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FIRST KIND r-WHITNEY NUMBERS FOR COMPLEX ARGUMENTS

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Abstract. First kind r-Whitney numbers for complex arguments are defined and an asymptotic formula is obtained using Hankel identity.

1. Introduction

The first kind r-Whitney numbers of integral parameters denoted by $S_{n,m}^{\alpha,r}$ introduced in [4] and [6], may be defined by the following vertical generating function,

$$(1+\alpha z)^{\frac{r}{\alpha}}[log(1+\alpha z)^{\frac{1}{\alpha}}]^m = m!\sum_{n=0}^{\infty} S_{n,m}^{\alpha,r}\frac{z^n}{n!}, \tag{1.1}$$

where arguments n, m are positive integers and α, r are rational numbers with $\alpha \neq 0$.

Applying Cauchy's integral formula to (1.1)

$$S_{n,m}^{\alpha,r} = \frac{n!}{m!2\pi i}\int_C \frac{(1+\alpha z)^{\frac{r}{\alpha}}[log(1+\alpha z)^{\frac{1}{\alpha}}]^m}{z^{n+1}}dz, \tag{1.2}$$

where C is any contour enclosing the origin and lying within the circle $|\alpha z| = 1$.

Following what is done in [10] let $\nu = \frac{r}{\alpha}$ and

$$f(qw) = \frac{log(1+qw) - qw}{qw},$$

$$q = \frac{2}{m}, \qquad \alpha z = qw.$$

Then (1.2) becomes

$$S_{n,m}^{\alpha,r} = \binom{n}{m}\left(\frac{\alpha}{q}\right)^{n-m}\frac{(n-m)!}{2\pi i}\int_C \frac{(1+qw)^{\nu}(f(qw)+1)^{\frac{2}{q}}}{w^{n-m+1}}dw.$$

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If we let

$$T(q,w,\nu)=exp\left\{w+\nu log(1+qw)+\frac{2}{q}log[f(qw)+1]\right\}, \tag{1.3}$$

then

$$S_{n,m}^{\alpha,r}=\binom{n}{m}\left(\frac{\alpha}{q}\right)^{n-m}\frac{(n-m)!}{2\pi i}\int_C\frac{e^{-w}T(q,w,\nu)}{w^{n-m+1}}dw. \tag{1.4}$$

On the other hand, the Hankel identity gives an integral representation of the reciprocal Gamma function. This integral is given by

$$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_{\mathcal{H}}(-t)^{-z}e^{-t}dt,$$

where the contour $\mathcal{H}$ is a contour that starts at $+\infty$ above the positive real axis with $arg(t)=0$, encircles the origin in the positive direction and goes back to $+\infty$ below the positive real axis with $arg(t)=2\pi$.

Observe that we can write the right hand side of the Hankel identity into the form

$$\frac{i}{2\pi}(-1)^{-z}\int_{\mathcal{H}}e^{-t}t^{-z}dt. \tag{1.5}$$

So we consider the function $f(t)=e^{-t}t^{-z}$ and the logarithm is taken to be the branch defined on $(|t|>0, 0<\mathit{phase}\, t<2\pi)$. On this branch, $(-1)^{-z}=e^{-i\pi z}$. Thus, from the Hankel identity, we obtain

$$\int_{\mathcal{H}}e^{-t}t^{-z}dt=\frac{-2\pi i e^{-i\pi z}}{\Gamma(z)}. \tag{1.6}$$

A proof of Hankel identity may be done by applying integration through a branch cut to the integral in (1.6) which is tantamount to deforming the Hankel contour $\mathcal{H}$ into the two sides of the interval $[r,+\infty)$ together with the circle $|t|=r$. A similar deformation is done in [7]. For a discussion on the use of integration through a branch cut, see [3].

Equation (1.6) will be used in section 2 to obtain an asymptotic formula for the first kind r-Whitney numbers with complex arguments. In section 3 the range of validity of the expansion obtained in section 2 when used as an asymptotic approximation is established.

2. First Kind r-Whitney Numbers with Complex Arguments

Following [5] we use Hankel contour to extend (1.4) to define the r-Whitney numbers with complex arguments x and y.

