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APPLICATIONS OF FRACTIONAL CALCULUS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. Fractional calculus is a very useful and simple means in obtaining particular solutions to certain nonhomogeneous linear differential equations. Our aim in this work is to obtain fractional solutions of the second order nonhomogeneous differential equation with Nishimoto's operator.

1. Introduction

The topic of fractional calculus, which has been widely studied, has gained considerable and popularity over the last three decades, due to its practices in many different areas of science and engineering ([6],[7],[9],[11]). Chemical analysis of fluids, heat transfer, diffusion, the Schrödinger equation, and material science are some areas where fractional calculus is used.

The fractional calculus operators and their generalizations ([1]-[3],[5],[8],[12]-[16]) have been used to solve some types of differential equations and fractional differential equations.

Riemann-Liouville fractional integration and fractional differentiation,

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \quad (t > a, \alpha > 0), $$

and

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t f(\tau)(t-\tau)^{n-\alpha-1} d\tau \quad (n-1 \leq \alpha < n), $$

where $n \in N$, N is the set of positive integers, Γ is Euler's gamma function.

Definition 1.1. (cf. ([8],[10])) Let $D = \{D^-, D^+\}$, $C = \{C^-, C^+\}$, C^- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$, C^+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$, D^- be a domain surrounded by C^-, D^+ be a domain surrounded by C^+ (here D contains the points over the curve C).

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Moreover, let $f(z)$ be a regular function in D $(z \in D)$,

$$f_\eta = (f)_\eta = \frac{\Gamma(\eta+1)}{2\pi i} \int_C \frac{f(t)}{(t-z)^{\eta+1}} dt \quad (\eta \neq -1, -2, ...), \tag{1.1}$$

$$f_{-n} = \lim_{\eta \to -n} f_\eta \quad \left(n \in \mathbb{Z}^+,\ \eta \in \mathbb{R}\right), \tag{1.2}$$

where $t \neq z$, $z \in C$,

$$-\pi \leq \arg(t-z) \leq \pi \text{ for } C^-,$$
$$0 \leq \arg(t-z) \leq 2\pi \text{ for } C^+,$$

then f_η $(\eta > 0)$ is said to be the fractional derivative of $f(z)$ of order η and f_η $(\eta < 0)$ is said to be the fractional integral of $f(z)$ of order $-\eta$, provided (in each case) that

$$|f_\eta| < \infty \quad (\eta \in \mathbb{R}). \tag{1.3}$$

Finally, let the fractional calculus operator N^η be defined by (cf. [8])

$$N^\eta = \left(\frac{\Gamma(\eta+1)}{2\pi i} \int_C \frac{dt}{(t-z)^{\eta+1}} \right) \quad (\eta \neq -1, -2, ...)$$

with

$$N^{-n} = \lim_{\eta \to -n} N^\eta \quad \left(n \in \mathbb{Z}^+,\ \eta \in \mathbb{R}\right).$$

We find it to be worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration which is defined above (cf.e.g. ([4],[8])).

Lemma 1.2. *(Linearity). Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_η and g_η exist, then*

$$(k_1 f + k_2 g)_\eta = k_1 f_\eta + k_2 g_\eta \tag{1.4}$$

where k_1 and k_2 are constants and $\eta \in \mathbb{R}; z \in \mathbb{C}$.

Lemma 1.3. *(Index law). Let $f(z)$ be an analytic and single-valued function. If $(f_\xi)_\eta$ and $(f_\eta)_\xi$ exist, then*

$$(f_\xi)_\eta = f_{\xi+\eta} = (f_\eta)_\xi \tag{1.5}$$

where $\xi, \eta \in \mathbb{R}; z \in \mathbb{C}$ and $\left| \frac{\Gamma(\xi+\eta+1)}{\Gamma(\xi+1)\Gamma(\eta+1)} \right| < \infty$.

