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DERIVATIVE AND INTEGRAL OF LIMIT SUMMAND FUNCTIONS WITH RELATIONS TO THE EULER TYPE CONSTANTS

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Abstract. Gamma type and limit summability of functions were studied and introduced by Webster and Hooshmand in 1997 and 2001, respectively. It is shown that Γ-type functions can be considered as a sub-topic of the limit summability. In this paper, by studying derivative and integral of limit summand functions f_σ, we state and prove several criteria for convergence of functional sequences $f'_{\sigma_n}(x)$ and $F_{\sigma_n}(x)$. Also, some necessary or sufficient conditions for existence of related Euler-type constants are proved. At last, we present some inequalities and applications of this topic with many examples about the gamma, Riemann zeta, digamma and some other special functions.

1. Introduction and Preliminaries

The Euler-Mascheroni constant is limit of the sequence $\sum_{k=1}^{n} \frac{1}{k} - \log n$ (denoted by $\gamma = 0.5772215...$, see [1]). Also, there are some generalizations of the constant γ, which some of them are studied in [9] and [3]. On can see some connections between many Euler-type constants and the digamma function (that is the derivative of $\log \Gamma$). On the other hand, there is a generalization of the gamma function (see [2]), namely Γ-type functions , which satisfy the functional equation $f(x+1) = g(x)f(x)$ (see [11]). In 2001, the topic of limit summability of real functions [4] was introduced by M.H. Hooshmand and he proved that Γ-type functions can be considered as an its sub-topic. Also, he introduced and studied another type of such summabilities entitled analytic summability of functions in 2016 [6]. It is worth noting that Muller and Schleicher [8] in 2010 used a similar functional sequence for a type of fractional sums while they were not aware of the limit summability topic. The following is a summary of limit summability of functions from [4]. Let $f : D_f \to \mathbb{C}$ or $\mathbb{R}$ be a function with $\mathbb{N}^* \subseteq D_f \subseteq \mathbb{C}$ (or $\mathbb{N}^* \subseteq D_f \subseteq \mathbb{R}$), where $\mathbb{N}^*$ denotes the set of natural numbers and $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. Then put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\} = \{x | \{x+1, x+2, x+3, \cdots\} \subseteq D_f\}.$$

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Also, for any positive integer n and $x \in \Sigma_f$ set

$$R_n(f,x) := R_n(x) = f(n) - f(x+n),$$

$$f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n} R_k(x).$$

When $x \in D_f$, we may use the notation $\sigma_n\big(f(x)\big)$ instead of $f_{\sigma_n}(x)$.

The function f is called limit summable at $x_0 \in \Sigma_f$ if the sequence $\{f_{\sigma_n}(x_0)\}$ is convergent. The function f is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all points of S. Also, we have

$$D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is limit summable at } x\},$$

and the function f is called limit summable if it is summable on its domain and $R(1) = 0$ (where $R(f,x) = R(x) = \lim\limits_{n\to\infty} R_n(x)$). In this case, the function f_σ is referred to the limit summand function of f.

It is shown that $1 \in D_{f_\sigma}$ if and only if $R_n(1)$ is convergent and if and only if $D_{f_\sigma} \cap D_f = D_{f_\sigma} + 1$. A necessary condition for limit summability of f at x is

$$\lim_{n\to\infty}\Big(R_n(x) - xR_{n-1}(1)\Big) = 0.$$

If $R(1) = 0$, then $R(f,x) = R(f,1)x$ on D_{f_σ} and

$$f_\sigma(x) = f(x) + f_\sigma(x-1); \qquad x \in D_{f_\sigma} + 1.$$

Thus

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{i=1}^{m} f(i); \quad \forall m \in \mathbb{N}^*. \tag{1.1}$$

It is proved that the following conditions are equivalent:

a) f is limit summable;

b) $D_{f_\sigma} = \Sigma_f$, $R(1) = 0$;

c) $f_\sigma(x) = f(x) + f_\sigma(x-1)$ for all $x \in D_f$.

Note. In this paper, mostly we consider the functions f with domains of the form $D_f = [M, +\infty)$ or $(M, +\infty)$. For such functions we have $\Sigma_f = [M-1, +\infty)$ or $(M-1, +\infty)$ and $D_f \subseteq \Sigma_f$. Hence, for their limit summability, it is enough to show that $D_{f_\sigma} = \Sigma_f$ and $R(1) = 0$.