Definition 2.1. The r-Whitney numbers with complex arguments x and y are defined by

$$S_{y,x}^{\alpha,r}=\binom{y}{x}\left(\frac{\alpha}{q}\right)^{y-x}\frac{(y-x)!}{2\pi i}\int_{\mathcal{H}}\frac{e^{-w}T(q,w,\nu)}{w^{n-m+1}}dw, \tag{2.1}$$

where $\mathcal{H}$ is the Hankel contour used in the Hankel identity and $T(q, w, \nu)$ is defined in (1.3) with

$$q = \frac{2}{x}, \qquad \alpha z = qw, \qquad \nu = \frac{r}{\alpha},$$

$$f(qw) = \frac{log(1 + qw) - qw}{qw}$$

and all factorials are defined via the Gamma function. The branch of the multivalued-function $w^{-(y-x+1)}$ is taken to be the branch defined on $D = \{w : |w| > 0, \quad 0 < phase\ w < 2\pi\}$.

We then recall the following lemma which is proved in [10].

Lemma 2.2. [10] *$T(q, w, \nu)$ defined in (1.3) has a Maclaurin expansion given by*

$$T(q, w, \nu) = 1 + \sum_{k=1}^{\infty} T_k(w, \nu) q^k, \tag{2.2}$$

where $T_k(w, \nu)$ is a polynomial in w whose lowest power in w is at least k.

For $k = 0, 1, 2$ the values of $T_k(w, \nu)$ are

$$T_0(w, \nu) = T(0, w, \nu) = 1,$$

$$T_1(w, \nu) = \nu w + \frac{5}{12} w^2,$$

$$T_2(w, \nu) = \frac{1}{2} \left\{ [\nu^2 - \nu] w^2 + \left[\frac{5}{6}\nu - \frac{1}{2} \right] w^3 + \frac{25}{144} w^4 \right\}.$$

From (1.4) and using (2.2), we have

$$S^{\alpha,r}_{n,m} = \binom{n}{m} \left(\frac{\alpha}{q}\right)^{n-m} \frac{(n-m)!}{2\pi i} \int_C \frac{e^{-w}}{w^{n-m+1}} dw + \int_C \sum_{k=1}^{\infty} \frac{T_k(w, \nu) q^k e^{-w}}{w^{n-m+1}} dw.$$

We are now ready to state our theorem.

Theorem 2.3. *For complex arguments y, x, α, r with $\alpha \neq 0$, the r-Whitney numbers of the first kind $S^{\alpha,r}_{y,x}$ satisfy the expansion*

$$\begin{aligned} S^{\alpha,r}_{y,x} = \binom{y}{x} \left(\frac{\alpha x}{2}\right)^{y-x} e^{-i\pi(y-x)} \Bigg[1 &+ \frac{1}{x} \left(-2\nu(y-x) + \frac{5}{6}(y-x)_2 \right) \\ &+ \frac{1}{x^2} \left(2(\nu^2 - \nu)(y-x)_2 - \left(\frac{5}{3}\nu - 1 \right)(y-x)_3 + \frac{25}{72}(y-x)_4 \right) \\ &+ \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{\sum_{k=3}^{\infty} T_k(w, \nu) q^k e^{-w}}{w^{y-x+1}} dw \Bigg], \end{aligned}$$

where $(y-x)_k = \frac{(y-x)!}{(y-x-k)!} = \frac{\Gamma(y-x+1)}{\Gamma(y-x-k+1)}$ and $T_k(w, \nu)$ is a polynomial in w whose lowest power in w is at least k.

Proof. It is observed from the proof in [10] that Lemma 2.2 holds even for complex qw. Moreover, the Maclaurin expasion converges for $0 < |qw| < \infty$. Consequently, as in the case of integer arguments we have

$$\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{-w}T(q,w,\nu)}{w^{y-x+1}}dw = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{-w}}{w^{y-x+1}}dw + \frac{1}{2\pi i}\int_{\mathcal{H}} \sum_{k=1}^{\infty} \frac{T_k(w,\nu)q^k e^{-w}}{w^{y-x+1}}dw.$$

Using (1.6) , we have

$$\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{-w}}{w^{y-x+1}}dw = \frac{1}{2\pi i}\int_{\mathcal{H}} e^{-w}w^{-(y-x+1)}dw = \frac{e^{-i\pi(y-x)}}{\Gamma(y-x+1)};$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{T_1(w,\nu)qe^{-w}}{w^{y-x+1}}dw &= \frac{q}{2\pi i}\int_{\mathcal{H}} \frac{(\nu w + \frac{5}{12}w^2)e^{-w}}{w^{y-x+1}}dw \\ &= q\left(\nu\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{-w}}{w^{y-x}}dw + \frac{5}{12}\cdot\frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{-w}}{w^{y-x-1}}dw\right) \\ &= q\left(\nu\cdot\frac{-e^{-i\pi(y-x)}}{\Gamma(y-x)} + \frac{5}{12}\cdot\frac{-e^{-i\pi(y-x-1)}}{\Gamma(y-x-1)}\right) \\ &= qe^{-i\pi(y-x)}\left(\frac{-\nu}{\Gamma(y-x)} + \frac{5}{12}\cdot\frac{1}{\Gamma(y-x-1)}\right);\end{aligned}$$