Lemma 1.4. *(Generalized Leibniz rule). Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_η and g_η exist, then*

$$(fg)_\eta = \sum_{n=0}^{\infty} \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)\Gamma(n+1)} f_{\eta-n} g_n \tag{1.6}$$

where $\eta \in \mathbb{R}; z \in \mathbb{C}$ and $\left| \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)\Gamma(n+1)} \right| < \infty$.

Property 1.5. *For a constant λ,*

$$\left(e^{\lambda z}\right)_\eta = \lambda^\eta e^{\lambda z} \quad (\lambda \neq 0; \eta \in \mathbb{R}; z \in \mathbb{C}), \tag{1.7}$$

$$\left(e^{-\lambda z}\right)_\eta = e^{-i\pi\eta} \lambda^\eta e^{-\lambda z} \quad (\lambda \neq 0; \eta \in \mathbb{R}; z \in \mathbb{C}), \tag{1.8}$$

$$\left(z^\lambda\right)_\eta = e^{-i\pi\eta} \frac{\Gamma(\eta-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\eta} \quad \left(\eta \in \mathbb{R}; z \in \mathbb{C}; \left| \frac{\Gamma(\eta-\lambda)}{\Gamma(-\lambda)} \right| < \infty \right). \tag{1.9}$$

2. Main results

With the help of above lemmas, we have the following main results of this paper.

Theorem 2.1. *Let* $\psi \in \{\psi : 0 \neq |\psi_\gamma| < \infty\}$ *and* $\phi \in \{\phi : 0 \neq |\phi_\gamma| < \infty\}$. *Then the nonhomogeneous second order differential equation*

$$(r-\alpha)(r-\beta)\psi_2 + (\kappa + \mu r)\psi_1 + \ell\psi = \phi \quad (r \neq \{\alpha, \beta\}) \tag{2.1}$$

has particular solutions of the forms;

$$\psi = \left(\left\{ \phi_\gamma \left[(r-\alpha)(r-\beta)\right]^{\gamma-1} (r-\alpha)^{\frac{\mu\alpha+\kappa}{\alpha-\beta}} (r-\beta)^{-\frac{\mu\beta+\kappa}{\alpha-\beta}} \right\}_{-1} \times \left[(r-\alpha)(r-\beta)\right]^{-\gamma} (r-\alpha)^{-\frac{\mu\alpha+\kappa}{\alpha-\beta}} (r-\beta)^{\frac{\mu\beta+\kappa}{\alpha-\beta}} \right)_{-(1+\gamma)} \tag{2.2}$$

and

$$\psi = \left\{ \left[\phi_\gamma (r-\alpha)^{(2\gamma+\mu-2)} e^{-\frac{\kappa+\mu\alpha}{r-\alpha}} \right]_{-1} (r-\alpha)^{-(2\gamma+\mu)} e^{\frac{\kappa+\mu\alpha}{r-\alpha}} \right\}_{-(1+\gamma)} \tag{2.3}$$

where $\psi_n = d^n\psi/dr^n$ $(n = 0, 1, 2)$, $\psi_0 = \psi = \psi(r)$, $\phi = \phi(r)$ *is a given function,* α, β, κ, μ *and* ℓ *constants,* $\gamma = \frac{(1-\mu)\pm\sqrt{(\mu-1)^2-4\ell}}{2}$ *with* $(\mu-1)^2 \geq 4\ell$.

Proof. ***i)*** In the case of $\alpha \neq \beta$;

Applying the operator N^γ to both members of (2.1), we then obtain

$$\left[\psi_2 (r-\alpha)(r-\beta)\right]_\gamma + \left[\psi_1 (\kappa+\mu r)\right]_\gamma + (\psi\ell)_\gamma = \phi_\gamma. \tag{2.4}$$

