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AN EXTENSION OF JENSEN-MERCER INEQUALITY FOR FUNCTIONS WITH NONDECREASING INCREMENTS

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Abstract. We present an extension of Jensen-Mercer inequality for functions with nondecreasing increments along with its refinements. A Mercer type variant is given for the Burkill-Mirsky-Pečarić's result and related inequalities for the finite sequence of $k-$tuples monotone in means.

1. Introduction and Preliminaries

In order to recall the definition of functions with nondecreasing increments we adopt the following notations and assumptions: Throughout this article we suppose that $\mathbf{U}$ is an interval in a $k-$dimensional vector lattice $\mathbb{R}^k$ and for real weights $w_1, \ldots, w_n$ we define $W_i = \sum_{j=1}^{i} w_j,\ i \in \{1, \ldots, n\}$ with $W_n = \sum_{j=1}^{n} w_j$. Let us define the partial ordering for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ is given as $\mathbf{x} = (x_1, \ldots, x_k) \leq \mathbf{y} = (y_1, \ldots, y_k)$ if and only if $x_i \leq y_i$,where x_i, y_i are real for each $i \in \{1, \ldots, k\}$. Furthermore a set $\{\mathbf{z} \in \mathbb{R}^k : \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\}$ is called an *interval* $[\mathbf{x}, \mathbf{y}]$ and the $k-$tuple $(0, \ldots, 0)$ is denoted by $\mathbf{0}$. Also for any $r, s \in \mathbb{R}$

$$r\mathbf{x} + s\mathbf{y} = (rx_1 + sy_1, \ldots, rx_k + sy_k).$$

In 1964 Brunk [2] introduced the notion of functions with nondecreasing increments and gave the following definition:

Definition 1. *A function* $f : \mathbf{U} \to \mathbb{R}$ *is said to have* nondecreasing increments *if*

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \leq f(\mathbf{b} + \mathbf{h}) - f(\mathbf{b}) \tag{1.1}$$

holds for each $\mathbf{a}, \mathbf{b} + \mathbf{h} \in \mathbf{U}$, $0 \leq \mathbf{h} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$.

Brunk observed that even if $k = 1$, inequality (1.1) does not imply continuity. It is of interest to note that such a function is convex along positively oriented lines, i.e., lines whose direction cosines are nonnegative, with equations of the form $\mathbf{x} = \mathbf{a}t + \mathbf{b}$ where $\mathbf{0} \leq \mathbf{a}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Moreover, in one dimension case these functions are known as *Wright-convex.* The class of Wright-convex functions is properly contained in class of $J-$convex functions and it properly contains the class of convex functions.

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All continuous functions with nondecreasing increments have a useful property: the well known Jensen-Steffensen inequality holds for them (see [9]). Jensen's inequality for functions with nondecreasing increments can be obtained as a special case of the Jensen-Steffensen inequality, but it is also a simple consequence of its integral variant proved in [2].
In [1] following important results were stated for functions with nondecreasing increments:

Theorem 1.1. *Let $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments, let $\mathbf{w}$ be a nonnegative n-tuple such that $W_n > 0$ and let $\xi^{(i)} \in \mathbf{U}, i \in \{1, \ldots, n\}$ be such that*

$$\xi^{(1)} \leq \cdots \leq \xi^{(n)} \quad \textbf{or} \quad \xi^{(1)} \geq \cdots \geq \xi^{(n)},$$

then

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \xi^{(i)}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\xi^{(i)}\right). \tag{1.2}$$

Remark 1. *Let f be as in Theorem 1.1 and let $\mathbf{w}$ be a real n-tuple such that*

$$w_1 > 0, w_i \leq 0, i \in \{2, \ldots, n\}, W_n > 0.$$

Let $\xi^{(i)}$ for $i \in \{1, \ldots, n\}$ be a sequence of k-tuples in $\mathbf{U}$ such that

$$\xi^{(1)} \leq \cdots \leq \xi^{(n)} \quad \textbf{or} \quad \xi^{(1)} \geq \cdots \geq \xi^{(n)},$$

and

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \xi^{(i)} \in \mathbf{U},$$

then the reversed inequality in (1.2) is valid.

Definition 2. *Let $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments and $\mathbf{a}$ and $\mathbf{b}$ be two specially chosen k-tuples in $\mathbf{U}$. Then for any $\mathbf{x} \in \mathbf{U}$ such that $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$*

$$f(\mathbf{a} + \mathbf{b} - \mathbf{x}) \leq f(\mathbf{a}) + f(\mathbf{b}) - f(\mathbf{x}). \tag{1.3}$$

In the same article [1], the following variant of Theorem 1.1 was proved. We will refer to this inequality as Jensen-Mercer inequality for functions with nondecreasing increments.

