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GENERALIZATION OF OSTROWSKI KIND INEQUALITY FOR DOUBLE INTEGRALS ON TIME SCALES

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Abstract. In this paper, we use the newly developed technique of parameter function to obtain some new Ostrowski type inequalities for double integrals. For this, we first obtain a generalized Montgomery identity for double integral and then use it obtain two Ostrowski type inequalities. Our results generalize some of the results of Zheng in [27] and Liu et al. in [22]. Furthermore, we apply our results to different time scales to obtain some interesting results that may be applied in the study of difference and differential equations.

1. Introduction

The following result due to A. Ostrowski is known in the literature as the Ostrowski inequality [8].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\ dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M,$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Ever since the above result was established in 1938, many researchers have developed loads of generalizations and extensions. See the survey article [1]. In 1998, Barnett and Dragomir [2], proved the following double integral version of Theorem 1.1.

Theorem 1.2. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ continuous on $[a,b] \times [c,d]$, $f''_{x,y} = \dfrac{\partial^2 f}{\partial x \partial y}$ exists on $(a,b) \times (c,d)$ and is bounded, i.e, $||f''_{x,y}||_\infty := \sup\limits_{(x,y) \in (a,b)\times(c,d)} \left| \dfrac{\partial^2 f(x,y)}{\partial x \partial y} \right| <$*

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∞. Then we have the inequality

$$\left|\int_a^b\int_c^d f(s,t)dsdt - \left[(b-a)\int_c^d f(x,t)dt + (d-c)\int_a^b f(s,y)ds - (d-c)(b-a)f(x,y)\right]\right|$$

$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right]||f''_{x,y}||_\infty \tag{1.1}$$

for all $(x,y) \in [a,b]\times[c,d]$.

With the advent of time scales in 1988, many research papers (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24]) have been published with regards to the extension of Theorem 1.2 to arbitrary time scale (see Section 2 for definition). Worthy of mention is the result due to Liu et al. [22] which extends Theorem 1.2 to time scales. They, in fact, proved the following result:

Theorem 1.3. *Let $x,t \in \mathbb{T}_1$, $y,s \in \mathbb{T}_2$ and $f : [a,b]\times[c,d] \to \mathbb{R}$ be such that $f(\cdot,\cdot)$ is integrable on $[a,b]\times[c,d]$, $f(x,\cdot)$ is integrable on $[c,d]$ for any $x \in [a,b]$ and $f(\cdot,y)$ is integrable on $[a,b]$ for any $y \in [c,d]$, the partial derivative $\dfrac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}$ exists and is continuous on $[a,b]\times[c,d]$. Then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s\right.$$
$$\left. - \frac{1}{d-c}\int_c^d f(x,\sigma_2(s))\Delta_2 s - \frac{1}{b-a}\int_a^b f(\sigma_1(t),y)\Delta_1 t + f(x,y)\right|$$
$$\leq \frac{M}{(b-a)(d-c)}\Big(h_2(x,a)+h_2(x,b)\Big)\Big(h_2(y,c)+h_2(y,d)\Big) \tag{1.2}$$

for all $(x,y) \in [a,b]\times[c,d]$, where $M = \sup\limits_{a<t<b;c<s<d}\left|\dfrac{\partial^2 f(t,s)}{\Delta_1 t\Delta_2 s}\right| < \infty$.

Motivated by the idea used in [25, 27] and using the technique of parameter functions initiated in [26], we give a somewhat generalization of Theorem 1.3 which in turn contains Theorem 1.2. For this, we first prove a generalized Montgomery identity for double integral and then use it to obtain two Ostrowski type inequalities for double integrals. Our generalized Montgomery identity generalizes Lemma 2.1 in [27].

This work is arranged as follows: we give a brief overview of time scale essentials in Section 2. In Section 3, our results are formulated and proved.

2. Preliminaries

We start this section by presenting the following definitions and results about time scales. To know more about the subject of one variable time scale calculus, we refer the interested reader to the books [3, 4].

Definition 2.1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \in \mathbb{T}$, respectively. Clearly, we see that $\sigma(t) \geq t$ and $\rho(t) \leq t$ for all $t \in \mathbb{T}$. If*

$\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. If $\sigma(t) = t$, then t is called right dense, and if $\rho(t) = t$ then t is called left dense. Points that are both right dense and left dense are called dense. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - m$; otherwise, $\mathbb{T}^k = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$, we define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals and half-open intervals are defined in the same manner.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with delta differential operators Δ_1 and Δ_2, respectively. Suppose $a, b \in \mathbb{T}_1$ and $c, d \in \mathbb{T}_2$, with $a < b$ and $c < d$, we define the rectangle $[a, b] \times [c, d]$ by

$$[a, b] \times [c, d] := \{(u, v) : u \in [a, b], v \in [c, d]\}.$$

For the sake of convenience, we will assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ have a uniform jump operator σ.

For more on two-variable time scales calculus and multiple integration, see [5, 6, 7].

Definition 2.2. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to differentiable at $t \in \mathbb{T}^k$, with delta derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\epsilon > 0$ there exist a neighborhood U of t such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \ \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \dfrac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t+1) - f(t)$.

Definition 2.3. *The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd$-$continuous if it is continuous at all dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left dense points $t \in \mathbb{T}$.*

Next, we state the following known properties of the $\Delta-$integral.

