geq 1$ for all $n \in \mathbb{N}$. By applying $(B3)$, (1) and the above inequality, we obtain

$$\begin{aligned} d(Tu, u) &\leq d(Tu, Ty_n) + d(Tx_n, Ty_n) + d(Tx_n, u) \\ &\leq \alpha(u, y_n) d(Tu, Ty_n) + \alpha(x_n, y_n) d(Tx_n, Ty_n) + d(x_{n+1}, u) \\ &\leq \psi(d(u, y_n)) + \psi(d(x_n, y_n)) + d(x_{n+1}, u) \\ &\leq \psi(d(u, y_n)) + \psi(d(x_n, u) + d(u, u) + d(u, y_n)) + d(x_{n+1}, u). \end{aligned}$$

Letting $n \to \infty$ in above inequality and using the continuity of ψ at $t = 0$, we get $d(Tu, u) = 0 \Rightarrow Tu = u$. Therefore u is a fixed point of T. □

Theorem 3.6. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be an α-ψ-contractive contravariant mapping. Suppose that the following conditions are satisfied;*

(i) *T is an α-admissible,*

(ii) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *If (x_n, y_n) is a bisequence such that $\alpha(x_n, y_n) \geq 1$ for all n and $y_n \to a \in X \cap Y$ as $n \to \infty$, then $\alpha(x_n, a) \geq 1$ for all n.*

Then T has a fixed point.

Proof. Following the proof of Theorem 3.3, we obtain the sequence (x_n, y_n) defined by $y_n = Tx_n,\ x_{n+1} = Ty_n$ for all $n \geq 0$, which is Cauchy bisequence in the complete metric space (X, Y, d) and converges to some $u \in X \cap Y$ such that $x_n \to u$, $y_n \to u$ as $n \to \infty$. From (4) and condition (iii), we get $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$. By applying $(B3)$, (2) and above inequality, we obtain

$$\begin{aligned} d(Tu, u) &\leq d(Tu, Tx_n) + d(Ty_n, Tx_n) + d(Ty_n, u) \\ &\leq \alpha(x_n, u)d(Tu, Tx_n) + \alpha(x_n, y_n)d(Ty_n, Tx_n) + d(x_{n+1}, u) \\ &\leq \psi(d(x_n, u)) + \psi(d(x_n, y_n)) + d(x_{n+1}, u) \\ &\leq \psi(d(x_n, u)) + \psi(d(x_n, u) + d(u, u) + d(u, y_n)) + d(x_{n+1}, u) \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the continuity of ψ at $t = 0$, we get $d(Tu, u) = 0 \Rightarrow Tu = u$. Therefore u is a fixed point of T. □

In the following, we give a hypothesis to obtain the uniqueness of the fixed point.
H : There exists $z \in X \cap Y$ such that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ for all $x \in X$ and $y \in Y$.

Theorem 3.7. *If the condition* **H** *is added to the hypotheses of Theorem 3.3 or Theorem 3.5 (resp., Theorem 3.4 or Theorem 3.6), we obtain that u is a unique fixed point of the covariant mapping (resp., the contravariant mapping) T.*

Proof. We want to show the uniqueness of a fixed point of the covariant mapping (resp., the contravariant mapping) T. We suppose the contrary, that is, v is another fixed point of T with $u \neq v$. Then, from the condition **H**, there exists $z \in X \cap Y$ such that

$$\alpha(u, z) \geq 1 \text{ and } \alpha(z, v) \geq 1. \tag{5}$$

Since T is α-admissible, using (5), we obtain

$$\alpha(u, T^n z) \geq 1 \text{ and } \alpha(T^n z, v) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{6}$$

By combining the inequalities (6) and (1), we have

$$\begin{aligned} d(u, T^n z) &= d(Tu, T(T^{n-1}z)) \\ &\leq \alpha(u, T^{n-1}z)d(Tu, T(T^{n-1}z)) \\ &\leq \psi(d(u, T^{n-1}z)) \\ \Rightarrow d(u, T^n z) &\leq \psi^n(d(u, z)) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Similarly, we easily obtain that

$$d(T^n z, v) \leq \psi^n(d(z, v)) \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we observe that

$$T^n z \to u \text{ and } T^n z \to v$$

which is contradicts with the uniqueness of the limit. Hence $u = v \in X \cap Y$. Then T has a unique fixed point.