One of the most important criteria for limit summability which was introduced in [5] stated that convexity or concavity of f together with boundedness of $R_n(f,1)$ imply limit summability of f. The following theorem is one of its results.

Theorem 1.1. *Suppose that $f : [M, +\infty) \to \mathbb{R}$ is a real function for which $R_n(1)$ is bounded. If f is convex or concave on $D_f = [M, +\infty)$ from a number on, then f is uniformly summable on every bounded subset of $\Sigma_f = [M-1, +\infty)$.*

Another important criteria implies the next result for limit summability of monotonic functions.

Theorem 1.2. *Let $f : [M, +\infty) \to \mathbb{R}$ be a real function that the sequence f_n is bounded. If f is monotone on $[M, +\infty)$ from a number on, then f is absolutely and uniformly summable on every bounded subset of $[M-1, +\infty)$.*

2. Derivative of limit summand functions and related Euler- type constants

In order to study of the derivative of the limit summand function f_σ, Hooshmand considered a topic which has several properties, application and connections. For the limit summand function f_σ, we can consider the functional sequences $f_{\sigma'_n}(x)$ and $f'_{\sigma_n}(x)$ (if f is differentiable). In [3], the functional sequence $f_{\sigma'_n}(x)$ and some of its properties is studied. Now, we study the functional sequence $f'_{\sigma_n}(x)$ and its relations to $f_{\sigma'}$, $(f_\sigma)'$ and also related topics such as generalized Euler-type constants. Let f be a differentiable real function on $x+\mathbb{N}^*$ (i.e., $x\in\Sigma_{f'}$), we put

$$f_{\sigma'_n}(x) := \big(f_{\sigma_n}(x)\big)' = f(n) - \sum_{k=1}^{n} f'(k+x).$$

Also, if $\mathbb{N}^*\cup\mathbb{N}^*+x\subseteq D_{f'}$ $\big($i.e., $\{0,x\}\subseteq\Sigma_{f'}\big)$, then we set

$$f'_{\sigma_n}(x) := \big(f'\big)_{\sigma_n}(x) = xf'(n) + \sum_{k=1}^{n}\Big(f'(k) - f'(k+x)\Big).$$

Notice that if $0\in\Sigma_{f'}$, then both functional sequences $f_{\sigma'_n}(x)$ and $f'_{\sigma_n}(x)$ are defined on $\Sigma_{f'}$. It is interesting to know that the functional sequence $(\log)_{\sigma'_n}$ is convergent to $-\gamma$. This fact lead us to the notations $\gamma_n(f,x) := -f_{\sigma'_n}(x)$ and $\gamma_n(f) := \gamma_n(f,0) = -f_{\sigma'_n}(0)$, which have also some connections to a class of Euler-type constants [3, 9]. If $f(n)\to 0$ as $n\to\infty$, then $\gamma(f) = \sum_{n=1}^{\infty} f'_n$, where $f'_n := f'(n)$. We call a (real or complex) function f "at most linear" if it is a polynomial of degree≤ 1 (including the zero function).

A basic question is a raised that, what is the relationship between f'_σ, $f_{\sigma'}$ and $(f_\sigma)'$?. For answering this question, at first, we state some preliminary properties in the next lemma.

Lemma 2.1. *Let f be a real (or complex) function with $\mathbb{N}^*\subseteq D_{f'}$ (thus $f_{\sigma'_n}(x)$ is defined on $\mathbb{N}$), then*

(a) If the functional sequence $f'_{\sigma_n}(x) - f_{\sigma'_n}(x)$ is convergent on a subset E such that $\mathbb{N}\subseteq E\subseteq\Sigma_{f'}$, then the sequences $f'(n)$ and $f_{\sigma'_n}(0)$ are convergent. Hence, if the functional sequences $f'_{\sigma_n}(x)$ and $f_{\sigma'_n}(x)$ are convergent, then the difference function of f'_σ and $f_{\sigma'}$ is at most linear, $D_{f'_\sigma} = D_{f_{\sigma'}}$ and

$$f'_\sigma(x) = f_{\sigma'}(x) + f'(\infty)x + \gamma(f);\quad x\in D_{f'_\sigma}. \tag{2.1}$$

(b) Conversely, if the functional sequence $xf'(n)+\gamma_n(f)$ is convergent on $\Sigma_{f'}$, then $D_{f'_\sigma} = D_{f_{\sigma'}}$ and the relation (2.1) is valid.