Theorem 2.4. *If $a, b, c \in \mathbb{T}$ with $a < c < b$, $\alpha \in \mathbb{R}$ and f, g are rd-continuous, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$
(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
(v) $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\Delta t;$
(vi) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$

Definition 2.5. *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}$ be functions that are recursively defined as*

$$h_0(t, s) = 1$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \ for \ all \ s, t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$, then for all $s, t \in \mathbb{T}$, $h_k(t, s) = \frac{(t-s)^k}{k!}$.

3. Main results

To prove our main results, we will need the following lemma.

Lemma 3.1 (A generalized Montgomery Identity for Double Integrals)**.** *Let* $a, b, s, x, \in \mathbb{T}_1,\ c, d, t, y, \in \mathbb{T}_2,$ *with* $a < b,\ c < d,\ f : [a, b] \times [c, d] \to \mathbb{R}$ *be* $\Delta_1\Delta_2$ *differentiable,* ψ *be a function of* $[0, 1]$ *into* $[0, 1]$*. Then, we have the following equation*

$$\begin{aligned}
&\Phi_1(\lambda)\Phi_2(\lambda)f(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d) - \Phi_6(\lambda)f(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y) - \Phi_4(\lambda)f(a,y)\Big)\\
&+ \Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d) - \Phi_4(\lambda)f(a,d)\Big) + \Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c) - \Phi_3(\lambda)f(b,c)\Big)\\
&- \Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s - \Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t\\
&+ \Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t - \Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s + \Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s\\
&+ \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&= \int_a^b\int_c^d K_1(s,x)K_2(t,y)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s, \qquad (3.1)
\end{aligned}$$

where $\lambda \in [0, 1]$,

$$K_1(s,x) = \begin{cases} s - \left(a + \psi(\lambda)\frac{b-a}{2}\right) = s - A_1(\lambda), & s \in [a,x),\\ s - \left(a + (1+\psi(1-\lambda))\frac{b-a}{2}\right) = s - B_1(\lambda), & s \in [x,b], \end{cases} \qquad (3.2)$$

$$K_2(t,y) = \begin{cases} t - \left(c + \psi(\lambda)\frac{d-c}{2}\right) = t - A_2(\lambda), & t \in [c,y),\\ t - \left(c + (1+\psi(1-\lambda))\frac{d-c}{2}\right) = t - B_2(\lambda), & t \in [y,d], \end{cases} \qquad (3.3)$$

$$\begin{aligned}
\Phi_1(\lambda) &= B_1(\lambda) - A_1(\lambda),\\
\Phi_2(\lambda) &= B_2(\lambda) - A_2(\lambda),\\
\Phi_3(\lambda) &= b - B_1(\lambda),\\
\Phi_4(\lambda) &= a - A_1(\lambda),\\
\Phi_5(\lambda) &= d - B_2(\lambda),\\
\Phi_6(\lambda) &= c - A_2(\lambda).
\end{aligned}$$

Proof. Using item (iv) of Theorem 2.4 gives

$$\begin{aligned}
&\int_a^b\int_c^d K_1(s,x)K_2(t,y)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s = \int_a^x\int_c^y \big(s - A_1(\lambda)\big)\big(t - A_2(\lambda)\big)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&+ \int_a^x\int_y^d \big(s - A_1(\lambda)\big)\big(t - B_2(\lambda)\big)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s + \int_x^b\int_c^y \big(s - B_1(\lambda)\big)\big(t - A_2(\lambda)\big)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&+ \int_x^b\int_y^d \big(s - B_1(\lambda)\big)\big(t - B_2(\lambda)\big)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s. \qquad (3.4)
\end{aligned}$$

We now consider each integral on the right hand side of Equation (3.4). Applying item (vi) of Theorem 2.4, we get

$$\int_a^x\int_c^y \big(s - A_1(\lambda)\big)\big(t - A_2(\lambda)\big)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s$$