If T is the contravariant mapping, then proof is similar. □

Now, we derive coupled fixed point theorems from the our obtained results.

Definition. [16] Let (X, Y, d) be a bipolar metric space, $a \in X$, $p \in Y$ and $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping. (a, p) is said to be a coupled fixed point of F if

$$F(a, p) = a \text{ and } F(p, a) = p.$$

The following Lemma can be easily proved.

Lemma 3.8. *Let $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping. If we define the covariant mapping $T : (X \times Y, Y \times X) \rightrightarrows$ $(X \times Y, Y \times X)$ with*

$$T(x, y) = (F(x, y), F(y, x)) \text{ for all } (x, y) \in X \times Y, \tag{7}$$

then (x, y) is a coupled fixed point of F if only if (x, y) is a fixed point of T.

Theorem 3.9. *Let (X, Y, d) be a complete bipolar metric space and*

$$F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$$

be a covariant mapping. We suppose that there exist functions $\psi \in \Psi$ and $\alpha : (X \times Y) \times (Y \times X) \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(v, y)) \tag{8}$$

for all $(x, y),\ (v, u) \in X \times Y$ and the following conditions are satisfied;

(i) *$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)) \geq 1$ for all $(x, y),\ (v, u) \in X \times Y$.*

(ii) *There exists $(x_0, y_0) \in X \times Y$ such that*

$$\alpha((x_0, y_0), (F(y_0, x_0), F(x_0, y_0))) \quad \geq \quad 1,$$

and

$$\alpha((F(x_0, y_0), F(y_0, x_0)), (y_0, x_0)) \geq 1.$$

(iii) *F is continuous.*

Then F has a coupled fixed point. Namely, there exists $(u, v) \in X \times Y$ such that $u = F(u, v)$ and $v = F(v, u)$.

Proof. We consider the complete bipolar metric space (A, B, δ) where $A = X \times Y$, $B = Y \times X$ and

$$\delta((x, y), (u, v)) = d(x, u) + d(v, y)$$

for all $(x, y) \in A$, $(u, v) \in B$. Using (8), we get

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \quad \leq \quad \frac{1}{2}\psi(\delta((x, u) + (v, y))) \tag{9}$$

$$\alpha((v, u), (y, x))d(F(x, y), F(u, v)) \quad \leq \quad \frac{1}{2}\psi(\delta((x, u) + (v, y))). \tag{10}$$

Combining (9) and (10), we obtain

$$\beta(\varepsilon, \eta)\delta(T\varepsilon, T\eta) \leq \psi(\delta(\varepsilon, \eta))$$

for all $\varepsilon = (\varepsilon_1, \varepsilon_2) \in A$, $\eta = (\eta_1, \eta_2) \in B$ where $\beta : A \times B \to [0, +\infty)$ is the function defined by

$$\beta(\varepsilon, \eta) = \min\{\alpha((\varepsilon_1, \varepsilon_2), (\eta_1, \eta_2)), \alpha((\eta_2, \eta_1), (\varepsilon_2, \varepsilon_1))\}$$

and $T : (A, B) \rightrightarrows (A, B)$ is defined as (7). Then T is continuous and β-ψ-contractive covariant mapping. We take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in A$, $\eta = (\eta_1, \eta_2) \in B$ such that $\beta(\varepsilon, \eta) \geq 1$. From conditions (i) and (8), we see that $\beta(T\varepsilon, T\eta) \geq 1$. So, T is β-admissible.

On the other hand, from conditions (ii) and (8), we obtain that there exists $(x_0, y_0) \in A$ (or $(y_0, x_0) \in B$) such that

$$\beta((x_0, y_0), T(x_0, y_0)) \geq 1$$

(or $\beta(T(x_0, y_0), (y_0, x_0)) \geq 1$). So, we observe that Theorem 3.3 is satisfied. Then T has a fixed point. Also, from Lemma 3.8, this fixed point is a coupled fixed point of F. □

In the next theorem, we omit the continuity hypothesis of F.