Theorem 1.2. *Let $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments and let $\xi^{(i)} \in \mathbf{U}, i \in \{1, \ldots, n\}$ satisfy the condition*

$$\xi^{(1)} \leq \cdots \leq \xi^{(n)} \quad \textbf{or} \quad \xi^{(1)} \geq \cdots \geq \xi^{(n)}$$

if $\mathbf{w}$ is a nonnegative n-tuple such that $W_n > 0$, then

$$f\left(\mathbf{a} + \mathbf{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \xi^{(i)}\right) \leq f(\mathbf{a}) + f(\mathbf{b}) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\xi^{(i)}), \tag{1.4}$$

where $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ are two specially chosen k-tuples related to $\mathbf{U}$. If $\mathbf{w}$ is a real n-tuple such that

$$w_1 > 0, w_i \leq 0, i \in \{2, \ldots, n\}, W_n > 0, \tag{1.5}$$

and if

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \xi^{(i)} \in \mathbf{U}. \tag{1.6}$$

Then the inequality (1.4) *remains valid.*

Now, let us state definition of majorization from [6] as follows. Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ denote two m-tuples and $x_{[1]} \geq \cdots \geq x_{[m]}, \quad y_{[1]} \geq \cdots \geq y_{[m]}$ be their ordered components.

Definition 3. *For* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$*,*

$$\mathbf{x} \prec \mathbf{y} \quad \textit{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} & , \quad k \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i & , \quad k = m. \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$*,* $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ *or* $\mathbf{y}$ majorizes $\mathbf{x}$.

This notion and notation of majorization was first introduced by Hardy et al. in [3]. We can find the well-known majorization theorem in the same book [11]. In order to use majorization for functions with nondecreasing increments let us recall partial ordering from beginning as $\mathbf{x} = (x_1, \ldots, x_k) \leq \mathbf{y} = (y_1, \ldots, y_k)$ if and only if $x_i \leq y_i$ for each $i \in \{1, \ldots, k\}$ where $\mathbf{x}$ and $\mathbf{y}$ are two $k-$dimensional vector lattice of points in $\mathbf{U}$.
Now for this partial ordering the following majorizaion can be defined. Let two m-tuples $\acute{\mathbf{x}} = \left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right)$ and $\acute{\mathbf{y}} = \left(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(m)}\right)$ with ordered components $\mathbf{x}_{[1]} \geq \cdots \geq \mathbf{x}_{[m]}, \quad \mathbf{y}_{[1]} \geq \cdots \geq \mathbf{y}_{[m]}$ such that every $\mathbf{x}^{(i)}, \mathbf{y}^{(i)} \in \mathbf{U} \subset \mathbb{R}^k$ is a k-tuple for $i \in \{1, \ldots, m\}$.

Definition 4.

$$\acute{\mathbf{x}} \prec \acute{\mathbf{y}} \quad \textit{if} \quad \begin{cases} \sum_{i=1}^{l} \mathbf{x}_{[i]} \leq \sum_{i=1}^{l} \mathbf{y}_{[i]} & , \quad l \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^{m} \mathbf{x}_i = \sum_{i=1}^{m} \mathbf{y}_i & , \quad l = m. \end{cases}$$

We now go through an interesting result given by M. Niezgoda in [8], which is actually an extension of Jensen-Mercer inequality [7] for convex functions.

Theorem 1.3. *Suppose that* $\mathbf{a} = (a_1, \ldots, a_m)$ *be an* m*-tuple with* $a_i \in J \subset \mathbb{R}$ *and* $\mathbf{X} = (\mathbf{x}_j) = (x_{ij})$ *is a real* $n \times m$ *matrix such that* $x_{ij} \in J$*,* $\forall i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$*. Let* f *be continuous convex function on* J*. If* $\mathbf{a}$ *majorizes each row of* $\mathbf{X}$*, that is,*

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \mathbf{a} \textit{ for each } i \in \{1, \ldots, n\},$$

then we have inequality

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}), \tag{1.7}$$

where $\sum_{i=1}^{n} w_i = 1$ *with* $w_i \geq 0$*.*

The following Lemma will play a key role in our main results.

Lemma 1.4. *Suppose that $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments. Let for any real m, we have elements $\acute{\mathbf{a}} = \left(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}\right)$ and $\acute{\mathbf{x}} = \left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right)$ in a way that every $\mathbf{a}^{(i)}, \mathbf{x}^{(i)} \in \mathbf{U}$ is a k-tuple for $i \in \{1, \ldots, m\}$. If $\acute{\mathbf{a}}$ majorizes $\acute{\mathbf{x}}$ with $\mathbf{a}^{(i)} = \mathbf{a}_{[i]}$ and $\mathbf{x}^{(i)} = \mathbf{x}_{[i]}$ for $i \in \{1, \ldots, m\}$, then*

$$f\left(\sum_{i=1}^{m} \mathbf{a}^{(i)} - \sum_{i=1}^{m-1} \mathbf{x}^{(i)}\right) \leq \sum_{i=1}^{m} f\left(\mathbf{a}^{(i)}\right) - \sum_{i=1}^{m-1} f\left(\mathbf{x}^{(i)}\right) \tag{1.8}$$

is valid.