$$
\begin{aligned}
&= \int_a^x \big(s - A_1(\lambda)\big) \left[\int_c^y \big(t - A_2(\lambda)\big) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t\right] \Delta_1 s \\
&= \int_a^x \big(s - A_1(\lambda)\big) \left[\big(y - A_2(\lambda)\big) \frac{\partial f(s,y)}{\Delta_1 s} - \big(c - A_2(\lambda)\big) \frac{\partial f(s,c)}{\Delta_1 s} - \int_c^y \frac{\partial f(s,\sigma(t))}{\Delta_1 s} \Delta_2 t\right] \Delta_1 s \\
&= \big(y - A_2(\lambda)\big) \int_a^x \big(s - A_1(\lambda)\big) \frac{\partial f(s,y)}{\Delta_1 s} \Delta_1 s - \big(c - A_2(\lambda)\big) \int_a^x \big(s - A_1(\lambda)\big) \frac{\partial f(s,c)}{\Delta_1 s} \Delta_1 s \\
&- \int_c^y \left[\int_a^x \big(s - A_1(\lambda)\big) \frac{\partial f(s,\sigma(t))}{\Delta_1 s} \Delta_1 s\right] \Delta_2 t \\
&= \big(y - A_2(\lambda)\big)\big(x - A_1(\lambda)\big) f(x,y) - \big(y - A_2(\lambda)\big)\big(a - A_1(\lambda)\big) f(a,y) - \big(y - A_2(\lambda)\big) \int_a^x f(\sigma(s),y) \Delta_1 s \\
&- \big(c - A_2(\lambda)\big)\big(x - A_1(\lambda)\big) f(x,c) + \big(c - A_2(\lambda)\big)\big(a - A_1(\lambda)\big) f(a,c) + \big(c - A_2(\lambda)\big) \int_a^x f(\sigma(s),c) \Delta_1 s \\
&- \int_c^y \big(x - A_1(\lambda)\big) f(x,\sigma(t)) \Delta_2 t + \int_c^y \big(a - A_1(\lambda)\big) f(a,\sigma(t)) \Delta_2 t + \int_a^x \int_c^y f(\sigma(s),\sigma(t)) \Delta_2 t \Delta_1 s.
\end{aligned} \tag{3.5}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_a^x \int_y^d \big(s - A_1(\lambda)\big)\big(t - B_2(\lambda)\big) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \\
&= \big(d - B_2(\lambda)\big)\big(x - A_1(\lambda)\big) f(x,d) - \big(d - B_2(\lambda)\big)\big(a - A_1(\lambda)\big) f(a,d) - \big(d - B_2(\lambda)\big) \int_a^x f(\sigma(s),d) \Delta_1 s \\
&- \big(y - B_2(\lambda)\big)\big(x - A_1(\lambda)\big) f(x,y) + \big(y - B_2(\lambda)\big)\big(a - A_1(\lambda)\big) f(a,y) + \big(y - B_2(\lambda)\big) \int_a^x f(\sigma(s),y) \Delta_1 s \\
&- \int_y^d \big(x - A_1(\lambda)\big) f(x,\sigma(t)) \Delta_2 t + \int_y^d \big(a - A_1(\lambda)\big) f(a,\sigma(t)) \Delta_2 t + \int_a^x \int_y^d f(\sigma(s),\sigma(t)) \Delta_2 t \Delta_1 s,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&\int_x^b \int_c^y \big(s - B_1(\lambda)\big)\big(t - A_2(\lambda)\big) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \\
&= \big(y - A_2(\lambda)\big)\big(b - B_1(\lambda)\big) f(b,y) - \big(y - A_2(\lambda)\big)\big(x - B_1(\lambda)\big) f(x,y) - \big(y - A_2(\lambda)\big) \int_x^b f(\sigma(s),y) \Delta_1 s \\
&- \big(c - A_2(\lambda)\big)\big(b - B_1(\lambda)\big) f(b,c) + \big(c - A_2(\lambda)\big)\big(x - B_1(\lambda)\big) f(x,c) + \big(c - A_2(\lambda)\big) \int_x^b f(\sigma(s),c) \Delta_1 s \\
&- \int_c^y \big(b - B_1(\lambda)\big) f(b,\sigma(t)) \Delta_2 t + \int_c^y \big(x - B_1(\lambda)\big) f(x,\sigma(t)) \Delta_2 t + \int_x^b \int_c^y f(\sigma(s),\sigma(t)) \Delta_2 t \Delta_1 s,
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
&\int_x^b \int_y^d \big(s - B_1(\lambda)\big)\big(t - B_2(\lambda)\big) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \\
&= \big(d - B_2(\lambda)\big)\big(b - B_1(\lambda)\big) f(b,d) - \big(d - B_2(\lambda)\big)\big(x - B_1(\lambda)\big) f(x,d) - \big(d - B_2(\lambda)\big) \int_x^b f(\sigma(s),d) \Delta_1 s \\
&- \big(y - B_2(\lambda)\big)\big(b - B_1(\lambda)\big) f(b,y) + \big(y - B_2(\lambda)\big)\big(x - B_1(\lambda)\big) f(x,y) + \big(y - B_2(\lambda)\big) \int_x^b f(\sigma(s),y) \Delta_1 s
\end{aligned}
$$

$$-\int_y^d \big(b-B_1(\lambda)\big)f(b,\sigma(t))\Delta_2 t+\int_y^d \big(x-B_1(\lambda)\big)f(x,\sigma(t))\Delta_2 t+\int_x^b\int_y^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s. \tag{3.8}$$

Substituting Equations (3.5), (3.6), (3.7) and (3.8) into (3.4), we obtain

$$\begin{aligned}
&\int_a^b\int_c^d K_1(s,x)K_2(t,y)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&=\Big(B_1(\lambda)-A_1(\lambda)\Big)\Big(B_2(\lambda)-A_2(\lambda)\Big)f(x,y)+\Big(B_2(\lambda)-A_2(\lambda)\Big)\Big[(b-B_1(\lambda))f(b,y)-(a-A_1(\lambda))f(a,y)\Big]\\
&+\Big(B_1(\lambda)-A_1(\lambda)\Big)\Big[(d-B_2(\lambda))f(x,d)-(c-A_2(\lambda))f(x,c)\Big]+\Big(c-A_2(\lambda)\Big)\Big[(a-A_1(\lambda))f(a,c)\\
&-(b-B_1(\lambda))f(b,c)\Big]+\Big(d-B_2(\lambda)\Big)\Big[(b-B_1(\lambda))f(b,d)-(a-A_1(\lambda))f(a,d)\Big]\\
&-\Big(B_2(\lambda)-A_2(\lambda)\Big)\int_a^b f(\sigma(s),y)\Delta_1 s-\Big(B_1(\lambda)-A_1(\lambda)\Big)\int_c^d f(x,\sigma(t))\Delta_2 t+\\
&\Big(c-A_2(\lambda)\Big)\int_a^b f(\sigma(s),c)\Delta_1 s-\Big(d-B_2(\lambda)\Big)\int_a^b f(\sigma(s),d)\Delta_1 s+\Big(a-A_1(\lambda)\Big)\int_c^d f(a,\sigma(t))\Delta_2 t-\\
&\Big(b-B_1(\lambda)\Big)\int_c^d f(b,\sigma(t))\Delta_2 t+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s.
\end{aligned}$$