Theorem 3.10. *Let (X, Y, d) be a complete bipolar metric space and*

$$F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$$

be a covariant mapping. We suppose that there exist functions $\psi \in \Psi$ and $\alpha : (X \times Y) \times (Y \times X) \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \frac{1}{2}\psi(d(x, u) + d(v, y)) \tag{11}$$

for all $(x, y) \in X \times Y$, $(u, v) \in Y \times X$ and the following conditions are satisfied;

(i) *$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)) \geq 1$ for all $(x, y) \in X \times Y$, $(u, v) \in Y \times X$.*

(ii) *There exists $(x_0, y_0) \in X \times Y$ such that*

$$\alpha((x_0, y_0), (F(y_0, x_0), F(x_0, y_0))) \quad \geq \quad 1,$$

and

$$\alpha((F(x_0, y_0), F(y_0, x_0)), (y_0, x_0)) \geq 1.$$

(iii) *If (x_n, y_n) is a bisequence such that*

$$\alpha((x_n, y_n), (y_{n+1}, x_{n+1})) \geq 1 \text{ and } \alpha((x_{n+1}, y_{n+1}), (y_n, x_n)) \geq 1$$

for all n and $x_n \to y \in Y, y_n \to x \in X$ as $n \to \infty$, then

$$\alpha((x_n, y_n), (x, y)) \geq 1 \text{ and } \alpha((x, y), (y_n, x_n)) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Then F has a coupled fixed point.

Proof. We will give the proof using the same notations in the proof of Theorem 3.9. Let (x_n, y_n) be a bisequence in A and (y_n, x_n) be a bisequence in B such that $\beta((x_n, y_n), (y_{n+1}, x_{n+1})) \geq 1$ and $(x_n, y_n) \to (y, x)$ as $n \to \infty$. From the condition (iii), we get $\beta((x_n, y_n), (y, x)) \geq 1$. Then the conditions in Theorem 3.5 are satisfied. So, T has a fixed point. Moreover, from Lemma 3.8, this fixed point is a coupled fixed point of F. □

In the following, we give a hypothesis to obtain the uniqueness of the coupled fixed point.

$\mathbf{H}'$: There exists $(z_1, z_2) \in (X \times Y) \cap (Y \times X)$ such that

$$\alpha((x, y), (z_1, z_2)) \geq 1, \ \alpha((z_2, z_1), (y, x)) \geq 1$$

and

$$\alpha((v, u), (z_1, z_2)) \geq 1, \ \alpha((z_2, z_1), (u, v)) \geq 1$$

for all $(x, y) \in X \times Y$ and $(u, v) \in Y \times X$.

Theorem 3.11. *If the condition* $\mathbf{H}'$ *is added to the hypotheses of Theorem 3.9 (resp., Theorem 3.10), we obtain that* u *is a unique coupled fixed point of the covariant mapping* F.

Proof. We consider the hypothesis $\mathbf{H}'$, then we easily say that T and β satisfy the hypothesis $\mathbf{H}$. From Theorem 3.5 and Lemma 3.8, the result is obtained obviously. □

Example 3.12. Let $U_n(\mathbb{R})$ and $L_n(\mathbb{R})$ be the sets of all $n \times n$ upper and lower triangular matrices over $\mathbb{R}$, respectively. A function $d : U_n(\mathbb{R}) \times L_n(\mathbb{R}) \to \mathbb{R}^+$ be defined as

$$d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|$$

for all $A = (a_{ij})_{n\times n} \in U_n(\mathbb{R})$ and $B = (b_{ij})_{n\times n} \in L_n(\mathbb{R})$. Then it is apparent that $(U_n(\mathbb{R}), L_n(\mathbb{R}), d)$ is a complete bipolar metric space. We take a covariant mapping

$$F : (U_n(\mathbb{R}) \times L_n(\mathbb{R}), L_n(\mathbb{R}) \times U_n(\mathbb{R})) \rightrightarrows (U_n(\mathbb{R}), L_n(\mathbb{R}))$$

by $F(A, B) = \left(\frac{a_{ij}+b_{ij}}{4}\right)_{n\times n}$ where

$$A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}) \in ((U_n(\mathbb{R}) \times L_n(\mathbb{R})) \cup (L_n(\mathbb{R}) \times U_n(\mathbb{R})).$$