Therefore, in each of the cases (a) and (b), if $f'(\infty) = f_{\sigma'}(0) = 0$, then $f'_\sigma = f_{\sigma'}$.

Proof. (a) A simple calculation shows that

$$f'_{\sigma_n}(x) - f_{\sigma'_n}(x) = xf'(n) - f_{\sigma'_n}(0) = xf'(n) + \gamma_n(f);\quad x\in E. \tag{2.2}$$

Thus, the functional sequence $xf'(n) - f_{\sigma'_n}(0)$ is convergent (at least) for each $x\in\mathbb{N}$, and this shows that the sequences $f'(n)$ and $f_{\sigma'_n}(0)$ are convergent (because putting $x=0$, we conclude that $f_{\sigma'_n}(0)$ is convergent and then we get the result). Now, we obtain (a) easily.

(b) The relation (2.2) shows that, for every $x\in\Sigma_{f'}$, $f'_{\sigma_n}(x)$ is convergent if and only if $f_{\sigma'_n}(x)$ is so. Letting $n\to\infty$ in the equation we get the result. □

The following corollary states a test for limit summability of f' and gives a connection to the generalized Euler-type constants.

Corollary 2.2. *Suppose $f : [1, +\infty) \to \mathbb{C}$ is a differentiable function such that the sequences $f'(n)$ and $f_{\sigma'_n}(0)$ are convergent. Then f' is limit summable on $[0, +\infty)$ if and only if the functional sequence $\gamma_n(f, x)$ is convergent on it, and the relation* (2.1) *is valid for all $x \geq 0$.*

By using Theorem 2.2 of [3] and lemma 2.1, we can get the following useful result.

Corollary 2.3. *Let $f : [1, +\infty) \to \mathbb{R}$ be a real function. If f has monotonic derivative and $R(f, 1) = 0$, then $f'_{\sigma_n}(x)$ is convergent for all $x \geq 0$, and $f'_\sigma(x)$ satisfies the functional equation*

$$f'_\sigma(x) = f'_\sigma(x-1) + f'(x) + R(f', x); \quad x \geq 1. \tag{2.3}$$

Moreover, if f' is increasing, then $f'_\sigma(x)$ satisfies the inequalities

$$f'(1) - f'(\infty) \leq f'_\sigma(x) - f'(\infty)x \leq \frac{f_\sigma(x)}{x} - f_{\sigma'}(0) \leq 0; \;\; 0 < x \leq 1, \tag{2.4}$$

and in the case f' is decreasing, the above inequalities should be reversed.

Proof. By considering Theorem 2.2 of [3] and relation (2.2), it is clear. □

As we know, definition of the gamma function can be extended by using its functional equation for all complex values. It is defined and differentiable on $\mathbb{C} \setminus \mathbb{Z}_0^-$ and the derivative of its logarithm is called the digamma function (see [10]), which is denoted by Ψ. Indeed, we have

$$\psi(z) = \frac{d}{dx} \ln\Big(\Gamma(z)\Big) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+z}\Big); \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-,$$

where $\mathbb{Z}_0^- = \{0, -1, -2, ...\}$. Also, the digamma function is differentiable of every order and we have

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) = \frac{d^n}{dz^n}\psi(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}.$$

Also, there is a generalization of the Riemann zeta function under the name of Hurwitz zeta function [10], which for each fix complex q with $Re(q) > 0$ is defined as follows

$$\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}; \quad Re(s) > 1.$$

Now, we are ready to present some examples and applications of the study.

Example 2.4. *If $f(x) = \frac{1}{x}$, then $\lim_{n\to\infty} R_n(1) = R(f, 1) = 0$ and f' is an increasing function. Therefore, $f'_{\sigma_n}(x)$ is convergent for all $x \geq 0$ and*

$$f_\sigma(x) = \lim_{n\to\infty} \Big(\frac{x}{n} + \sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+x}\Big)\Big) = \Psi(x+1) + \gamma,$$

$$f'_\sigma(x) = \lim_{n\to\infty}\Big(\frac{-x}{n^2} + \sum_{k=1}^{n}\Big(\frac{1}{(k+x)^2} - \frac{1}{k^2}\Big)\Big) = \zeta(2, x+1) - \zeta(2),$$