$$\int_{\mathcal{H}} \frac{T_2(w,v)q^2e^{-w}}{w^{y-x+1}}dw = \frac{1}{2}q^2\int_{\mathcal{H}} \frac{\left([\nu^2-\nu]w^2 + \left(\frac{5}{6}\nu - \frac{1}{2}\right)w^3 + \frac{25}{144}w^4\right)e^{-w}}{w^{y-x+1}}dw,$$

where

$$\begin{aligned}\int_{\mathcal{H}} \frac{(\nu^2-\nu)e^{-w}}{w^{y-x-1}}dw &= \frac{(\nu^2-\nu)(-e^{-i\pi(y-x-1)})}{\Gamma(y-x-1)} \\ &= \frac{(\nu^2-\nu)e^{-i\pi(y-x)}}{\Gamma(y-x-1)},\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{H}} \frac{\left(\frac{5}{6}\nu-\frac{1}{2}\right)w^3}{w^{y-x+1}}e^{-w}dw &= \int_{\mathcal{H}} \frac{\left(\frac{5}{6}\nu-\frac{1}{2}\right)e^{-w}}{w^{y-x-2}}dw \\ &= \left(\frac{5}{6}\nu-\frac{1}{2}\right)\cdot\frac{-e^{-i\pi(y-x-2)}}{\Gamma(y-x-2)} \\ &= \frac{\left(\frac{5}{6}\nu-\frac{1}{2}\right)(-1)e^{-i\pi(y-x)}}{\Gamma(y-x-2)},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{H}} \frac{25}{144}w^4\frac{e^{-w}}{w^{y-x+1}}dw &= \frac{25}{144}\int \frac{e^{-w}}{w^{y-x-3}}dw \\ &= \frac{25}{144}\frac{-e^{-i\pi(y-x-3)}}{\Gamma(y-x-3)} \\ &= \frac{25}{144}\frac{e^{-i\pi(y-x)}}{\Gamma(y-x-3)}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathcal{H}} \frac{T_2(w,v)q^2e^{-w}}{w^{y-x+1}}dw &= \frac{1}{2}q^2e^{-i\pi(y-x)}\left(\frac{\nu^2-\nu}{\Gamma(y-x-1)} - \frac{\frac{5}{6}\nu-\frac{1}{2}}{\Gamma(y-x-2)}\right.\\ &\quad \left.+\frac{25}{144}\cdot\frac{1}{\Gamma(y-x-3)}\right)\\ &= \frac{e^{-i\pi(y-x)}}{x^2}\left(\frac{2(\nu^2-\nu)}{(y-x-2)!} - \frac{\frac{5}{3}\nu-1}{(y-x-3)!}\right.\\ &\quad \left.+\frac{25}{72(y-x-4)!}\right).\end{aligned}$$

The last equality above uses the property of the Gamma function $\Gamma(n+1) = n!$. For more properties of the Gamma function refer to [8], [9].