Using $(1.4) - (1.6)$ we have

$$\left[\psi_2 (r-\alpha)(r-\beta)\right]_\gamma = \psi_{2+\gamma} (r-\alpha)(r-\beta) + \gamma\left[2r - (\alpha+\beta)\right]\psi_{1+\gamma} + \gamma(\gamma-1)\psi_\gamma \tag{2.5}$$

and

$$\left[\psi_1 (\kappa+\mu r)\right]_\gamma = \psi_{1+\gamma}(\kappa+\mu r) + \gamma\mu\psi_\gamma. \tag{2.6}$$

Making of the relations (2.5) and (2.6), rewriting (2.4) in the following form;

$$\psi_{2+\gamma}(r-\alpha)(r-\beta) + \psi_{1+\gamma}\left\{\gamma\left[2r-(\alpha+\beta)\right] + \kappa + \mu r\right\} + \psi_\gamma\left[\gamma(\gamma-1) + \gamma\mu + \ell\right] = \phi_\gamma. \tag{2.7}$$

Choosing γ such that

$$\gamma = \frac{(1-\mu) \pm \sqrt{(\mu-1)^2 - 4\ell}}{2}, \quad (\mu-1)^2 \geq 4\ell \tag{2.8}$$

then we obtain

$$\psi_{2+\gamma}(r-\alpha)(r-\beta) + \psi_{1+\gamma}\left[(2\gamma+\mu)r + \kappa - \gamma(\alpha+\beta)\right] = \phi_\gamma \tag{2.9}$$

from (2.7).

Next, writing

$$\psi_{1+\gamma} = u = u(r), \quad \left(\psi = u_{-(1+\gamma)}\right) \tag{2.10}$$

we obtain

$$u_1 + u\left[\frac{(2\gamma+\mu)r + \kappa - \gamma(\alpha+\beta)}{(r-\alpha)(r-\beta)}\right] = \phi_\gamma \frac{1}{(r-\alpha)(r-\beta)} \tag{2.11}$$

from (2.9). A particular solution to this linear first order differential equation is given by

$$u=\left\{\phi_\gamma\left[(r-\alpha)(r-\beta)\right]^{\gamma-1}(r-\alpha)^{\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{-\frac{\mu\beta+\kappa}{\alpha-\beta}}\right\}_{-1}$$
$$\times\left[(r-\alpha)(r-\beta)\right]^{-\gamma}(r-\alpha)^{-\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{\frac{\mu\beta+\kappa}{\alpha-\beta}}. \tag{2.12}$$

Thus we obtain the solution (2.2) from (2.10), (2.12).

ii) In the case of $\alpha=\beta$;

In this case we have

$$(r-\alpha)^2\psi_2+(\kappa+\mu r)\psi_1+\ell\psi=\phi \quad (r\neq\alpha) \tag{2.13}$$

from (2.1).

Operate N^γ to the both sides of equation (2.13), we have then

$$\left[\psi_2(r-\alpha)^2\right]_\gamma+\left[\psi_1(\kappa+\mu r)\right]_\gamma+(\psi\ell)_\gamma=\phi_\gamma. \tag{2.14}$$

Using $(1.4)-(1.6)$ we have

$$\left[\psi_2(r-\alpha)^2\right]_\gamma=\psi_{2+\gamma}(r-\alpha)^2+2\gamma(r-\alpha)\psi_{1+\gamma}+\gamma(\gamma-1)\psi_\gamma. \tag{2.15}$$

Making use of the relations (2.6), (2.15) we may write (2.14) in the following form

$$\psi_{2+\gamma}(r-\alpha)^2+\psi_{1+\gamma}\left[2\gamma(r-\alpha)+\kappa+\mu r\right]+\psi_\gamma\left[\gamma(\gamma-1)+\gamma\mu+\ell\right]=\phi_\gamma. \tag{2.16}$$

Choose γ such that

$$\gamma=\frac{(1-\mu)\pm\sqrt{(\mu-1)^2-4\ell}}{2}, \quad (\mu-1)^2\geq 4\ell$$

we have then

$$\psi_{2+\gamma}(r-\alpha)^2+\psi_{1+\gamma}\left[(2\gamma+\mu)r+\kappa-2\gamma\alpha\right]=\phi_\gamma \tag{2.17}$$

from (2.16).