$$f_{\sigma'}(0) = \lim_{n\to\infty}\Big(\frac{1}{n} + \sum_{k=1}^{n}\frac{1}{k^2}\Big) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} = \zeta(2),$$

thus the inequalities (2.4) *hold and*

$$-1 \le \zeta(2, x+1) - \frac{\pi^2}{6} \le \frac{1}{x}\Big(\Psi(x+1) + \gamma\Big) - \frac{\pi^2}{6} \le 0; \quad 0 < x \le 1.$$

or

$$x\Big(\frac{\pi^2}{6} - 1\Big) - \gamma \le x\zeta(2, x+1) - \gamma \le \Psi(x+1) \le \frac{\pi^2}{6}x - \gamma.$$

Therefore, we obtain the upper (*resp. lower*) *bound* $\frac{\pi^2}{6}x - \gamma$ (*resp.* $x\zeta(2, x+1) - \gamma$) *for the shifted digamma function* $\Psi(x+1)$ *on* $(0, 1]$. *It is interesting to know that the above upper bound for* $0 < x < \frac{2}{\pi^2}$ *is stronger than* $1 - \gamma$ *that was introduced by A. Lafrogia and P. Natalini in Theorem* 4.2 *of* [7].

Example 2.5. *Consider the real function* $f(x) = \tan^{-1}(x)$ *on* $\mathbb{R}$. *Then* $f'_{\sigma_n}(x)$ *is convergent, for all* $x \ge 0$, *and*

$$f_{\sigma'}(x) = \frac{\pi}{2} - \frac{1}{2i}\Big(\Psi(x+1-i) - \Psi(x+1+i)\Big),$$

$$f'_\sigma(x) = \frac{1}{2i}\Big(\Psi(x+1+i) - \Psi(x+1-i) - \Psi(1+i) + \Psi(1-i)\Big).$$

Notice that corollary 2.2 *implies* $f'_\sigma - f_{\sigma'}$ *is a constant function* (*since* $f'(\infty) = 0$), *and we have*

$$f'_\sigma(x) - f_{\sigma'}(x) = -f_{\sigma'}(0) = \frac{1}{2i}\Big(\Psi(1-i) - \Psi(1+i)\Big) - \frac{\pi}{2} = -\frac{1}{2}\Big(\pi\coth(\pi) - 1\Big) - \frac{\pi}{2}.$$

If f is a limit summable function and f_σ is differentiable on its domain, then $(f_\sigma)'$ is not necessarily equal to $f_{\sigma'}$ or f'_σ. However, the next lemma states some conditions on f for which the identity holds.

Lemma 2.6. *Suppose that* f *is a real function with* $[a, b] + \mathbb{N} \subseteq D_f$ (*equivalent* $[a, b] \subseteq \Sigma_f + 1$) *such that*

(a) f *is differentiable on* $[a, b] + \mathbb{N}$;

(b) $f_{\sigma'_n}(x)$ *is uniformly convergent on* $[a, b]$;

(c) f *is summable at some points of* $[a, b]$.

Then

i) f *is uniformly summable on* $[a, b]$, f_σ *is differentiable on* $[a, b]$ *and* $\big(f_\sigma(x)\big)' = f_{\sigma'}(x)$. *Also, if* $b - a > 1$, *then* f *is limit summable on the whole* $[a, b] + \mathbb{N}$.

ii) In addition, if $0 \in [a, b]$ *and one of the sequences* $f'(n)$ *or* $f(n)$ *is convergent, then* f' *is uniformly limit summable on* $[a, b]$ *and we have*

$$\Big(f_\sigma(x)\Big)' = f_{\sigma'}(x) = f'_\sigma(x) + f_{\sigma'}(0) - xf'(\infty); \qquad a \le x \le b. \tag{2.5}$$

Thus, if $f_{\sigma'}(0) = f'(\infty) = 0$, *then* $f'_\sigma = f_{\sigma'} = (f_\sigma)'$.

Proof. (i): This part (i) is a direct result of a well-known theorem in classical analysis. Now, if $b-a>1$, then there exists $x_0 \neq 0$ such that $x_0, x_0-1 \in [a,b]$ and thus $f_{\sigma_n}(x_0)$ and $f_{\sigma_n}(x_0-1)$ are convergent. Hence, the identities

$$R_n(x_0) = f_{\sigma_n}(x_0) - f_{\sigma_n}(x_0-1) - f(x_0),$$

$$f_{\sigma_n}(x_0) - f_{\sigma_n-1}(x_0) = R_n(x_0) - x_0 R_{n-1}(1),$$

imply that $R_n(1)$ is convergent. Therefore, f is limit summable on the whole $[a,b]+\mathbb{N}$ by corollary 2.13 of [4].