Hence, Lemma 3.1 follows. □

Remark. *If we assume that* $\lambda\in[0,1]$ *is such that* $a+\psi(\lambda)\frac{b-a}{2},\ a+(1+\psi(1-\lambda))\frac{b-a}{2}\in\mathbb{T}_1$, $c+\psi(\lambda)\frac{d-c}{2},\ c+(1+\psi(1-\lambda))\frac{d-c}{2}\in\mathbb{T}_2$, *and* $x\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]$, $y\in\left[c+\psi(\lambda)\frac{d-c}{2},c+(1+\psi(1-\lambda))\frac{d-c}{2}\right]$, *then Lemma 3.1 reduces to Lemma 2.1 in* [27] *if we take* $\psi(\lambda)=\lambda$.

Theorem 3.2. *Let* $a,b,s,x,\in\mathbb{T}_1,\ c,d,t,y,\in\mathbb{T}_2$, *with* $a<b,\ c<d$, $f:[a,b]\times[c,d]\to\mathbb{R}$ *be* $\Delta_1\Delta_2$ *differentiable,* ψ *be a function of* $[0,1]$ *into* $[0,1]$. *Assume also that* $K:=\sup_{a<s<b;c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\right|<\infty$. *Then, we have the following inequality*

$$\begin{aligned}
&\Bigg|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y)+\Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d)-\Phi_6(\lambda)f(x,c)\Big)+\Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y)-\Phi_4(\lambda)f(a,y)\Big)\\
&+\Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d)-\Phi_4(\lambda)f(a,d)\Big)+\Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c)-\Phi_3(\lambda)f(b,c)\Big)\\
&-\Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t-\Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s-\Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t\\
&+\Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t-\Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s+\Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s\\
&+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg|\\
&\le K\left[\int_a^x|s-A_1(\lambda)|\Delta_1 s+\int_x^b|s-B_1(\lambda)|\Delta_1 s\right]\left[\int_c^y|t-A_2(\lambda)|\Delta_2 t+\int_y^d|t-B_2(\lambda)|\Delta_2 t\right],
\end{aligned} \tag{3.9}$$

where $A_i(\lambda)$, $B_i(\lambda)$ *for* $i \in \{1,2\}$ *and* $\Phi_i(\lambda)$ *for* $i \in \{1,2,3,4,5,6\}$ *are all defined in Lemma 3.1.*

Proof. By applying Lemma 3.1 and item (v) of Theorem 2.4, we get

$$\begin{aligned}
&\Bigg|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d) - \Phi_6(\lambda)f(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y) - \Phi_4(\lambda)f(a,y)\Big) \\
&+ \Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d) - \Phi_4(\lambda)f(a,d)\Big) + \Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c) - \Phi_3(\lambda)f(b,c)\Big) \\
&- \Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s - \Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t \\
&+ \Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t - \Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s + \Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s \\
&+ \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg| \\
&\le \int_a^b\int_c^d |K_1(s,x)||K_2(t,y)|\left|\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\right|\Delta_2 t\Delta_1 s. \qquad (3.10)
\end{aligned}$$

The desired result follows by using the definitions of $K_1(s,x)$ and $K_2(t,y)$ as given in (3.2) and (3.3). ☐

3.1. **Application I.** We now apply Theorem 3.2 to obtain the following results.

Corollary 3.3. *If* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, *then for all* $(x,y) \in [a,b]\times[c,d]$ *with* $K := \sup_{a<s<b;c<t<d}\left|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right| < \infty$, *we have that the inequality in Theorem 3.2 becomes*

$$\begin{aligned}
&\Bigg|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d) - \Phi_6(\lambda)f(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y) - \Phi_4(\lambda)f(a,y)\Big) \\
&+ \Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d) - \Phi_4(\lambda)f(a,d)\Big) + \Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c) - \Phi_3(\lambda)f(b,c)\Big) \\
&- \Phi_1(\lambda)\int_c^d f(x,t)dt - \Phi_2(\lambda)\int_a^b f(s,y)ds - \Phi_3(\lambda)\int_c^d f(b,t)dt \\
&+ \Phi_4(\lambda)\int_c^d f(a,t)dt - \Phi_5(\lambda)\int_a^b f(s,d)ds + \Phi_6(\lambda)\int_a^b f(s,c)ds + \int_a^b\int_c^d f(s,t)dtds\Bigg| \\
&\le K\left[\int_a^x |s - A_1(\lambda)|ds + \int_x^b |s - B_1(\lambda)|ds\right]\left[\int_c^y |t - A_2(\lambda)|dt + \int_y^d |t - B_2(\lambda)|dt\right], \qquad (3.11)
\end{aligned}$$

where $A_i(\lambda)$, $B_i(\lambda)$ *for* $i \in \{1,2\}$ *and* $\Phi_i(\lambda)$ *for* $i \in \{1,2,3,4,5,6\}$ *are all defined in Lemma 3.1.*

