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NUMERICAL CALCULATION OF THE REAL r-LAMBERT FUNCTION

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Abstract. The classical Lambert function and its generalizations are widely used in applied mathematics and physics. In this paper, an interesting approach for the numerical calculation of one particular generalization, the r-Lambert function, is provided.

1. Introduction

The Lambert $W(x)$ function [6] is the solution of the following transcendental equation

$$ye^y = x.$$

If $x \neq 0$, this equation has infinitely many solution; and, in particular, it has one or two real solutions, depending on the value of x [6]. There are plenty of methods developed in the literature to calculate $W(x)$ with arbitrary or fixed precision [1, 2, 7, 8, 13, 14], real or complex variable.

The r-Lambert function, $W_r(x)$, is the solution of the equation [9]

$$ye^y + ry = x. \tag{1.1}$$

Here the parameter r is real but otherwise arbitrary. This function has been studied and is being applied in a wide range of areas. See [4, 5, 9, 11, 12] for a list of references, and for the basic properties of $W_r(x)$.

In this report we discuss the numerical approximations of the *real* solutions of equation (1.1). This approximation was programmed in C [10]. The input is the argument x, and the parameter r, and the output will be one to three real numbers, depending on the number of real solutions.

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2. Numerical approximations of the solutions of (1.1)

First of all, we need to know how many real solutions (1.1) has. This question was studied in [11]; we repeat Theorem 4 of that paper.

Theorem 2.1. *Depending on the value of r, we can classify $W_r(x)$ as follows.*

(1) *If $r > 1/e^2$, then $W_r(x) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, everywhere differentiable function such that* $\mathrm{sgn}(W_r(x)) = \mathrm{sgn}(x)$.
(2) *If $r = 1/e^2$, then $W_r(x) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function which is differentiable on $\mathbb{R} \setminus \{-4/e^2\}$. Moreover,* $\mathrm{sgn}(W_r(x)) = \mathrm{sgn}(x)$.
(3) *If $0 < r < 1/e^2$, then there are three branches of W_r which we denote by $W_{r,-2}$, $W_{r,-1}$, $W_{r,0}$. Let*

$$\alpha_r = W_{-1}(-re) - 1, \quad \textit{and} \quad \beta_r = W_0(-re) - 1,$$

where W_{-1} and W_0 are the two branches of the Lambert function.
Then α_r and β_r determine the above branches as follows:
- *$W_{r,-2} :]-\infty, f_r(\alpha_r)] \to]-\infty, \alpha_r]$ is a strictly increasing function,*
- *$W_{r,-1} : [f_r(\alpha_r), f_r(\beta_r)] \to [\alpha_r, \beta_r]$ is a strictly decreasing function, and*
- *$W_{r,0} : [f_r(\beta_r), +\infty[\to [\beta_r, +\infty[$ is a strictly increasing function.*

These three functions are differentiable on the interior of their domains.
(4) *Finally, if $r < 0$, then W_r has two branches, $W_{r,-1}$ and $W_{r,0}$. Let*

$$\gamma_r = W(-re) - 1,$$

where W is the classical Lambert function.
Then for these branches we have that
- *$W_{r,-1} : [f_r(\gamma_r), +\infty[\to]-\infty, \gamma_r]$ is strictly decreasing, while*
- *$W_{r,0} : [f_r(\gamma_r), +\infty[\to [\gamma_r, +\infty[$ is a strictly increasing function.*

Both of them are differentiable in the interior of their domains.

Here sgn is the signum function such that $sgn(0) = 0$; and $f_r(x) = xe^x + rx$.

According to this theorem, we have three possibilities: $r \geq 1/e^2$, $0 < r < 1/e^2$, and $r < 0$. The $r = 0$ case is the classical Lambert function, for this we rely on the C code written by Keith Briggs [3].

In any cases, we use the classical Halley algorithm to approximate the root of the equation

$$ye^y + ry - x = 0.$$

The method says that in the $(n+1)$th approximation

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}.$$

Here $f(x) = xe^x + rx$. The initial point is to be chosen such that it is “close” to the solution assuring the convergence. This choice is made in the three different cases by an analysis of the asymptotics of the r-Lambert function (see Theorem 8 in [11] in particular).

2.1. The case when $r \geq 1/e^2$.

In this case there is only one real solution, since $f(x) = xe^x + rx$ is a strictly increasing function if $r > 1/e^2$. The following code of mathematica (computer software)

```
Plot[x Exp[x] + (1/E^2 + 2) x, {x, -5, 5}]
```

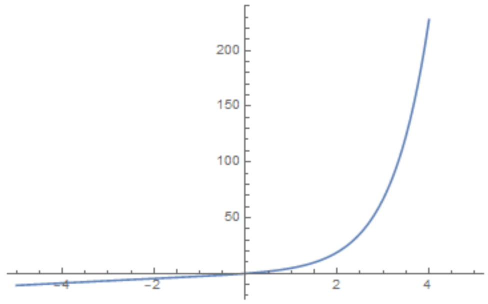

FIGURE 1. The Graph of $f(x) = xe^x + \left(\frac{1}{e^2} + 2\right) x$

gives the graph of the function $f(x)$ when $r = 1/e^2 + 2$ (see Figure 1), which is increasing and crosses the x-axis once. This implies that the inverse of the function $f^{-1}(x) = W_{1/e^2+2}(x)$ must also be increasing and must cross the x-axis once. If $r = 1/e^2$ then the r-Lambert function is not differentiable at $x = -4/e^2$ (see [11, p. 7923]). Here the algorithm gives back the exact value

$$W_{1/e^2}\left(-\frac{4}{e^2}\right) = -2$$

as the output. If x is close to but not equal to $-4/e^2$, there is no problem, the Halley algorithm converges, in the used accuracy of double precision (`double` variable in C).