(ii): Since $0 \in [a,b]$, then $f_{\sigma'_n}(x) - f_{\sigma'_n}(0)$ is uniformly convergent on $[a,b]$ and the relation (2.2) completes the proof. □

Example 2.7. *(a) The real function $f(x) = \sqrt{x+1} - \sqrt{x+2}$ satisfies the conditions of the Lemma 2.6(i) (for a=1, b=2). Therefore, f is uniformly limit summable on $[1,2]$ and*

$$\Big(f_\sigma(x)\Big)' = f_{\sigma'}(x) = \sum_{k=1}^{\infty}\Big(\frac{1}{2\sqrt{k+x+2}} - \frac{1}{2\sqrt{k+x+1}}\Big) = \frac{-1}{2\sqrt{x+2}}.$$

(b) If $f(x) = \log x + \gamma$, then $f'(\infty) = f_{\sigma'}(0) = 0$. Since $\Sigma_f = (-1,+\infty)$ then putting $a = -1+\epsilon$ and $b = M$, where ϵ and M are two arbitrary positive real numbers such that $-1+\epsilon < M$, then we have

$$\Big(f_\sigma(x)\Big)' = f_{\sigma'}(x) = f'_\sigma(x) = \Psi(x+1) + \gamma; \quad x > -1,$$

(by using Lemma 2.6(ii)). Hence, for this function we have $f_{\sigma'} = (f_\sigma)' = f'_\sigma$.

Corollary 2.8. *Let $f : [1,+\infty) \to \mathbb{R}$ be a differentiable function such that f is limit summable at some point of $[0,1]$ and $f_{\sigma'_n}(x)$ is uniformly convergent on $[0,1]$. Then f is uniformly limit summable and f_σ is differentiable on $[0,1]$. Also, if one of the sequences $f'(n)$ and $f(n)$ is convergent, then f' is uniformly limit summable and we have*

$$\Big(f_\sigma(x)\Big)' = f_{\sigma'}(x) = f'_\sigma(x) + f_{\sigma'}(0) - xf'(\infty); \quad 0 \le x \le 1.$$

Example 2.9. *The real function $f(x) = a^x$, where $0 < a < 1$, satisfies all conditions of Corollary 2.8, thus f is uniformly limit summable on $[0,1]$ and*

$$\Big(f_\sigma(x)\Big)' = \Big(\sigma(a^x)\Big)' = \frac{\ln(a)a^{x+1}}{a-1}.$$

f' is also uniformly limit summable on $[0,1]$ (since a^n is convergent), and

$$f_{\sigma'}(x) = f'_\sigma(x) + f_{\sigma'}(0) = \frac{a\ln(a)}{1-a} - \frac{a^{1+x}\ln(a)}{1-a} - \frac{a\ln(a)}{1-a} = \frac{a^{1+x}\ln(a)}{a-1}; \quad 0 \le x \le 1.$$

Thus $f_{\sigma'} = (f_\sigma)' \neq f'_\sigma$.

3. Primitive function of limit summand function and their integrals

Similar to the previous functional sequence in section 2, we can consider the functional sequences $F_{\sigma_n}(x)$ and $\int f_{\sigma_n}(x)dx$, where F is a primitive function of f with several useful properties and applications.

Theorem 3.1. *Let $f : [1, +\infty) \to \mathbb{C}$ or $\mathbb{R}$ be a function with a fixed primitive function F. Also, consider the primitive function of $f_{\sigma_n}(x)$ as follows*

$$\Phi_n(x) := \frac{1}{2} f(n)x^2 + \sum_{k=1}^{n} \Big(f(k)x - F(k+x) \Big); \qquad x \geq 0. \tag{3.1}$$

(a) If the functional sequences $\Phi_n(x)$ and $F_{\sigma_n}(x)$ are convergent on $[0, +\infty)$ then the series $\sum_{k=1}^{\infty} f(k)$ and $\sum_{k=1}^{\infty} F(k)$ are convergent, the difference function of $\Phi(x)$ and $F_\sigma(x)$ is at most linear, and we have

$$F_\sigma(x) - \Phi(x) = \sum_{k=1}^{\infty} \Big(F(k) - f(k)x \Big); \quad x \geq 0. \tag{3.2}$$