Corollary 3.4. *If* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, *then for all* $(x,y) \in [a,b-1]_{\mathbb{Z}}\times[c,d-1]_{\mathbb{Z}}$ *the inequality in Theorem 3.2 becomes*

$$\Bigg|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d) - \Phi_6(\lambda)f(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y) - \Phi_4(\lambda)f(a,y)\Big)$$

$$+\Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d)-\Phi_4(\lambda)f(a,d)\Big)+\Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c)-\Phi_3(\lambda)f(b,c)\Big)$$
$$-\Phi_1(\lambda)\sum_{t=c}^{d-1}f(x,t+1)-\Phi_2(\lambda)\sum_{s=a}^{b-1}f(s+1,y)-\Phi_3(\lambda)\sum_{t=c}^{d-1}f(b,t+1)$$
$$+\Phi_4(\lambda)\sum_{t=c}^{d-1}f(a,t+1)-\Phi_5(\lambda)\sum_{s=a}^{b-1}f(s+1,d)+\Phi_6(\lambda)\sum_{s=a}^{b-1}f(s+1,c)+\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg|$$
$$\leq K\left[\sum_{s=a}^{x-1}|s-A_1(\lambda)|+\sum_{s=x}^{b-1}|s-B_1(\lambda)|\right]\left[\sum_{t=c}^{y-1}|t-A_2(\lambda)|+\sum_{t=y}^{d-1}|t-B_2(\lambda)|\right], \tag{3.12}$$

where $A_i(\lambda),\ B_i(\lambda)$ for $i\in\{1,2\}$ and $\Phi_i(\lambda)$ for $i\in\{1,2,3,4,5,6\}$ are all defined in Lemma 3.1 and K denotes the maximum value of the absolute value of the difference $\Delta_1\Delta_2 f$ over $[a,b-1]_{\mathbb{Z}}\times[c,d-1]_{\mathbb{Z}}$.

Corollary 3.5. *If we assume that $\lambda\in[0,1]$ is such that $a+\psi(\lambda)\frac{b-a}{2},\ a+(1+\psi(1-\lambda))\frac{b-a}{2}\in\mathbb{T}_1,\ c+\psi(\lambda)\frac{d-c}{2},\ c+(1+\psi(1-\lambda))\frac{d-c}{2}\in\mathbb{T}_2$, and $x\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right],\ y\in\left[c+\psi(\lambda)\frac{d-c}{2},c+(1+\psi(1-\lambda))\frac{d-c}{2}\right]$, then Inequality (3.9) can be formulated as follows*

$$\Bigg|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y)+\Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d)-\Phi_6(\lambda)f(x,c)\Big)+\Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y)-\Phi_4(\lambda)f(a,y)\Big)$$
$$+\Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d)-\Phi_4(\lambda)f(a,d)\Big)+\Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c)-\Phi_3(\lambda)f(b,c)\Big)$$
$$-\Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t-\Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s-\Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t$$
$$+\Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t-\Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s+\Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s$$
$$+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg|$$
$$\leq K\Bigg[h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)$$
$$+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\Bigg]\Bigg[h_2\left(c,c+\psi(\lambda)\frac{d-c}{2}\right)+h_2\left(y,c+\psi(\lambda)\frac{d-c}{2}\right)$$
$$+h_2\left(y,c+(1+\psi(1-\lambda))\frac{d-c}{2}\right)+h_2\left(d,c+(1+\psi(1-\lambda))\frac{d-c}{2}\right)\Bigg], \tag{3.13}$$

where $\Phi_i(\lambda)$ for $i\in\{1,2,3,4,5,6\}$ are all defined in Lemma 3.1 and $h_2(\cdot,\cdot)$ is given in Definition 2.5.

Taking $\psi(\lambda)=\lambda$ in the above Corollary 3.5, we obtain

Corollary 3.6. *If we assume that $\lambda\in[0,1]$ is such that $a+\lambda\frac{b-a}{2},\ a+(2-\lambda)\frac{b-a}{2}\in\mathbb{T}_1,\ c+\lambda\frac{d-c}{2},\ c+(2-\lambda)\frac{d-c}{2}\in\mathbb{T}_2$, and $x\in\left[a+\lambda\frac{b-a}{2},a+(2-\lambda)\frac{b-a}{2}\right],\ y\in$*

$\left[c+\lambda\frac{d-c}{2}, c+(2-\lambda)\frac{d-c}{2}\right]$, *then*

$$\Big|\Phi_1(\lambda)\Phi_2(\lambda)f(x,y)+\Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d)-\Phi_6(\lambda)f(x,c)\Big)+\Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y)-\Phi_4(\lambda)f(a,y)\Big)$$
$$+\Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d)-\Phi_4(\lambda)f(a,d)\Big)+\Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c)-\Phi_3(\lambda)f(b,c)\Big)$$
$$-\Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t-\Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s-\Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t$$
$$+\Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t-\Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s+\Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s$$
$$+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big|$$
$$\leq K\left[h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+(2-\lambda)\frac{b-a}{2}\right)\right.$$
$$\left.+h_2\left(b,a+(2-\lambda)\frac{b-a}{2}\right)\right]\left[h_2\left(c,c+\lambda\frac{d-c}{2}\right)+h_2\left(y,c+\lambda\frac{d-c}{2}\right)\right.$$
$$\left.+h_2\left(y,c+(2-\lambda)\frac{d-c}{2}\right)+h_2\left(d,c+(2-\lambda)\frac{d-c}{2}\right)\right]. \tag{3.14}$$

Here $\Phi_i(\lambda)$ *for* $i\in\{1,2,3,4,5,6\}$ *are all defined in Lemma 3.1 with* $\psi(\lambda)=\lambda$.