It is clear that F is a continuous covariant mapping. On the other hand, we define

$$\alpha : (U_n(\mathbb{R}) \times L_n(\mathbb{R})) \times (L_n(\mathbb{R}) \times U_n(\mathbb{R})) \to [0, +\infty)$$

by

$$\alpha((A, B), (C, D)) = \begin{cases} 1, & a_{ij} \geq b_{ij},\ c_{ij} \geq d_{ij} \\ 0, & \text{otherwise} \end{cases}$$

where $A = (a_{ij})_{n\times n},\ D = (d_{ij})_{n\times n} \in U_n(\mathbb{R})$ and $B = (b_{ij})_{n\times n}, C = (c_{ij})_{n\times n} \in L_n(\mathbb{R})$. Then we get

$$\begin{aligned} d(F(A, B), F(C, D)) &= d\left(\left(\frac{a_{ij} + b_{ij}}{4}\right)_{n\times n}, \left(\frac{c_{ij} + d_{ij}}{4}\right)_{n\times n}\right) \\ &= \sum_{i,j=1}^{n} \left|\frac{a_{ij} + b_{ij} - c_{ij} - d_{ij}}{4}\right| \\ &\leq \sum_{i,j=1}^{n} \left|\frac{a_{ij} - c_{ij}}{4}\right| + \left|\frac{b_{ij} - d_{ij}}{4}\right| \\ &= \frac{1}{4}\left(d(A, C) + d(B, D)\right) \end{aligned}$$

for all $(A, B) \in U_n(\mathbb{R}) \times L_n(\mathbb{R})$ and $(C, D) \in L_n(\mathbb{R}) \times U_n(\mathbb{R})$. From definition of the function α, we observe that

$$\alpha((A, B), (C, D))d(F(A, B), F(C, D)) \leq \frac{1}{4}(d(A, C) + d(B, D)).$$

So, the inequality (8) is satisfied where $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Moreover, the conditions (i) and (ii) of Theorem 3.9 are satisfied for $(x_0, y_0) = (I_n, I_n)$. So F has a coupled fixed point. In particular, the coupled fixed point is $(0_{n\times n}, 0_{n\times n}) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$ where $0_{n\times n}$ is the null matrix.

In the following, we give some existing results which are obtained from our main results:

Corollary 3.13. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x \in X$ and $y \in Y$. Then T has a unique fixed point.

Proof. We take the mapping $\alpha : X \times Y \to [0, +\infty)$ as $\alpha(x, y) = 1$ for all $x \in X$, $y \in Y$ in Theorem 3.3 and Theorem 3.7. It is obvious that all the conditions of Theorem 3.3 and Theorem 3.7 are satisfied. Then the proof is completed. □

Now, we give a similar result for contravariant mappings.

Corollary 3.14. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Ty, Tx) \leq \psi(d(x, y))$$

for all $x \in X$ and. Then T has a unique fixed point.

Proof. By using similar method with proof of Corollary 3.13,we say that it suffices to take $\alpha(x, y) = 1$ for all $x \in X$ and $y \in Y$ in Theorem 3.4 and Theorem 3.7 to prove the corollary.

□

Corollary 3.15. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping such that*

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x \in X$ and $y \in Y$ where $\lambda \in [0, 1)$. Then T has a unique fixed point.

Proof. If we take the mapping $\psi : [0, +\infty) \to [0, +\infty)$ as $\psi(t) = \lambda t$ for all $x \in X$ and $y \in Y$ in Corollary 3.13, the proof is easily obtained. □

Corollary 3.16. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a covariant mapping such that*

$$d(Ty, Tx) \leq \lambda d(x, y)$$

for all $x \in X$ and $y \in Y$ where $\lambda \in [0, 1)$. Then T has a unique fixed point.

Proof. In Corollary 3.14, we take the mapping $\psi : [0, +\infty) \to [0, +\infty)$ as $\psi(t) = \lambda t$ for all $x \in X$ and $y \in Y$. It suffices for the proof. □

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[1] Mehmet Akif Ersoy University, Faculty of Science and Arts, Department of Mathematics, 15030, Burdur/TURKEY.

E-mail address: utkugurdal@gmail.com

[2] Manisa Celal Bayar University, Faculty of Science and Arts, Department of Mathematics, 45140, Manisa/TURKEY.

E-mail address: abgamutlu@gmail.com

E-mail address: kubra.ozkan@hotmail.com