(b) Conversely, if $f_{\sigma_n}(x)$ is uniformly convergent on every bounded subset of $[0, +\infty)$ and the series $\sum_{k=1}^{\infty} f(k)$ and $\sum_{k=1}^{\infty} F(k)$ are convergent on $[0, +\infty)$, then $F_{\sigma_n}(x)$ and $\Phi_n(x)$ are convergent on $[0, +\infty)$, Φ is a primitive function of f_σ and $F_\sigma - \Phi$ is at most linear on $[0, +\infty)$ and (3.2) holds.
(c) If the conditions of (b) hold, then

$$\int_a^b f_\sigma(x) = \sum_a^b F(x) + \big(b - a\big) \sum_{k=1}^{\infty} f(k), \tag{3.3}$$

for every $0 \leq a < b$, where $\sum_a^b F(x)$ denotes the subtraction $F_\sigma(b) - F_\sigma(a)$.

Proof. (a) A simple calculation shows that

$$F_{\sigma_n}(x) - \Phi_n(x) = -\frac{1}{2} f(n)x^2 + \Big(F(n) - \sum_{k=1}^{n} f(k) \Big) x + \sum_{k=1}^{n} F(k); \quad x \geq 0. \tag{3.4}$$

Since $F_{\sigma_n}(x) - \Phi_n(x)$ is convergent on $[0, +\infty)$, then the series $\sum_{k=1}^{\infty} f(k)$ and $\sum_{k=1}^{\infty} F(k)$ are convergent $\big($for if a functional sequence $a_n x^2 + b_n x + c_n$ is convergent at three distinct points, then the sequences a_n, b_n, c_n are convergent$\big)$ and we arrive at (a).

(b) Since the functional sequence $\Phi_n'(x) = f_{\sigma_n}(x)$ is uniformly convergent on every interval $[0, M]$ and $\Phi_n(2)$ is convergent $(\phi(2) = 2\sum_{k=1}^{\infty} f(k) - \sum_{k=3}^{\infty} F(k))$, then $\Phi_n(x)$ is uniformly convergent on $[0, M]$ and $\Phi'(x) = f_\sigma(x)$. Hence, Φ is a primitive function of f_σ on $[0, +\infty)$. Also, the relation (3.4) shows that $F_{\sigma_n}(x)$ is convergent on $[0, +\infty)$. Therefore, the conditions of (a) hold and proof of this part is complete.

(c) By putting $x = a$, $x = b$ in the relation (3.2) and subtracting them, we get the following relation

$$F_\sigma(b) - F_\sigma(a) - \Phi(b) + \Phi(a) = \big(a - b\big) \sum_{k=1}^{\infty} f(k).$$

Since $R(F, 1) = 0$, then the above identity together with the relation (1.1) impliy that

$$\int_a^b f_\sigma(x)dx = \sum_a^b F(k) + \big(b - a\big) \sum_{k=1}^{\infty} f(k).$$

□

Corollary 3.2. *Suppose that $f : [1, +\infty) \to \mathbb{R}$ is a convex or concave real function (thus the primitive function of f exists) such that the series $\sum_{k=1}^{\infty} f(k)$ and $\sum_{k=1}^{\infty} F(k)$ are convergent, then $\Phi_n(x)$ and $F_{\sigma_n}(x)$ are convergent on $[0, +\infty)$. Hence, all parts of Theorem 3.1 hold.*

Remark. *In the above corollary, the convexity on $[1, +\infty)$ can be replaced with the convexity on $[1, +\infty)$ from a number on, if primitive function F exists on the whole $[1, +\infty)$.*

Corollary 3.3. *Let f be an increasing function (resp. a decreasing) function from $[1, +\infty)$ to $\mathbb{R}^-$ (resp. $\mathbb{R}^+$). Also, let F be a positive (resp. negative) primitive function of f such that $\sum_{t=1}^{\infty} F(t)$ is convergent. Then F is uniformly limit summable on $[0, +\infty)$ and we obtain all results of Theorem 3.1.*

Corollary 3.4. *If the conditions of Theorem 3.1(b) hold, then we obtain the following interesting equality*

$$M_{[a,b]}(f_\sigma) = AM_{[a,b]}(F) + \gamma(F),$$

where $M_{[a,b]}(f_\sigma)$ and $AM_{[a,b]}$ denotes the "mean value of f_σ" and "arithmetic mean of F" on $[a, b]$, respectively. Also, we obtain the following formula

$$\gamma(F) = \frac{\int_a^b f_\sigma(x) - \sum_a^b F(k)}{b - a},$$

for evaluating the Euler-type constant of F.