Remark. *If we interchange the roles of* s *and* t *and set* $\lambda=0$, *then Corollary 3.6 reduces to Theorem 1.3 with* $\sigma_1=\sigma_2=\sigma$.

We now present another result in same direction.

Theorem 3.7. *Let* $a,b,s\in\mathbb{T}_1$, $c,d,t\in\mathbb{T}_2$, *with* $a<b$, $c<d$, $f,g:[a,b]\times[c,d]\to\mathbb{R}$ *be* $\Delta_1\Delta_2$ *differentiable,* ψ *be a function of* $[0,1]$ *into* $[0,1]$. *Assume also that* $K:=\sup_{a<s<b;c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\right|<\infty$ *and* $L:=\sup_{a<s<b;c<t<d}\left|\frac{\partial^2 g(s,t)}{\Delta_1 s\Delta_2 t}\right|<\infty$. *Then for all* $(x,y)\in[a,b]\times[c,d]$, *we have the following inequality*

$$\Big|2\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$$
$$+\Phi_1(\lambda)\Phi_2(\lambda)\left[f(x,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s+g(x,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right]$$
$$+\Phi_1(\lambda)\Phi_5(\lambda)\left[f(x,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s+g(x,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right]$$
$$-\Phi_1(\lambda)\Phi_6(\lambda)\left[f(x,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s+g(x,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right]$$
$$+\Phi_2(\lambda)\Phi_3(\lambda)\left[f(b,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s+g(b,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right]$$
$$-\Phi_2(\lambda)\Phi_4(\lambda)\left[f(a,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s+g(a,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right]$$

$$
\begin{aligned}
&+ \Phi_3(\lambda)\Phi_5(\lambda)\Big[f(b,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(b,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_4(\lambda)\Phi_5(\lambda)\Big[f(a,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(a,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&+ \Phi_4(\lambda)\Phi_6(\lambda)\Big[f(a,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(a,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_3(\lambda)\Phi_6(\lambda)\Big[f(b,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(b,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_1(\lambda)\Big[\int_c^d f(x,\sigma(t))\Delta_2 t\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(x,\sigma(t))\Delta_2 t\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_2(\lambda)\Big[\int_a^b f(\sigma(s),y)\Delta_1 s\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),y)\Delta_1 s\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_3(\lambda)\Big[\int_c^d f(b,\sigma(t))\Delta_2 t\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(b,\sigma(t))\Delta_2 t\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&+ \Phi_4(\lambda)\Big[\int_c^d f(a,\sigma(t))\Delta_2 t\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(a,\sigma(t))\Delta_2 t\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_5(\lambda)\Big[\int_a^b f(\sigma(s),d)\Delta_1 s\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),d)\Delta_1 s\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&+ \Phi_6(\lambda)\Big[\int_a^b f(\sigma(s),c)\Delta_1 s\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),c)\Delta_1 s\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\Bigg|\\
&\le \Big[K\int_a^b\int_c^d |g(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s + L\int_a^b\int_c^d |f(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s\Big] \qquad (3.15)\\
&\times \Big[\int_a^x |s-A_1(\lambda)|\Delta_1 s + \int_x^b |s-B_1(\lambda)|\Delta_1 s\Big]\Big[\int_c^y |t-A_2(\lambda)|\Delta_2 t + \int_y^d |t-B_2(\lambda)|\Delta_2 t\Big],
\end{aligned}
$$

where $A_i(\lambda)$, $B_i(\lambda)$ for $i \in \{1,2\}$ and $\Phi_i(\lambda)$ for $i \in \{1,2,3,4,5,6\}$ are all defined in Lemma 3.1.

Proof. By Lemma 3.1, we have

$$
\begin{aligned}
&\Phi_1(\lambda)\Phi_2(\lambda)f(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)f(x,d) - \Phi_6(\lambda)f(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)f(b,y) - \Phi_4(\lambda)f(a,y)\Big)\\
&+ \Phi_5(\lambda)\Big(\Phi_3(\lambda)f(b,d) - \Phi_4(\lambda)f(a,d)\Big) + \Phi_6(\lambda)\Big(\Phi_4(\lambda)f(a,c) - \Phi_3(\lambda)f(b,c)\Big)\\
&- \Phi_1(\lambda)\int_c^d f(x,\sigma(t))\Delta_2 t - \Phi_2(\lambda)\int_a^b f(\sigma(s),y)\Delta_1 s - \Phi_3(\lambda)\int_c^d f(b,\sigma(t))\Delta_2 t\\
&+ \Phi_4(\lambda)\int_c^d f(a,\sigma(t))\Delta_2 t - \Phi_5(\lambda)\int_a^b f(\sigma(s),d)\Delta_1 s + \Phi_6(\lambda)\int_a^b f(\sigma(s),c)\Delta_1 s\\
&+ \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&= \int_a^b\int_c^d K_1(s,x)K_2(t,y)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s, \qquad (3.16)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\Phi_1(\lambda)\Phi_2(\lambda)g(x,y) + \Phi_1(\lambda)\Big(\Phi_5(\lambda)g(x,d) - \Phi_6(\lambda)g(x,c)\Big) + \Phi_2(\lambda)\Big(\Phi_3(\lambda)g(b,y) - \Phi_4(\lambda)g(a,y)\Big)\\
&+ \Phi_5(\lambda)\Big(\Phi_3(\lambda)g(b,d) - \Phi_4(\lambda)g(a,d)\Big) + \Phi_6(\lambda)\Big(\Phi_4(\lambda)g(a,c) - \Phi_3(\lambda)g(b,c)\Big)\\
&- \Phi_1(\lambda)\int_c^d g(x,\sigma(t))\Delta_2 t - \Phi_2(\lambda)\int_a^b g(\sigma(s),y)\Delta_1 s - \Phi_3(\lambda)\int_c^d g(b,\sigma(t))\Delta_2 t\\
&+ \Phi_4(\lambda)\int_c^d g(a,\sigma(t))\Delta_2 t - \Phi_5(\lambda)\int_a^b g(\sigma(s),d)\Delta_1 s + \Phi_6(\lambda)\int_a^b g(\sigma(s),c)\Delta_1 s\\
&+ \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&= \int_a^b\int_c^d K_1(s,x)K_2(t,y)\frac{\partial^2 g(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s, \qquad (3.17)
\end{aligned}
$$