Therefore setting

$$\psi_{1+\gamma}=w=w(r), \quad \left(\psi=w_{-(1+\gamma)}\right) \tag{2.18}$$

we have

$$w_1+w\left[\frac{(2\gamma+\mu)r+\kappa-2\gamma\alpha}{(r-\alpha)^2}\right]=\phi_\gamma\frac{1}{(r-\alpha)^2}. \tag{2.19}$$

A particular solution to this linear first order differential equation is given by

$$w=\left[\phi_\gamma(r-\alpha)^{(2\gamma+\mu-2)}e^{-\frac{\kappa+\mu\alpha}{r-\alpha}}\right]_{-1}(r-\alpha)^{-(2\gamma+\mu)}e^{\frac{\kappa+\mu\alpha}{r-\alpha}}. \tag{2.20}$$

Thus we obtain the solution (2.3) from (2.18) and (2.20).

Theorem 2.2. *Let* $\psi\in\{\psi:0\neq|\psi_\gamma|<\infty;\gamma\in\mathbb{R}\}$. *Then the homogeneous second order linear ordinary differential equation*

$$(r-\alpha)(r-\beta)\psi_2+(\kappa+\mu r)\psi_1+\ell\psi=0, \quad (r\neq\{\alpha,\beta\}) \tag{2.21}$$

has solutions of the forms;

$$\psi=k\left\{\left[(r-\alpha)(r-\beta)\right]^{-\gamma}(r-\alpha)^{-\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{\frac{\mu\beta+\kappa}{\alpha-\beta}}\right\}_{-(1+\gamma)} \tag{2.22}$$

and

$$\psi = k\left\{(r-\alpha)^{-(2\gamma+\mu)}\, e^{\frac{\kappa+\mu\alpha}{r-\alpha}}\right\}_{-(1+\gamma)} \tag{2.23}$$

where $\psi_n = d^n\psi/dr^n$ $(n=0,1,2)$, $\psi_0 = \psi = \psi(r)$ *is a given function,* α, β, κ, μ *and* ℓ *constants,* $\gamma = \frac{(1-\mu)\pm\sqrt{(\mu-1)^2-4\ell}}{2}$ *with* $(\mu-1)^2 \geq 4\ell$.

Proof. When $\phi = 0$ in Theorem 2.1

$$u_1 + u\left[\frac{(2\gamma+\mu)\,r+\kappa-\gamma(\alpha+\beta)}{(r-\alpha)(r-\beta)}\right] = 0 \tag{2.24}$$

and

$$w_1 + w\left[\frac{(2\gamma+\mu)\,r+\kappa-2\gamma\alpha}{(r-\alpha)^2}\right] = 0 \tag{2.25}$$

for $\alpha \neq \beta$ and $\alpha = \beta$, instead of (2.11) and (2.19), respectively.

Therefore, we obtain (2.22) for (2.24) and (2.23) for (2.25).

Theorem 2.3. *Let* $\phi \in \{\phi : 0 \neq |\phi_\gamma| < \infty; \gamma \in \mathbb{R}\}$. *Then the fractional differintegrated functions*

$$\begin{aligned}\psi = &\left(\left\{\phi_\gamma\left[(r-\alpha)(r-\beta)\right]^{\gamma-1}(r-\alpha)^{\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{-\frac{\mu\beta+\kappa}{\alpha-\beta}}\right\}_{-1}\right.\\ &\left.\times\left[(r-\alpha)(r-\beta)\right]^{-\gamma}(r-\alpha)^{-\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{\frac{\mu\beta+\kappa}{\alpha-\beta}}\right)_{-(1+\gamma)}\\ &+k\left\{\left[(r-\alpha)(r-\beta)\right]^{-\gamma}(r-\alpha)^{-\frac{\mu\alpha+\kappa}{\alpha-\beta}}(r-\beta)^{\frac{\mu\beta+\kappa}{\alpha-\beta}}\right\}_{-(1+\gamma)}\end{aligned} \tag{2.26}$$