Example 3.5. *Let $q > 2$ be a fixed real number and we define the function $f : [1, +\infty) \to \mathbb{R}$ by $f(x) = \frac{1}{x^q}$. Since the series $\sum_{k=1}^{\infty} \frac{1}{k^q}$ and $\sum_{k=1}^{\infty} \frac{k^{1-q}}{1-q}$ are convergent and also $f_{\sigma_n}(x)$ is uniformly convergent on every bounded subset of $[0, +\infty)$, then*

$$f_\sigma(x) = \zeta(q) - \zeta(q, x + 1); \quad x \geq 0,$$

$$\Phi(x) = x\zeta(q) + \frac{1}{q-1}\zeta(q - 1, x + 1); \quad x \geq 0,$$

$$F_\sigma(x) = \frac{1}{q-1}\Big(\zeta(q - 1, x + 1) - \zeta(q - 1)\Big); \quad x \geq 0.$$

Hence, we have

$$F_\sigma(x) - \Phi(x) = \frac{1}{q-1}\Big(- \zeta(q - 1) - x\zeta(q)\Big), \quad x \geq 0,$$

that is (at most) linear (as it is expected).

Also, putting $a = 0$ and $b = 1$ in the relation (3.3), we obtain

$$\int_0^1 \zeta(q, x + 1)dx = \frac{1}{q-1}.$$

The following theorem not only gives a criteria for limit summablity of primitive functions of a given function, but also provide an inequality for it with some applications.

Theorem 3.6. *Let $f : [1, +\infty) \to \mathbb{R}^+$ be a continuously differentiable real function with negative derivative and F being its primitive function such that $\int_1^\infty F(t)dt$ is convergent. Also, $\Phi_n(x)$ be the primitive function of $f_{\sigma_n}(x)$ defined by (3.1). Then F is uniformly limit summable on $[0, M)$ for every $M > 0$, and we have*

$$\Big|F_\sigma(x) - \Phi(x)\Big| \leq xf(1) + \Big|F(1) + x\big(F(1) - f(1)\big)\Big| + \int_1^\infty \Big(f(t) + \big|F(t)\big|\Big)dt; \quad x \geq 0 \tag{3.5}$$

and

$$\Big|\Phi(x)\Big| \leq \Big|F_\sigma(x)\Big| + xf(1) + \Big|F(1) + x\big(F(1) - f(1)\big)\Big| + \int_1^\infty \Big(f(t) + \big|F(t)\big|\Big)dt. \tag{3.6}$$

Proof. According to the assumption, we find that F is positive or negative from a number on. Because, by putting $S := \sup\{F(x) : x \geq 1\}$ we can consider the following cases for S. If $0 < S < \infty$, then $F(x_0) > 0$ for some $x_0 \geq 1$, hence $F(x) > 0$ for all $x \geq x_0$. But, if $S \leq 0$, then $F(x) \leq S \leq 0$, hence $F(x) \leq 0$ for all $x \geq 1$. Notice that if $S = \infty$, then $\lim_{x\to\infty} F(x) = \infty$. Thus, we can find some $M > 0$ such that $F(x) > 1$, for all $x \geq M$, which contradicts the assumption of convergence $\int_1^\infty F(t)dt$. Therefore, $\sum_{k=1}^\infty F(k)$ is convergent and $\lim_{t\to\infty} F(t) = 0$. As a result $\int_1^\infty f(t)dt$ is convergent and $\sum_{k=1}^\infty f(k)$ is so. Since F is concave and f is decreasing on $[1, +\infty)$ and also $R(F, 1) = R(f, 1) = 0$, Theorem 1.1 and 1.2 show that the functions f and F are uniformly limit summable on every interval $[0, M)$. Therefore, according to the equation (3.1), the functional sequence $\Phi_n(x)$ is convergent to $\Phi(x)$. Now, fix a real number $x \geq 0$, if F is positive from a number on, then there exists positive integer k_0 such that