Consequently,

$$\begin{aligned}S^{\alpha,r}_{y,x} &= \binom{y}{x}\left(\frac{\alpha}{q}\right)^{y-x}\frac{(y-x)!}{2\pi i}\int_{\mathcal{H}}\frac{e^{-w}T(q,w,\nu)}{w^{y-x+1}}dw\\ &= \binom{y}{x}\left(\frac{\alpha}{q}\right)^{y-x}(y-x)!\left[\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{e^{-w}}{w^{y-x+1}}dw + \frac{1}{2\pi i}\int_{\mathcal{H}}\frac{T_1(w,v)qe^{-w}}{w^{y-x+1}}\right.\\ &\quad \left.+\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{T_2(w,\nu)q^2e^{-w}}{w^{y-x+1}}dw + \frac{1}{2\pi i}\int_{\mathcal{H}}\sum_{k=3}^{\infty}\frac{T_k(w,\nu)q^ke^{-w}}{w^{y-x+1}}dw\right]\\ &= \binom{y}{x}\left(\frac{\alpha x}{2}\right)^{y-x}(y-x)!\left[\frac{e^{-i\pi(y-x)}}{(y-x)!}\right.\\ &\quad + \frac{2}{x}e^{-i\pi(y-x)}\left(\frac{-\nu}{(y-x-1)!} + \frac{5}{12(y-x-2)!}\right)\\ &\quad + \frac{e^{-i\pi(y-x)}}{x^2}\left(\frac{2(\nu^2-\nu)}{(y-x-2)!} - \frac{\frac{5}{3}\nu-1}{(y-x-3)!} + \frac{25}{72(y-x-4)!}\right)\\ &\quad \left.\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{\sum_{k=3}^{\infty}T_k(w,\nu)q^ke^{-w}}{w^{y-x+1}}dw\right]\\ &= \binom{y}{x}\left(\frac{\alpha x}{2}\right)^{y-x}e^{-i\pi(y-x)}\left[1+\frac{1}{x}\left(-2\nu(y-x)+\frac{5}{6}(y-x)_2\right)\right.\\ &\quad + \frac{1}{x^2}\left(2(\nu^2-\nu)(y-x)_2 - \left(\frac{5}{3}\nu-1\right)(y-x)_3 + \frac{25}{72}(y-x)_4\right)\\ &\quad \left.+\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{\sum_{k=3}^{\infty}T_k(w,\nu)q^ke^{-w}}{w^{y-x+1}}dw\right],\end{aligned}$$

where $(y-x)_k = \frac{(y-x)!}{(y-x-k)!} = \frac{\Gamma(y-x+1)}{\Gamma(t-x-k+1)}$. That $T_k(w,\nu)$ is a polynomial in w whose lowest power in w is at least k follows from Lemma 2.2. □

The formula in Theorem 2.3 can be used as an exact formula. For example,

$$\begin{aligned}S^{\alpha,r}_{y,y-1} &= \binom{y}{y-1}\frac{\alpha(y-1)}{2}e^{-i\pi}\left[1+\frac{1}{y-1}(-2\nu)\right]\\ &= -\frac{\alpha}{2}y(y-1)+ry.\end{aligned}$$

$$
\begin{aligned}
S^{\alpha,r}_{y,y-2} &= \binom{y}{y-2}\left(\frac{\alpha}{2}\right)^2 e^{-2\pi i}\left[1+\frac{1}{y-2}(-4\nu+\frac{5}{6}(2)_2)+\frac{1}{(y-2)^2}(4(\nu^2-\nu))\right] \\
&= \frac{1}{8}y(y-1)\alpha^2\left[(y-2)^2+(y-2)(-4\nu+\frac{5}{3})+4(\nu^2-\nu)\right].
\end{aligned}
$$

3. Range of Validity

To use the formula in Theorem 2.3 as an exact formula beyond the displayed terms will involve tedious computation. Thus the need to compute the range of validity of the formula when used as an asymptotic approximation. Let

$$
E_s = \frac{(y-x)!q^k}{2\pi i}\int_{\mathcal{H}}\frac{\sum_{k=s+1}^{\infty}T_k(w,\nu)e^{-w}}{w^{y-x+1}}dw. \tag{3.1}
$$

Note that

$$
\begin{aligned}
\frac{(y-x)!}{2\pi i}q^k\int_{\mathcal{H}}\frac{T_k(w,\nu)e^{-w}}{w^{y-x+1}}dw &= \frac{(y-x)!}{2\pi i}q^k\int_{\mathcal{H}}\frac{e^{-w}\sum_{j=k}^{2k}a_jw^j}{w^{y-x+1}}dw \\
&= (y-x)!q^k\sum_{j=k}^{2k}a_j\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{e^{-w}}{w^{y-x+1-j}}dw \\
&= (y-x)!q^k\sum_{j=k}^{2k}\frac{-e^{-i\pi(y-x)}}{\Gamma(y-x+1-j)}a_j \\
&= (y-x)!\left(-e^{-i\pi(y-x)}\right)q^k\sum_{j=k}^{2k}\frac{a_j}{\Gamma(y-x+1-j)} \\
&= -e^{-i\pi(y-x)}q^k\sum_{j=k}^{2k}a_j\frac{\Gamma(y-x+1)}{\Gamma(y-x+1-j)}.
\end{aligned}
$$

Since a_j is a complex number for all $j, j=k,k+1,\cdots,2k$, there is a constant M such that $|a_j|\le M$. Thus, we have

$$
\left|\frac{(y-x)!q^k}{2\pi i}\int_{\mathcal{H}}\frac{T_k(w,\nu)e^{-w}}{w^{y-x+1}}dw\right| \le M|q^k|\sum_{j=k}^{2k}\left|\frac{\Gamma(y-x+1)}{\Gamma(y-x+1-j)}\right|.
$$