Now, multiplying both sides of (3.16) by $\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$ and (3.17) by $\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s$, adding the resulting equations and taking absolute values, we get

$$
\begin{aligned}
&\Bigg|2\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&+ \Phi_1(\lambda)\Phi_2(\lambda)\Bigg[f(x,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(x,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&+ \Phi_1(\lambda)\Phi_5(\lambda)\Bigg[f(x,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(x,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&- \Phi_1(\lambda)\Phi_6(\lambda)\Bigg[f(x,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(x,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&+ \Phi_2(\lambda)\Phi_3(\lambda)\Bigg[f(b,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(b,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&- \Phi_2(\lambda)\Phi_4(\lambda)\Bigg[f(a,y)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(a,y)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&+ \Phi_3(\lambda)\Phi_5(\lambda)\Bigg[f(b,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(b,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&- \Phi_4(\lambda)\Phi_5(\lambda)\Bigg[f(a,d)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(a,d)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&+ \Phi_4(\lambda)\Phi_6(\lambda)\Bigg[f(a,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(a,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&- \Phi_3(\lambda)\Phi_6(\lambda)\Bigg[f(b,c)\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + g(b,c)\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]\\
&- \Phi_1(\lambda)\Bigg[\int_c^d f(x,\sigma(t))\Delta_2 t\int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(x,\sigma(t))\Delta_2 t\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&- \Phi_2(\lambda)\Big[\int_a^b f(\sigma(s),y)\Delta_1 s \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),y)\Delta_1 s \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_3(\lambda)\Big[\int_c^d f(b,\sigma(t))\Delta_2 t \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(b,\sigma(t))\Delta_2 t \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&+ \Phi_4(\lambda)\Big[\int_c^d f(a,\sigma(t))\Delta_2 t \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_c^d g(a,\sigma(t))\Delta_2 t \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&- \Phi_5(\lambda)\Big[\int_a^b f(\sigma(s),d)\Delta_1 s \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),d)\Delta_1 s \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\\
&+ \Phi_6(\lambda)\Big[\int_a^b f(\sigma(s),c)\Delta_1 s \int_a^b\int_c^d g(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s + \int_a^b g(\sigma(s),c)\Delta_1 s \int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Big]\Bigg|\\
&\leq \left[K\int_a^b\int_c^d |g(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s + L\int_a^b\int_c^d |f(\sigma(s),\sigma(t))|\Delta_2 t\Delta_1 s\right] \times \int_a^b\int_c^d |K_1(s,x)||K_2(t,y)|\Delta_2 t\Delta_1 s.
\end{aligned}
\tag{3.18}
$$

The desired result follows by using the definitions of $K_1(s,x)$ and $K_2(t,y)$ as given in (3.2) and (3.3). □

3.2. **Application II.** In this section, we apply Theorem 3.7 to the real and discrete case.

Corollary 3.8. *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 3.7, then we obtain the inequality*