According to the known approximations of the r-Lambert function (see again Theorem 8 in [11]), we can start the Halley iteration from

$$x_0 = \begin{cases} \log x - \log\log x, & \text{if } x > 1; \\ \frac{1}{r}x, & \text{if } x < -1; \\ 0, & -1 \leq x \leq 1. \end{cases}$$

The following code of mathematica

```
rLambert[x_, r_] :=If[r > 1/E^2, FindRoot[y*Exp[y] + r*y == x,
          {y, If[x > 1, Log[x]- Log[Log[x]], 1/r x]}]]
```

gives the approximate value of the r-Lambert function $W_r(x)$ when $r > 1/e^2$. For instance,

$$W_3(5) \approx \text{rLambert}[5, 3] = 0.911196.$$

One can also verify that

$$0.911196 \exp[0.911196] + 3(0.911196) \approx 5.$$

2.2. **The case when $0 < r < 1/e^2$.** In this case $W_r(x)$ has three branches intersecting the real line, so we must find three solution. The following code of mathematica

```
Plot[x Exp[x] + (1/(E^2 + 5)) x, {x, -7, 1}]
```

gives the graph of the function $f(x)$ when $r = 1/(e^2+5)$ (see Figure 2). We observe that the graph of the function $f(x)$ and the line $y = -0.42$ intersect at three points. This implies that the inverse of the function $f^{-1}(-0.42) = W_{1/(e^2+5)}(x)$ must cross the x-axis three times.

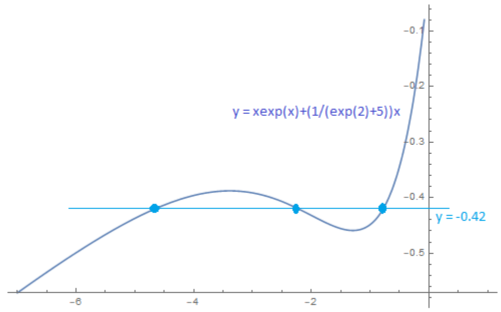

FIGURE 2. The Graph of $f(x) = xe^x + \left(\frac{1}{e^2+5}\right)x$

The α_r and β_r numbers (functions of r) determine where to look for the branch limit points. These are read out from the above Theorem, and they are important, because the three functions belonging to the three different branches ($W_{r,-2}, W_{r,-1}$, and $W_{r,0}$) are taking real values only on given intervals (see again the Theorem above). Thus the argument x itself determines which solution it belongs to.

Let

$$\begin{aligned}\alpha &= W_{-1}(-re) - 1,\\ \beta &= W_0(-re) - 1,\\ f_\alpha &= f(\alpha, r),\\ f_\beta &= f(\beta, r).\end{aligned}$$

Here $f(x, r) = xe^x + rx$, as above

Then, if $x < f_\beta$, then we are on the leftmost branch, and the output will belong to the leftmost solution. In this case the approximation can be started from

$$x_0 = x/r,$$

according to the asymptotic behavior of $W_r(x)$ when x is small.

One issue occurs here: if $x < 40.54374295204823$ (so that for this value $xe^x < 10^{-16}$) the double precision accuracy is insufficient to calculate the f function. In this case the r-Lambert function is substituted by $W_r(x) \approx x/r$.

If $f_\beta < x < f_\alpha$ then the initial points can taken to be $(x/r, -3, -1)$, respectively, the solutions are close to these numbers.

2.3. **The case when** $r < 0$**.** Finally, when $r < 0$, we have two real solutions, if $x \not< f(W(-re) - 1)$. Such x is not in the real part of the domain of the r-Lambert function. In addition, if $x = \log(-r)$ then the solution is 0. The following code of mathematica

```
Plot[x Exp[x] - 10 x, {x, -5, 5}]
```

gives the graph of the function $f(x)$ when $r = -10$ (see Figure 3). We observe that the graph of the function $f(x)$ and the line $y = 20$ intersect at two points. This implies that the inverse of the function $f^{-1}(20) = W_{-10}(x)$ must cross the x-axis twice.

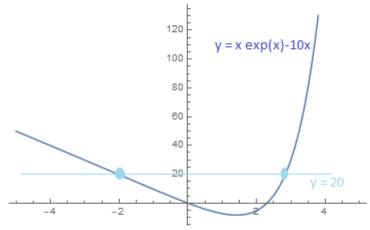

FIGURE 3. The Graph of $f(x) = xe^x - 10x$

Moreover, if $x < 0$ the two initial values fall into the two sides of $W(-re) - 1$; so we start the iteration from $W(-re)-2$ and $W(-re)$, these seemed to be appropriate during the checking of the performance of the code.

If, in turn, $x > 0$, we initiate the iteration according to the value of r. On the left branch we choose

$$x_0 = \begin{cases} \log(-r) - 1, & \text{if } r > -1; \\ -1, & \text{otherwise.} \end{cases},$$

while on the right branch we take

$$x_0 = \begin{cases} 0, & \text{if } r > -1; \\ \log(-r) + 1, & \text{otherwise.} \end{cases}.$$

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