Proof. If $\acute{\mathbf{a}} = \left(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}\right)$ majorizes $\acute{\mathbf{x}} = \left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right)$ and $\mathbf{a}_{[i]} = \mathbf{a}^{(i)}$, $\mathbf{x}_{[i]} = \mathbf{x}^{(i)}$ for $i \in \{1, \ldots, m\}$, then

$$\sum_{i=1}^{m-1} \mathbf{x}^{(i)} \leq \sum_{i=1}^{m-1} \mathbf{a}^{(i)},$$

which leads to,

$$\mathbf{x}^{(m-1)} \leq \sum_{i=1}^{m-1} \mathbf{a}^{(i)} - \sum_{i=1}^{m-2} \mathbf{x}^{(i)}$$

obviously

$$\mathbf{a}^{(m)} \leq \mathbf{x}^{(m-1)},$$

then we have,

$$\mathbf{a}^{(m)} \leq \mathbf{x}^{(m-1)} \leq \sum_{i=1}^{m-1} \mathbf{a}^{(i)} - \sum_{i=1}^{m-2} \mathbf{x}^{(i)}. \tag{1.9}$$

Now consider

$$f\left(\sum_{i=1}^{m} \mathbf{a}^{(i)} - \sum_{i=1}^{m-1} \mathbf{x}^{(i)}\right) = f\left[\left(\sum_{i=1}^{m-1} \mathbf{a}^{(i)} - \sum_{i=1}^{m-2} \mathbf{x}^{(i)}\right) + \mathbf{a}^{(m)} - \mathbf{x}^{(m-1)}\right]$$

applying (1.3) by taking $\mathbf{a} := \mathbf{a}^{(m)}$, $\mathbf{b} := \left(\sum_{i=1}^{m-1} \mathbf{a}^{(i)} - \sum_{i=1}^{m-2} \mathbf{x}^{(i)}\right)$ and $\mathbf{x} := \mathbf{x}^{(m-1)}$ where $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$ by (1.9), we get,

$$f\left(\sum_{i=1}^{m} \mathbf{a}^{(i)} - \sum_{i=1}^{m-1} \mathbf{x}^{(i)}\right) \leq f\left(\sum_{i=1}^{m-1} \mathbf{a}^{(i)} - \sum_{i=1}^{m-2} \mathbf{x}^{(i)}\right) + f\left(\mathbf{a}^{(m)}\right) - f\left(\mathbf{x}^{(m-1)}\right)$$

$$= f\left[\left(\sum_{i=1}^{m-2} \mathbf{a}^{(i)} - \sum_{i=1}^{m-3} \mathbf{x}^{(i)}\right) + \mathbf{a}^{(m-1)} - \mathbf{x}^{(m-2)}\right] + f\left(\mathbf{a}^{(m)}\right) - f\left(\mathbf{x}^{(m-1)}\right)$$

again applying (1.3) by taking $\mathbf{a} := \mathbf{a}^{(m-1)}$, $\mathbf{b} := \left(\sum_{i=1}^{m-2} \mathbf{a}^{(i)} - \sum_{i=1}^{m-3} \mathbf{x}^{(i)}\right)$ and $\mathbf{x} := \mathbf{x}^{(m-2)}$ we get,

$$\begin{aligned} f\left(\sum_{i=1}^{m} \mathbf{a}^{(i)} - \sum_{i=1}^{m-1} \mathbf{x}^{(i)}\right) \quad \leq \quad & f\left(\sum_{i=1}^{m-2} \mathbf{a}^{(i)} - \sum_{i=1}^{m-3} \mathbf{x}^{(i)}\right) \\ & + f\left(\mathbf{a}^{(m-1)}\right) - f\left(\mathbf{x}^{(m-2)}\right) + f\left(\mathbf{a}^{(m)}\right) - f\left(\mathbf{x}^{(m-1)}\right) \end{aligned}$$

continuing in this way we finally get,

$$f\left(\sum_{i=1}^{m}\mathbf{a}^{(i)}-\sum_{i=1}^{m-1}\mathbf{x}^{(i)}\right)\leq f\left(\mathbf{a}^{(1)}+\mathbf{a}^{(2)}-\mathbf{x}^{(1)}\right)+\sum_{i=3}^{m}\left(\mathbf{a}^{(i)}\right)-\sum_{i=2}^{m-1}\left(\mathbf{x}^{(i)}\right)$$

which clearly summarizes the proof under (1.3) as,

$$f\left(\sum_{i=1}^{m}\mathbf{a}^{(i)}-\sum_{i=1}^{m-1}\mathbf{x}^{(i)}\right)\leq \sum_{i=1}^{m}f\left(\mathbf{a}^{(i)}\right)-\sum_{i=1}^{m-1}f\left(\mathbf{x}^{(i)}\right).$$

□

In [5] we can find the following expressions for functions with nondecreasing increments of order n.
If we write $\Delta_{\mathbf{h_1}}f(\mathbf{x})=f(\mathbf{x}+\mathbf{h_1})-f(\mathbf{x})$ and inductively,

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}\cdots\Delta_{\mathbf{h_n}}f(\mathbf{x})=\Delta_{\mathbf{h_1}}(\Delta_{\mathbf{h_2}}\cdots\Delta_{\mathbf{h_n}}f(\mathbf{x}))\quad\text{for}\quad n\geq 2,$$

where $\mathbf{x},\mathbf{x}+\mathbf{h_1}+\cdots+\mathbf{h_n}\in\mathbf{U}$, $\mathbf{h_i}\in\mathbb{R}_*^k$ for $i\in\{1,\ldots,n\}$, where $\mathbb{R}_*^k$ denotes set of nonnegative $\mathbb{R}^k$. Using this notation with $\mathbf{h}=\mathbf{h_1}$, $\mathbf{s}=\mathbf{h_2}$, $\mathbf{b}=\mathbf{a}+\mathbf{s}$, condition (1.1) becomes

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}f(\mathbf{a})\geq 0.$$

The Definition 1 can also to be extended to the following form.