It is known that $\frac{\Gamma(y-x+1)}{\Gamma(y-x+1-j)} \sim (y-x)^j$, as $(y-x) \to \infty$ (see [1] p. 36). Hence,

$$\begin{aligned}
\left|\frac{(y-x)!q^k}{2\pi i}\int_{\mathcal{H}} \frac{T_k(w,\nu)e^{-w}}{w^{y-x+1}}dw\right| &\leq M\left|q^k\sum_{j=k}^{2k}(y-x)^j\right| \\
&= M\left|q^k(y-x)^{2k}\sum_{j=1}^{k}\frac{1}{(y-x)^j}\right| \\
&= M\left|q^k(y-x)^{2k}\sum_{j=1}^{k}\frac{1}{(y-x)^j}\right| \\
&\leq M|q^k||y-x|^{2k}\sum_{j=0}^{k}\left(\frac{1}{|y-x|}\right)^j.
\end{aligned}$$

Then,

$$\begin{aligned}
|E_s| &\leq \sum_{k=s+1}^{\infty} M\left|q^k(y-x)^{2k}\right|\left|\frac{1}{1-\left|\frac{1}{y-x}\right|}\right| \\
&\leq M\frac{1}{1-\left|\frac{1}{y-x}\right|}\left|\sum_{k=s+1}^{\infty} q^k(y-x)^{2k}\right| \\
&\leq M\frac{1}{1-\left|\frac{1}{y-x}\right|}\sum_{k=s+1}^{\infty}\left|\frac{2(y-x)^2}{x}\right|^k \\
&= M\frac{1}{1-\left|\frac{1}{y-x}\right|}\left|\frac{2(y-x)^2}{x}\right|^{s+1}\sum_{k=0}^{\infty}\left|\frac{2(y-x)^2}{x}\right|^k. \qquad (3.2)
\end{aligned}$$

The series in (3.2) converges provided that

$2(y-x)^2 = o(x) \Leftrightarrow (y-x)^2 = o(x)$ or $(y-x) = o(\sqrt{x}) = o(\sqrt{y}) \Leftrightarrow y - o(\sqrt{y}) = x$.

We have just proved the following theorem.

Theorem 3.1. *The first few terms in the expansion in Theorem 2.3 can be used as an asymptotic approximation for $S^{\alpha,r}_{y,x}$ valid when $x = y - o(\sqrt{y})$.*

Similar computation for integral values of y and x is done in [10] where $y = n$ and $x = m$. For the second kind r-Whitney numbers with real parameters, a similar computation is done in [2].

4. Conclusion

The Hankel contour is used to define the r-Whitney numbers with complex arguments as being the case here and that in [5] while a form of Hankel identity is used to obtain an asymptotic approximation of the numbers under consideration. In doing this research the authors revisited the Hankel identity and the required deformation of the Hankel contour to be able to derive the Hankel identity. The idea of the deformation of the Hankel contour is so clever as can be seen in [7] and [3]. As the authors continue to search for approximations for number sequences and polynomials with complex arguments, these tools are so handy for use whenever applicable.

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References

[1] B.C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, 1977.

[2] C.B.Corcino, R.B. Corcino, N.G. Acala, Asymptotic estimates for r-Whitney numbers of the second kind, *Journal of Applied Mathematics*, **2014**, Article ID 354053.

[3] R. Churchill and J. Brown, *Complex Variables and Applications*, McGraw-Hill, 4th ed.

[4] R.B. Corcino, C.B. Corcino and M.L. Herrera, On the limiting form of the differences of the generalized factorial, *Journal of Research in Science and Engineering*, **2**(3) (2005), pp. 1-7.

[5] Flajolet, P. and Prodinger, H.: Stirling Numbers for Complex Arguments and Hankel Contour, SIAM J. Discrete Math., **12**(2) (1999), 155-159.

[6] I. Mezo, A new formula for the Bernoulli polynomials, *Results in Mathematics*, **58**(3) (2010), pp. 329-335.

[7] F.W.J. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press, 1974.

[8] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[9] N.M. Temme, *Asymptotic Methods for Integrals*, World Scientific Publishing Co. Pte. Ltd., 2014.

[10] M.A.R.P. Vega and C.B. Corcino, More asymptotic formulas for the generalized Stirling numbers of the first kind, *Utilitas Mathematica*, **75**(2008), pp. 259-272.

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