satisfies (2.1) *for* $\alpha \neq \beta$. *And*

$$\begin{aligned}\psi = &\left\{\left[\phi_\gamma(r-\alpha)^{(2\gamma+\mu-2)}\, e^{-\frac{\kappa+\mu\alpha}{r-\alpha}}\right]_{-1}(r-\alpha)^{-(2\gamma+\mu)}\, e^{\frac{\kappa+\mu\alpha}{r-\alpha}}\right\}_{-(1+\gamma)}\\ &+k\left\{(r-\alpha)^{-(2\gamma+\mu)}\, e^{\frac{\kappa+\mu\alpha}{r-\alpha}}\right\}_{-(1+\gamma)}\end{aligned} \tag{2.27}$$

satisfies (2.13) *for* $\alpha = \beta$.

Proof. It is clear by Theorems 2.1 and 2.2.

3. Conclusion

In this paper, we apply the fractional calculus operator method for homogeneous and nonhomogeneous second order differential equations. The most important advantage of the method is that it can be applied for singular equations.

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References

[1] A. Akgül, M. Inc and D. Baleanu, *On Solutions of Variable-Order Fractional Differential Equations*, An International Journal of Optimization and Control: Theories & Applications, **7 1** (2017) 112-116.

[2] E. K. Akgül, *A Novel Method for the Space and Time Fractional Bloch-Torrey Equations*, Thermal Science, **22 1** (2018) S253-S258.

[3] D. Baleanu, O.G. Mustafa, R.P. Agarwal, *On the Solution set for a Class of Sequential Fractional Differential Equations*, Journal of Physics A-Mathematical and Theoretical **43** 385209 (2010) 1-7.

[4] L.M.B.C. Campos, *On the Solution of Some Simple Fractional Differential Equations*, Internat. J. Math. & Math. Sci., **13 3** (1990) 481-496.

[5] M.S. Hashemi, A. Akgül, *Solitary wave solutions of time–space nonlinear fractional Schrödinger's equation: Two analytical approaches*, Journal of Computational and Applied Mathematics **339** (2018) 147–160.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, (2006).

[7] K. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, (1993).

[8] K. Nishimoto, *Fractional Calculus, vols. I, II, III, IV and V*, Descartes Press, Koriyama, 1984, 1987, 1989, 1991 and (1996).

[9] K. Oldham, J. Spanier, *The Fractional Calculus; Theory and Applications of Differentiation and Integration to Arbitrary Order* (Mathematics in Science and Engineering, V), Academic Press (1974).

[10] M.D.Ortigueira, *Fractional Calculus for Scientists and Engineers*, Springer Dordrecht Heidelberg London New York (2011).

[11] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Enginering, vol. 198, Academic Press, New York, London, Tokyo and Toronto (1999).

[12] B. Ross, *Fractional Calculus and Its Applications*, Conference Proceedings held at the University of New Haven, June 1974, Springer, New York (1975).

[13] S.-T. Tu, D.-K. Chyan, H.M. Srivastava, *Some Families of Ordinary and Partial Fractional Differintegral Equations*, Integral Transform. Spec. Funct. **11** (2001) 291-302.

[14] R. Yilmazer, *N-Fractional Calculus Operator N^{μ} method to a Modified Hydrogen Atom Equation*, Math. Commun. **15** (2010) 489-501.

[15] R. Yilmazer, M. Inc, F. Tchier and D. Baleanu, *Particular Solutions of the Confluent Hypergeometric Differential Equation by Using the Nabla Fractional Calculus Operator*. Entropy, **18 2** 49 (2016) 1-6.

[16] R. Yilmazer, O. Ozturk, *On Nabla Discrete Fractional Calculus Operator for a Modified Bessel Equation*, Thermal Science, **22 1** (2018) S203-S209.

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