Definition 5. *Let* $f:\mathbf{U}\to\mathbb{R}$ *is said to be a* function with nondecreasing increments of order n *if*

$$\Delta_{\mathbf{h_1}}\cdots\Delta_{\mathbf{h_n}}f(\mathbf{x})\geq 0$$

holds, whenever $\mathbf{x},\mathbf{x}+\mathbf{h_1}+\cdots+\mathbf{h_n}\in\mathbf{U}$*,* $\mathbf{h_i}\in\mathbb{R}_*^k$ *for* $i\in\{1,\ldots,n\}$*.*

Now, let us describe an interesting concept of monotonicity in means (see [10]).

Definition 6. *A finite sequence* $(\mathbf{X_1},\ldots,\mathbf{X_n})\in\mathbf{U}^n$ *is said to be* nondecreasing in means *with respect to weights* $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbb{R}_+^n$ *if the inequalities*

$$\mathbf{X_1}\leq A_2(\mathbf{X};\mathbf{w})\leq\cdots\leq A_n(\mathbf{X};\mathbf{w})\tag{1.10}$$

hold, where

$$A_j(\mathbf{X};\mathbf{w})=\frac{1}{W_j}\sum_{i=1}^{j}w_i\mathbf{X_i},\qquad W_j=\sum_{i=1}^{j}w_i.$$

If the directions of inequalities are reversed in (1.10)*, then the sequence* $(\mathbf{X_1},\ldots,\mathbf{X_n})$ *is said to be* nonincreasing in means.

The following result gives us a Jensen type inequality for functions with nondecreasing increments when the finite sequence of $k-$tuples $(\mathbf{X_1},\ldots,\mathbf{X_n})$ is monotone in means. This is in fact a Pečarić's generalization of Burkill-Mirsky's result which we refer as to Burkill-Mirsky-Pečarić's result (see [10]).

Theorem 1.5. *Let* $f:\mathbf{U}\to\mathbb{R}$ *be a continuous function with nondecreasing increments and* $\mathbf{w}=(w_1,\ldots,w_n)\in\mathbb{R}_+^n$*. If* $(\mathbf{X_1},\ldots,\mathbf{X_n})\in\mathbf{U}^n$ *is nondecreasing or nonincreasing in means with respect to weights* w_i *for* $i\in\{1,\ldots,n\}$*, then the inequality*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\mathbf{X_i}\right)\leq\frac{1}{W_n}\sum_{i=1}^{n}w_if(\mathbf{X_i})\tag{1.11}$$

holds.

In present article, we give extension of Jensen-Mercer inequality for functions with nondecreasing increments in the form of Neizgoda's inequality. We also give some refinements of our results in such a way that many results of [1] become special cases of our results. In addition, we will prove Mercer type variant of Burkill-Mirsky-Pečarić's result for the finite sequence of $k-$tuples $(\mathbf{X_1}, \dots, \mathbf{X_n})$, monotone in means.
This article is organized in following manner. The first section is devoted to preliminaries and introduction. In second section we give extension of Jensen-Mercer inequality for functions with nondecreasing increments in the form of Neizgoda's inequality. In third section we use index set functions to give various refinements of result proved in the second section. And the fourth section we define Mercer type variant for the Burkill-Mirsky-Pečarić's result and give related inequalities.

2. Neizgoda's Inequality For Functions With Nondecreasing Increments

Theorem 2.1. *Suppose that $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments. Let there are m elements in $\mathbf{\acute{a}} = \left(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}\right)$ with $\mathbf{a}^{(j)} \in \mathbf{U}$ for $j \in \{1, \dots, m\}$. Also there are $n \times m$ elements represented as $\mathbf{x}^{(ij)}$ such that every $\mathbf{x}^{(ij)}$ is a k-tuple for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$ and all these $\mathbf{x}^{(ij)}$ can be arranged in a $n \times m$ matrix $\mathbf{X} = (\mathbf{x}^{(ij)})$. Let $\mathbf{w}$ be a nonnegative n-tuple such that $W_n > 0$ for $i \in \{1, \dots, n\}$ and If $\mathbf{\acute{a}}$ majorizes each row of $\mathbf{X}$, that is,*

$$\mathbf{x}^{(i.)} = \left(\mathbf{x}^{(i1)}, \dots, \mathbf{x}^{(im)}\right) \prec \left(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}\right) = \mathbf{\acute{a}} \text{ for each } i \in \{1, \dots, n\},$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \mathbf{x}^{(ij)}\right) \leq \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right), \tag{2.1}$$

this inequality also holds for $\mathbf{w}$ satisfying (1.5) *and* (1.6).

Proof. Let $\mathbf{w}$ be a nonnegative n-tuple with $W_n > 0$, we have

$$\begin{aligned} & f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \mathbf{x}^{(ij)}\right) \\ & = f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)}\right)\right). \end{aligned}$$

Now using majorization for each $i \in \{1, \dots, n\}$

$$\sum_{j=1}^{m} \mathbf{a}^{(j)} = \sum_{j=1}^{m} \mathbf{x}^{(ij)}$$

we take

$$\mathbf{x}^{(im)} = \sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)} = \xi^{(i)}$$

let for $i \in \{1, \ldots, n\}$

$$\xi^{(1)} \leq \cdots \leq \xi^{(n)} \quad \text{or} \quad \xi^{(1)} \geq \cdots \geq \xi^{(n)},$$

therefore from Theorem 1.1 we get

$$f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \mathbf{x}^{(ij)}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(ij)}\right).$$