$$
\begin{aligned}
&\Bigg|2\int_a^b\int_c^d f(s,t)dtds\int_a^b\int_c^d g(s,t)dtds\\
&+ \Phi_1(\lambda)\Phi_2(\lambda)\left[f(x,y)\int_a^b\int_c^d g(s,t)dtds + g(x,y)\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_1(\lambda)\Phi_5(\lambda)\left[f(x,d)\int_a^b\int_c^d g(s,t)dtds + g(x,d)\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_1(\lambda)\Phi_6(\lambda)\left[f(x,c)\int_a^b\int_c^d g(s,t)dtds + g(x,c)\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_2(\lambda)\Phi_3(\lambda)\left[f(b,y)\int_a^b\int_c^d g(s,t)dtds + g(b,y)\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_2(\lambda)\Phi_4(\lambda)\left[f(a,y)\int_a^b\int_c^d g(s,t)dtds + g(a,y)\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_3(\lambda)\Phi_5(\lambda)\left[f(b,d)\int_a^b\int_c^d g(s,t)dtds + g(b,d)\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_4(\lambda)\Phi_5(\lambda)\left[f(a,d)\int_a^b\int_c^d g(s,t)dtds + g(a,d)\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_4(\lambda)\Phi_6(\lambda)\left[f(a,c)\int_a^b\int_c^d g(s,t)dtds + g(a,c)\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_3(\lambda)\Phi_6(\lambda)\left[f(b,c)\int_a^b\int_c^d g(s,t)dtds + g(b,c)\int_a^b\int_c^d f(s,t)dtds\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \Phi_1(\lambda)\left[\int_c^d f(x,t)dt \int_a^b\int_c^d g(s,t)dtds + \int_c^d g(x,t)dt\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_2(\lambda)\left[\int_a^b f(s,y)ds \int_a^b\int_c^d g(s,t)dtds + \int_a^b g(s,y)ds\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_3(\lambda)\left[\int_c^d f(b,t)dt \int_a^b\int_c^d g(s,t)dtds + \int_c^d g(b,t)dt\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_4(\lambda)\left[\int_c^d f(a,t)dt \int_a^b\int_c^d g(s,t)dtds + \int_c^d g(a,t)dt\int_a^b\int_c^d f(s,t)dtds\right]\\
&- \Phi_5(\lambda)\left[\int_a^b f(s,d)ds \int_a^b\int_c^d g(s,t)dtds + \int_a^b g(s,d)ds\int_a^b\int_c^d f(s,t)dtds\right]\\
&+ \Phi_6(\lambda)\left[\int_a^b f(s,c)ds \int_a^b\int_c^d g(s,t)dtds + \int_a^b g(s,c)ds\int_a^b\int_c^d f(s,t)dtds\right]\Bigg|\\
&\le \left[K\int_a^b\int_c^d |g(s,t)|dtds + L\int_a^b\int_c^d |f(s,t)|dtds\right]\\
&\times\left[\int_a^x |s-A_1(\lambda)|ds + \int_x^b |s-B_1(\lambda)|ds\right]\left[\int_c^y |t-A_2(\lambda)|dt + \int_y^d |t-B_2(\lambda)|dt\right],
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $K := \sup_{a<s<b;c<t<d}\left|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right|$ *and* $L := \sup_{a<s<b;c<t<d}\left|\frac{\partial^2 g(s,t)}{\partial s\partial t}\right|$.

Corollary 3.9. *If* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ *in Theorem 3.7, then we obtain the inequality*

$$
\begin{aligned}
&\Bigg|2\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1)\\
&+ \Phi_1(\lambda)\Phi_2(\lambda)\left[f(x,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(x,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&+ \Phi_1(\lambda)\Phi_5(\lambda)\left[f(x,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(x,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&- \Phi_1(\lambda)\Phi_6(\lambda)\left[f(x,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(x,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&+ \Phi_2(\lambda)\Phi_3(\lambda)\left[f(b,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(b,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&- \Phi_2(\lambda)\Phi_4(\lambda)\left[f(a,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(a,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&+ \Phi_3(\lambda)\Phi_5(\lambda)\left[f(b,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(b,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]\\
&- \Phi_4(\lambda)\Phi_5(\lambda)\left[f(a,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} g(s+1,t+1) + g(a,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1} f(s+1,t+1)\right]
\end{aligned}
$$

$$+\Phi_4(\lambda)\Phi_6(\lambda)\Bigg[f(a,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+g(a,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$-\Phi_3(\lambda)\Phi_6(\lambda)\Bigg[f(b,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+g(b,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$-\Phi_1(\lambda)\Bigg[\sum_{t=c}^{d-1}f(x,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{t=c}^{d-1}g(x,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$-\Phi_2(\lambda)\Bigg[\sum_{s=a}^{b-1}f(s+1,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{s=a}^{b-1}g(s+1,y)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$-\Phi_3(\lambda)\Bigg[\sum_{t=c}^{d-1}f(b,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{t=c}^{d-1}g(b,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$+\Phi_4(\lambda)\Bigg[\sum_{t=c}^{d-1}f(a,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{t=c}^{d-1}g(a,t+1)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$-\Phi_5(\lambda)\Bigg[\sum_{s=a}^{b-1}f(s+1,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{s=a}^{b-1}g(s+1,d)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]$$
$$+\Phi_6(\lambda)\Bigg[\sum_{s=a}^{b-1}f(s+1,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}g(s+1,t+1)+\sum_{s=a}^{b-1}g(s+1,c)\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}f(s+1,t+1)\Bigg]\Bigg|$$
$$\leq\left[K\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}|g(s+1,t+1)|+L\sum_{s=a}^{b-1}\sum_{t=c}^{d-1}|f(s+1,t+1)|\right]$$
$$\times\left[\sum_{s=a}^{x-1}|s-A_1(\lambda)|+\sum_{s=x}^{b-1}|s-B_1(\lambda)|\right]\left[\sum_{t=c}^{y-1}|t-A_2(\lambda)|+\sum_{t=y}^{d-1}|t-B_2(\lambda)|\right],$$

for all $(x,y)\in[a,b-1]\times[c,d-1]$*, where* K *denotes the maximum value of the absolute value of the difference* $\Delta_2\Delta_1 f$ *over* $[a,b-1]_{\mathbb{Z}}\times[c,d-1]_{\mathbb{Z}}$ *and* L *denotes the maximum value of the absolute value of the difference* $\Delta_2\Delta_1 g$ *over* $[a,b-1]_{\mathbb{Z}}\times[c,d-1]_{\mathbb{Z}}$.

Remark. *More results can be obtained from Theorem 3.7 by taking* $\psi(\lambda)=\lambda$ *or* $\psi(\lambda)=\lambda^2$ *and then consider different values of* λ *in* $[0,1]$*, for example,* $\lambda=0,1/2$ *and* 1

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