$$0 \leq F(k + x) \leq F(k + [x] + 1); \quad k \geq k_0.$$

Since the series $\sum_{k=1}^\infty F(k)$ is convergent, then $\sum_{k=1}^\infty F(k + x)$ is so $\big($for every $x \geq 0\big)$. Hence, due to the convergence of $\sum_{k=1}^\infty f(k)$, we have

$$\Big|\Phi(x)\Big| = \Big|\sum_{k=1}^\infty \Big(f(k)x - F(k + x)\Big)\Big| \leq \sum_{k=1}^\infty f(k)x + \sum_{k=1}^\infty \Big|F(k + x)\Big|; \quad x \geq 0.$$

Now, by using the Euler's summation formula for the function $g_x(t) = F(t) - f(t)x$ $\big(t \geq 0\big)$, we obtain

$$\Big|\sum_{1\leq t\leq n} g_x(t)\Big| = \Big|g_x(1) - x\Big(F(n) - F(1)\Big) + \int_1^n F(t)dt + \int_1^n \Big(f(t) - xf'(t)\Big)\{t\}dt\Big|$$

$$\leq \Big|F(1) - f(1)x - x\Big(F(n) - F(1)\Big)\Big| + \int_1^n \Big(f(t) + \big|F(t)\big|\Big)dt - x\Big(f(n) - f(1)\Big).$$

Applying Theorem 3.1 (a) we arrive at

$$\Big|F_\sigma(x) - \Phi(x)\Big| \leq xf(1) + \Big|F(1) + x\big(F(1) - f(1)\big)\Big| + \int_1^\infty \Big(f(t) + \big|F(t)\big|\Big)dt; \;\; x \geq 0.$$

Finally, the above inequality implies (3.6). □

Example 3.7. *For any real number $r < -2$, put $f(x) = x^r$. The function f with $D_f = [1, +\infty)$ satisfies the conditions of Theorem 3.6, then F is uniformly limit summable on $[0, M)$, for every $M > 0$, and*

$$F_\sigma(x) = \frac{1}{r+1}\sigma\Big(x^{r+1}\Big) = \frac{1}{r+1}\Big(\zeta(-r-1) - \zeta(-r-1, x+1)\Big),$$

$$\Phi(x) = x\zeta(-r) - \frac{1}{r+1}\zeta(-r-1, x+1).$$

Therefore, according to the relation (3.5), we have

$$\Big|F_\sigma(x) - \Phi(x)\Big| = \Big|\frac{1}{r+1}\zeta(-r-1) - x\zeta(-r)\Big| \leq x + \frac{rx-1}{r+1} - \frac{1}{r+2}; \quad x \geq 0,$$

thus

$$\Big|\frac{\zeta(-r-1) - (r+1)\zeta(-r)x}{r+1}\Big| \leq \frac{2xr^2 + 5rx - 2r + 2x - 3}{(r+1)(r+2)}; \quad x \geq 0.$$

In particular if $x = 0$ and $t = -r-1$, then $\zeta(t) \leq \frac{2t-1}{t-1}$ for $t > 1$.

Example 3.8. *Consider the real function $f(x) = a^x$ in which $0 < a < 1$. By applying Theorem 3.6, we conclude that F is uniformly limit summable on $[0, M)$, for every $M > 0$, and*

$$F_\sigma(x) = \frac{1}{\ln a}\sigma\Big(a^x\Big) = \frac{a^{x+1} - a}{\ln a(a-1)},$$

$$\Phi(x) = \lim_{n\to\infty}\Phi_n(x) = \frac{a^{x+1} - ax\ln a}{\ln a(a-1)}.$$

Therefore

$$\Big|F_\sigma(x) - \Phi(x)\Big| = \frac{a - ax\ln a}{\ln a(a-1)}.$$

Thus, the inequalities (3.5) hold, and we get

$$0 \leq \frac{a - ax\ln a}{\ln a(a-1)} \leq ax - \frac{a(1 + x - x\ln a)}{\ln a} + \frac{a(1 - \ln a)}{(\ln a)^2}; \quad x \geq 0,$$

or

$$\Big((2a-1)(\ln a)^2 + (1-a)\ln a\Big)x + \big(1 - 2a\big)\ln a + a - 1 \leq 0; \qquad x \geq 0.$$

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