Now by using Lemma 1.4 for each $i \in \{1, \ldots, n\}$ we obtain

$$\begin{aligned} & f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \mathbf{x}^{(ij)}\right) \\ & \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \sum_{j=1}^{m-1} f\left(\mathbf{x}^{(ij)}\right)\right] \\ & = \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right) \end{aligned}$$

which proves (2.1)
Now if $\mathbf{w}$ is a real n-tuple satisfying (1.5) and (1.6), then from Lemma 1.4 and Remark 1, we proceed as

$$\begin{aligned} & f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \mathbf{x}^{(ij)}\right) \\ & \leq \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - f\left(\frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=1}^{m-1} w_i \mathbf{x}^{(ij)}\right) \\ & \leq \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right). \end{aligned}$$

which completes the proof. □

Remark 2. *By putting* $m = 2$ *,* $\mathbf{a}^{(1)} = \mathbf{a}, \mathbf{a}^{(2)} = \mathbf{b}$ *with* $\mathbf{a}^{(1)} \leq \mathbf{a}^{(2)}$ *and* $\mathbf{x}^{(i1)} = \mathbf{x}^{(i)}$ *we get* (1.4) *as a special case of our result.*

3. Index Set Functions and Refinements of Niezgoda's Inequality For Functions With Nondecreasing Increments

Let all the assumptions of Theorem 2.1 are valid and for a nonempty finite set of positive integers I, let $\mathbf{w} = \{w_i | i \in I\}$ be a real sequence such that $W_I = \sum_{i\in I} w_i$ and $\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \mathbf{x}^{(ij)} \in \mathbf{U}$. Moreover we define $A_I(\mathbf{x}_j; \mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i \mathbf{x}^{(ij)}$ for some fixed $j \in \{1, \ldots, m\}$. For the sake of simplicity we will sometimes write A_I instead of $A_I(\mathbf{x}_j; \mathbf{w})$.

Now we are able to define the index set function for the Neizgoda's inequality for functions with nondecreasing increments given in (2.1).

$$F(I) = W_I \left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_I} \sum_{j=1}^{m-1} \sum_{i \in I} w_i f\left(\mathbf{x}^{(ij)}\right) - f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_I} \sum_{j=1}^{m-1} \sum_{i \in I} w_i \mathbf{x}^{(ij)} \right) \right] \quad (3.1)$$

Now, by using techniques of article [1] we proof the following results here. Throughout this section $I_n = \{1, \ldots, n\}$.

Theorem 3.1. *Let all the assumptions of the Theorem* 2.1 *are true. Let* I *and* I' *be finite nonempty sets of positive integers such that* $I \cup I' = I_n$ *and* $I \cap I' = \emptyset$ *and* $\mathbf{w} = (w_i),\ i \in I \cup I'$ *such that* $W_{I \cup I'} > 0$. *Let* $A_S(\mathbf{x}_j; \mathbf{w}) \in \mathbf{U}$ *for* $(S \in \{I, I', I \cup I'\})$. *If* $W_I > 0$ *and* $W_{I'} > 0$, *and if*

$$\sum_{j=1}^{m} A_I(\mathbf{x}_j; \mathbf{w}) \leq \sum_{j=1}^{m} A_{I'}(\mathbf{x}_j; \mathbf{w}) or \sum_{j=1}^{m} A_I(\mathbf{x}_j; \mathbf{w}) \geq \sum_{j=1}^{m} A_{I'}(\mathbf{x}_j; \mathbf{w}) \quad (3.2)$$

then

$$F(I \cup I') \geq F(I) + F(I'). \quad (3.3)$$

If $W_I \cdot W_{I'} < 0$, *and if* (3.2) *holds then inequality* (3.3) *is reversed.*

Proof. For $W_I > 0$ and $W_I^{'} > 0$ let

$$\xi^{(1)} = \sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A_I, \xi^{(2)} = \sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A_{I'}, w_1 = W_I, w_2 = W_{I'}$$

we have $\xi^{(1)} \leq \xi^{(2)}$ or $\xi^1 \geq \xi^2$ and $w_i > 0$ for $i \in \{1, 2\}$ from Theorem 1.1 we have

$$f\left(\frac{1}{W_2} \sum_{i=1}^{2} w_i \xi^{(i)} \right) \leq \frac{1}{W_2} \sum_{i=1}^{2} w_i f\left(\xi^{(i)}\right), \quad (3.4)$$

that is,

$$f\left[\frac{1}{W_I \cap I'} \left(W_I \left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A_I \right) + W_{I'} \left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A'_I \right) \right) \right]$$
$$\leq \frac{1}{W_{I \cap I'}} \left(W_I f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A_I \right) + W_{I'} f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} A'_I \right) \right).$$

Since, $W_{I \cap I'} > 0$, the above inequality becomes

$$W_{I \cap I'} f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_{I \cap I'}} \sum_{j=1}^{m-1} \sum_{i \in I \cap I'} w_i \mathbf{x}^{(ij)} \right)$$
$$\leq W_I f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_I} \sum_{j=1}^{m-1} \sum_{i \in I} w_i \mathbf{x}^{(ij)} \right) + W_{I'} f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \frac{1}{W_{I'}} \sum_{j=1}^{m-1} \sum_{i \in I'} w_i \mathbf{x}^{(ij)} \right),$$

now if we multiply the above inequality by (-1) and add following term to the both sides

$$W_{I\cap I'}\left(\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right)-\sum_{j=1}^{m-1}\sum_{i\in I\cap I'} w_i f\left(\mathbf{x}^{(ij)}\right)\right)$$

we obtain

$$\begin{aligned}
&W_{I\cap I'}\left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right)-\sum_{j=1}^{m-1}\sum_{i\in I\cap I'} w_i f\left(\mathbf{x}^{(ij)}\right)-f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_{I\cap I'}}\sum_{j=1}^{m-1}\sum_{i\in I\cap I'} w_i\mathbf{x}^{(ij)}\right)\right]\\
&\geq W_I\left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right)-\sum_{j=1}^{m-1}\sum_{i\in I} w_i f\left(\mathbf{x}^{(ij)}\right)-f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I} w_i\mathbf{x}^{(ij)}\right)\right]\\
&+W_{I'}\left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right)-\sum_{j=1}^{m-1}\sum_{i\in I'} w_i f\left(\mathbf{x}^{(ij)}\right)-f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_{I'}}\sum_{j=1}^{m-1}\sum_{i\in I'} w_i\mathbf{x}^{(ij)}\right)\right],
\end{aligned}$$

which concludes the proof of (3.3).
In case $W_I.W_{I'}<0$ say $W_I>0$ and $W_{I'}<0$, we define

$$\xi^{(1)}=\sum_{j=1}^{m}\mathbf{a}^{(j)}-\sum_{j=1}^{m-1}A_I,\xi^{(2)}=\sum_{j=1}^{m}\mathbf{a}^{(j)}-\sum_{j=1}^{m-1}A_{I'},w_1=W_I,w_2=W_{I'},$$

and the reversed (1.2) is obtained from Remark 1. □

Following results give us refinements of (2.1).

Corollary 3.2. *Let $f:\mathbf{U}\to\mathbb{R}^k$ be a function with nondecreasing increments under the assumptions of* (2.1) *let* $\mathbf{w}$ *be a real n-tuple such that*

$$w_1>0,w_i\geq 0,i\in\{2,\ldots,n\},\tag{3.5}$$

then

$$F(I_n)\geq F(I_{n-1})\geq\cdots\geq F(I_2)\geq F(I_1)\geq 0,\tag{3.6}$$

where $I_k=\{1,\ldots,k\}$. If

$$w_i\leq 0,i\in\{2,\ldots,n\},W_{I_n}>0\tag{3.7}$$

and $A_{I_n}(\mathbf{x}_j,\mathbf{w})\in\mathbf{U}$, then

$$0\leq F(I_n)\leq F(I_{n-1})\leq\cdots\leq F(I_2)\leq F(I_1).\tag{3.8}$$

Proof. Without loss of generality we may assume

$$\sum_{j=1}^{m-1}\mathbf{x}^{(1j)}\leq\sum_{j=1}^{m-1}\mathbf{x}^{(2j)}\leq\cdots\leq\sum_{j=1}^{m-1}\mathbf{x}^{(nj)}.$$

Suppose $\mathbf{w}$ satisfies (3.5) with

$$W_1=w_1>0,\ldots,W_n>0.$$

First we show that $F(\{t\})\geq 0$ for any $t\in I_n$. From the definition of F we have

$$
\begin{aligned}
F(\{t\} &= w_t \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - w_t \sum_{j=1}^{m-1} f\left(\mathbf{x}^{(tj)}\right) - w_t f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(tj)}\right) \\
&= w_t \left[\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \sum_{j=1}^{m-1} f\left(\mathbf{x}^{(tj)}\right) - f\left(\sum_{j=1}^{m} \mathbf{a}^{(j)} - \sum_{j=1}^{m-1} \mathbf{x}^{(tj)}\right)\right]
\end{aligned}
$$

$\Rightarrow F(\{t\}) \geq 0$ from Lemma 1.4.
We also have

$$
\sum_{i=1}^{n-1} w_i \left(\mathbf{x}^{(i,j)} - \mathbf{x}^{(n,j)}\right) = W_{n-1}\left(\mathbf{x}^{(n-1,j)} - \mathbf{x}^{(n,j)}\right) + \sum_{i=1}^{n-2} W_i \left(\mathbf{x}^{(i,j)} - \mathbf{x}^{(i+1,j)}\right) \leq 0
$$

which leads to

$$
A_{I_{n-1}} = \frac{1}{W_{n-1}} \sum_{i=1}^{n-1} w_i \mathbf{x}^{(i,j)} \leq \mathbf{x}^{(n,j)}
$$

now if we take $I = A_{I_{n-1}}$ and $I' = n$, then

$$
A_I \leq A_{I'}, W_I > 0, W_{I'} > 0, W_{I \cup I' > 0}
$$

and all other conditions of Theorem 3.1 also satisfied, so we have

$$
F(I_n) \geq F(I_{n-1}) + F(\{n\}) \geq F(I_{n-1})
$$

by iteration we get (3.6), where the case $F(I_1) \geq 0$ follows from Lemma 1.4.

If $\mathbf{w}$ satisfies the conditions (3.7) and (1.6), then again we have

$$
W_1 = w_1 > 0, \ldots, W_n > 0.
$$

Similarly as before we get $F(\{t\}) \leq 0$ for any $t \in \{2, \ldots, n\}$.

If we take $I = I_{n-1}$ and $I' = n$, then we have

$$
A_I \leq A_{I'}, W_I > 0, W_{I'} < 0, W_{I \cup I' > 0}
$$

and from the second part of Theorem 3.1 we conclude

$$
F(I_n) \leq F(I_{n-1}) + F(\{n\})) \leq F(I_{n-1}).
$$

Since by assumption we know that $A_{I_n} \in \mathbf{U}$, we may deduce that all the inequalities in (3.8). We further proceed by iteration, and since from second part of Theorem 2.1 we also have $F(I_n) \geq 0$ which implies (3.8). □

Remark 3. *The Theorem* 2 *and related results in* [1] *becomes a special case of our result if we take* $m = 2$, $\mathbf{a}^{(1)} = \mathbf{a}, \mathbf{a}^{(2)} = \mathbf{b}$ *with* $\mathbf{a}^{(1)} \leq \mathbf{a}^{(2)}$ *and* $\mathbf{x}^{(i1)} = \mathbf{x}^{(i)}$.

Throughout this section we assume that $I \subseteq I_n$ unless stated otherwise. Now we give refinement of (2.1) as follows.

Theorem 3.3. *Let all assumptions of Theorem 2.1 be valid. Then following refinement hold:*

$$f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\mathbf{x}^{(ij)}\right)$$
$$\leq D(\mathbf{w},\mathbf{X},f;I)\leq\sum_{j=1}^{m}f\left(\mathbf{a}^{(j)}\right)-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_if\left(\mathbf{x}^{(ij)}\right),\quad(3.9)$$

where $W_I=\sum_{i\in I}w_i,\ W_{I'}=\sum_{i\in I'}w_i,\ I'=I_n\backslash I$ *and*

$$\begin{aligned}D(\mathbf{w},\mathbf{X},f;I) &= \frac{W_I}{W_n}f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I}w_i\mathbf{x}^{(ij)}\right)\\ &+ \frac{W_{I'}}{W_n}f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_{I'}}\sum_{j=1}^{m-1}\sum_{i\in I'}w_i\mathbf{x}^{(ij)}\right).\end{aligned}$$

Proof. From Theorem 1.1 we get

$$\begin{aligned}&f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\mathbf{x}^{(ij)}\right)\leq\frac{1}{W_n}\sum_{i=1}^{n}w_if\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\sum_{j=1}^{m-1}\mathbf{x}^{(ij)}\right)\\ &=\frac{1}{W_n}\left[W_I\left(\frac{1}{W_I}\sum_{i\in I}w_if\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\sum_{j=1}^{m-1}\mathbf{x}^{(ij)}\right)\right)\right.\\ &\left.+W_{I'}\left(\frac{1}{W_{I'}}\sum_{i\in I'}w_if\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\sum_{j=1}^{m-1}\mathbf{x}^{(ij)}\right)\right)\right]\\ &=\frac{W_I}{W_n}f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I}w_i\mathbf{x}^{(ij)}\right)+\\ &\frac{W_{I'}}{W_n}f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)}-\frac{1}{W_{I'}}\sum_{j=1}^{m-1}\sum_{i\in I'}w_i\mathbf{x}^{(ij)}\right)\\ &=D(\mathbf{w},\mathbf{X},f;I).\end{aligned}$$

For any I, which proves the first inequality in (3.9).

By inequality (2.1) we also have

$$
\begin{aligned}
D(\mathbf{w},\mathbf{X},f;I) &= \frac{W_I}{W_n} f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)} - \frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I} w_i \mathbf{x}^{(ij)}\right) \\
&+\frac{W_{I'}}{W_n} f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)} - \frac{1}{W_{I'}}\sum_{j=1}^{m-1}\sum_{i\in I'} w_i \mathbf{x}^{(ij)}\right) \\
&\leq \frac{W_I}{W_n}\left(\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right)\right) \\
&+\frac{W_{I'}}{W_n}\left(\sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_{I'}}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right)\right) \\
&= \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right)
\end{aligned}
$$

for any I, which proves the second inequality in (3.9). □

Remark 4.

$$
f\left(\sum_{j=1}^{m}\mathbf{a}^{(j)} - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \mathbf{x}_{(ij)}\right) \leq \min_I D(\mathbf{w},\mathbf{X},f;I)
$$

$$
\max_I D(\mathbf{w},\mathbf{X},f;I) \leq \sum_{j=1}^{m} f\left(\mathbf{a}^{(j)}\right) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f\left(\mathbf{x}^{(ij)}\right).
$$

4. Functions with Nondecreasing Increments of Order 3

In this section we will proceed through the same technique used to generalize the Burkill-Mirsky-Pečarić result in [5]. Throughout this section, we will use the notation $\mathbf{X_i} = (x_{i1}, \ldots, x_{ik})$ for $i \in \{1, \ldots, n\}$, where $\mathbf{X_i} \in \mathbb{R}^k$.

Theorem 4.1. *Let $f : \mathbf{U} \to \mathbb{R}$ be a continuous function with nondecreasing increments and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$. If $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in \mathbf{U}^n$ is nondecreasing or nonincreasing in means with respect to weights w_i for $i \in \{1, \ldots, n\}$, then the inequality*

$$
f\left(\mathbf{a}+\mathbf{b} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \leq f(\mathbf{a}) + f(\mathbf{b}) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i})
$$

holds, where $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ are two specially chosen k-tuples related to $\mathbf{U}$.

Proof. In [1] the following index set is defined

$$
F(I) = W_I\left[f\left(\mathbf{a}\right) + f\left(\mathbf{b}\right) - \frac{1}{W_I}\sum_{i\in I} w_i f\left(\mathbf{x}^{(i)}\right) - f\left(\mathbf{a}+\mathbf{b} - \frac{1}{W_I}\sum_{i\in I} w_i \mathbf{x}^{(i)}\right)\right]. \tag{4.1}
$$

In Theorem 3 of [1] it was also proved that this index function with its refinements holds for any sequence of $(\mathbf{x^{(1)}}, \ldots, \mathbf{x^{(n)}}) \in \mathbf{U}^n$ which is nondecreasing or nonincreasing in means with respect to weights w_i for $i \in \{1, \ldots, n\}$. The said theorem indicated the following interesting result

$$F(I_n) \geq 0$$

which directly leads to (1.11) by using (4.1) and replacing $(\mathbf{x^{(1)}}, \ldots, \mathbf{x^{(n)}})$ by $(\mathbf{X_1}, \ldots, \mathbf{X_n})$. □

Theorem 4.2. *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{0}, \mathbf{d}]^n,\ (\mathbf{d} > \mathbf{0})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i \in \{1, \ldots, n\}$*. If* f *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$*, then the inequality*

$$\begin{aligned}\frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}) \quad &- \quad f\left(\mathbf{a}+\mathbf{b}-\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \\ &\leq \quad \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(2\mathbf{d}-\mathbf{X_i}\right) - f\left(\mathbf{a}+\mathbf{b}-\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(2\mathbf{d}-\mathbf{X_i}\right)\right)\end{aligned}$$

holds.

Proof. If f is a function with nondecreasing increments of order three on $\mathbf{J}$, then the following inequality holds

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}\Delta_{\mathbf{h_3}} f(\mathbf{x}) \geq 0 \quad \text{for} \quad \mathbf{x}, \mathbf{x}+\mathbf{h_1}+\mathbf{h_2}+\mathbf{h_3} \in \mathbf{J}, \quad \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} \in \mathbb{R}^k_*,$$

i.e.,

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}\left(f(\mathbf{x}+\mathbf{h_3}) - f(\mathbf{x})\right) \geq 0. \tag{4.2}$$

If $\mathbf{x} \in \mathbf{I}$ and $\mathbf{h_3} = 2\mathbf{d} - 2\mathbf{x}$, we have

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}\left(f(2\mathbf{d}-\mathbf{x}) - f(\mathbf{x})\right) \geq 0,$$

i.e., the function $\mathbf{x} \mapsto f(2\mathbf{d}-\mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments of order two, i.e., it is a function with nondecreasing increments. Now, using Theorem 4.1, we obtain Theorem 4.2. □ □

Theorem 4.3. *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{r}, \mathbf{d}-\mathbf{p}]^n,\ (\mathbf{0} < \mathbf{p} < \mathbf{d}-\mathbf{r})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i \in \{1, \ldots, n\}$*. If* f *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{r}, \mathbf{d}]$*, then the following inequality holds*

$$\begin{aligned}\frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}) \quad &- \quad f\left(\mathbf{a}+\mathbf{b}-\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \\ &\leq \quad \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\mathbf{p}+\mathbf{X_i}\right) - f\left(\mathbf{a}+\mathbf{b}-\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\mathbf{p}+\mathbf{X_i}\right)\right)\end{aligned}$$

Proof. Using $\mathbf{h_3} = \mathbf{p} = \text{constant} \in \mathbb{R}^k$, we have that $\mathbf{x} \mapsto f(\mathbf{p}+\mathbf{x}) + f(\mathbf{x})$ is a function with nondecreasing increments, so from the Theorem 4.1, we get Theorem 4.3. □

Corollary 4.4. *(a) Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 4.2*. Then the inequalities*

$$0 \leq W_n^{k-1} \sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} x_{ij} \right) - \prod_{j=1}^{k} \left(\mathbf{a} + \mathbf{b} - \sum_{i=1}^{n} w_i x_{ij} \right)$$
$$\leq W_n^{k-1} \sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} (2d_j - x_{ij}) \right) - \prod_{j=1}^{k} \left(\mathbf{a} + \mathbf{b} - \sum_{i=1}^{n} w_i (2d_j - x_{ij}) \right)$$

hold.

(b) If $\mathbf{X}$ *satisfies the assumptions of Theorem* 4.3*. Then the inequalities*

$$0 \leq W_n^{k-1} \sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} x_{ij} \right) - \prod_{j=1}^{k} \left(\mathbf{a} + \mathbf{b} - \sum_{i=1}^{n} w_i x_{ij} \right)$$
$$\leq W_n^{k-1} \sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} (p_j + x_{ij}) \right) - \prod_{j=1}^{k} \left(\mathbf{a} + \mathbf{b} - \sum_{i=1}^{n} w_i (p_j + x_{ij}) \right)$$

hold, where all components of $\mathbf{X}$ *are nonnegative.*

Proof. We consider the function $f(\mathbf{x}) = x_1 \cdots x_k$ which is a function with nondecreasing increments of orders two and three for $\mathbf{x} \in \mathbb{R}_*^k = [0, +\infty)^k$. So, using Theorems 4.2 and 4.3 we obtain Corollary 4.4. □

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